A CONSTRUCTIVE CONVERSE OF THE MEAN VALUE THEOREM.

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ABSTRACT. Consider the following converse of the Mean Value Theorem.

Let f be a differentiable function on $[a, b]$. If $c \in (a, b)$, then there are α and β in $[a, b]$ such that $(f(\beta) - f(\alpha))/(\beta - \alpha) = f'(c)$. Assuming some weak conditions to be mentioned in Section 3, Tong and Braza [3] were able to prove this statement. Unfortunately their proof does not provide a method to compute α and β. We give a constructive proof.

1. INTRODUCTION

Constructive mathematics tries to determine the constructive or computational content of mathematics. One sometimes distinguishes several varieties of constructive mathematics [2]. We prove a result that is acceptable to all of them, as we avoid non-constructive steps, but do not assume axioms that are classically false. We have to warn those who are familiar with [1] or [2]. Unlike Bishop we do not demand by definition that continuous functions are uniformly continuous on compact sets. Nor do we demand by definition that differentiable functions are uniformly continuous or continuously differentiable.

In constructive mathematics 'there exists an x' is interpreted as 'there is an effective construction for x'. A constructive proof of 'A or B' is a proof of A or a proof of B. In order to prove 'A or not A' we have to prove or refute A. As there will always be unsolved problems, we do not recognize the scheme $A \vee \neg A$, *Tertium non datur*, as a valid principle.

A *real number* x is a sequence of rational numbers $x(0), x(1), \ldots$, such that for all k there exists(!) an N satisfying $|x(N) - x(n)| < 1/k$, for all $n > N$. Let x and y be real numbers. x and y are *equal* ($x = y$) if for all k there exists an N such that $|x(n) - y(n)| < 1/k$, for all $n > N$.

x is *greater than* y ($x > y$) if there are k and N such that $|x(N) - y(N+n)| > 1/k$, for all n. Notice that if $x < y$ then $x < z$ or $z < y$ for all z.

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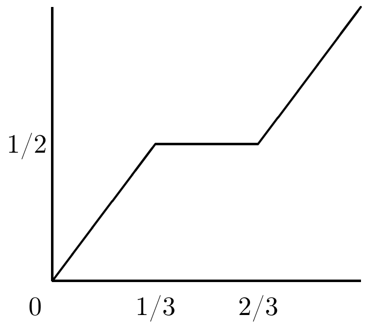

FIGURE 2.1. The function f

x is *not-greater-than* y ($x \leq y$) if not $x > y$. Finally, x is *apart from* y ($x \# y$) if $x > y$ or(!) $x < y$. Now $x = y$ if and only if not $x \# y$, but conversely it is not true in general that if not $x = y$ then $x \# y$. Addition, subtraction, multiplication, etc. are defined in the usual way.

2. A Constructive Mean Value Theorem

We start by giving weak counterexample in the style of L.E.J. Brouwer for the Intermediate Value Theorem. Let f be the function (Figure 2.1) from [0,1] to [0,1] given by:

$$f(x) := \inf(3x/2, 1/2) + \sup(3x/2 - 1, 0).$$

Define a function k_{99} from $\mathbb{N}$ to $\mathbb{N}$ by:

$$k_{99}(n) := \begin{cases} k & \text{if the first block of 99 nines starts at position } k \text{ in the} \\ & \text{decimal expansion of } \pi \text{ and } k < n, \\ n & \text{if such } k \text{ does not exist.} \end{cases}$$

Define a function t from $\mathbb{N}$ to $\mathbb{N}$ by: $t(n) := \frac{1}{2} + (-1/2)^{k_{99}(n)}$. Observe that t is a real number. Suppose we find x such that $f(x) = t$; then we are able to decide either $x < \frac{2}{3}$ or $x > \frac{1}{3}$. If $x < \frac{2}{3}$, then, if there exists a block of 99 nines in the decimal expansion of π the first one will start at an odd position. Similarly if $x > \frac{1}{3}$, then, if there exists a block of 99 nines in the decimal expansion of π, the first one will start at an even position. Both conclusions are unjustified.

Observe that this difficulty arises as soon as a function is constant on an interval.

Definition 2.1. Let f be a function on $[a, b]$ and let y be a real number. f is called *densely apart from* y if in every interval there exists a real number x such that $f(x) \# y$. If f is densely apart from all $y \in \mathbb{R}$, then f is called *locally nonconstant.*

If p is a polynomial function of degree at least one, then p is locally nonconstant (Cf. [1, Problem 17, p63]). The function f in Figure 2.1 is not densely apart from $1/2$.

Lemma 2.2. *If f is continuous on $[a,b]$, then there is a countable $T \subset \mathbb{R}$, such that if $s \# t$ for all $t \in T$, then f is densely apart from s.*

We express this fact as follows: f is densely apart from *all but countably many* real numbers.

Proof. Take $T := \{f(x) : x \in \mathbb{Q} \cap [a,b]\}$. $\square$

Lemma 2.3. [Intermediate Value Lemma] *Let f be continuous on $[a,b]$. If $f(a) < t < f(b)$ and f is densely apart from t, then there exists c in $[a,b]$ such that $f(c) = t$.*

Proof. We use successive bisection. Choose $x \in (a+\frac{b-a}{4}, a+\frac{3(b-a)}{4})$ for which $f(x) \# t$. This means that either $f(x) < t$ or $f(x) > t$. If $f(x) < t$ let $a_1 := x$ and $b_1 := b$, otherwise let $a_1 := a$ and $b_1 := x$. Now $f(a_1) < t < f(b_1)$ and $b_1 - a_1 < 3/4$. This process, applied recursively, produces sequences $a_0 < a_1 < \ldots$ and $b_0 > b_1 > \ldots$, such that for each i, $0 < b_i - a_i < (\frac{3}{4})^i(b-a)$ and $f(a_i) < t < f(b_i)$.
Therefore $c := \lim_{i\to\infty} a_i = \lim_{i\to\infty} b_i$ satisfies $f(c) = t$. $\square$

A direct consequence of Lemma 2.2 and Lemma 2.3 is the following constructive version of the Intermediate Value Theorem.

Theorem 2.4. [Intermediate Value Theorem] *Let f be continuous on $[a,b]$. For all but countably many t: if $f(a) < t < f(b)$ then there is c in $[a,b]$ satisfying $f(c) = t$.*

A countable set of exceptions may indeed occur. Consider Cantor's function (Figure 2.2). This is the unique continuous and nondecreasing function f, which is constant on every interval outside Cantor's discontinuum and satisfies $f(x) = 1/2$ for x in $[1/3, 2/3]$, $f(x) = 1/4$ for x in $[1/9, 2/9]$, $f(x) = 3/4$ for x in $[7/9, 8/9]$, etc.

We will obtain the Mean Value Theorem as a corollary of the following theorem.

Theorem 2.5. [Rolle] *Let f be differentiable on $[a,b]$. If f is locally nonconstant and $f(a) = f(b)$, then there exists $c \in (a,b)$, such that $f'(c) = 0$.*

Proof. Choose x close to $\frac{a+b}{2}$, such that $f(x) \# f(a)$. We assume $f(x) > f(a)$.
Let $\alpha_0 = a$, $\beta_0 = b$ and $\gamma_0 = x$. Choose y close to $\frac{3a+b}{4}$ and z close to $\frac{a+3b}{4}$, such that $f(y) \# f(\gamma_0)$ and $f(z) \# f(\gamma_0)$.
If $f(y) > f(\gamma_0)$, define $\alpha_1 = \alpha_0$, $\beta_1 = \gamma_0$ and $\gamma_1 = y$.

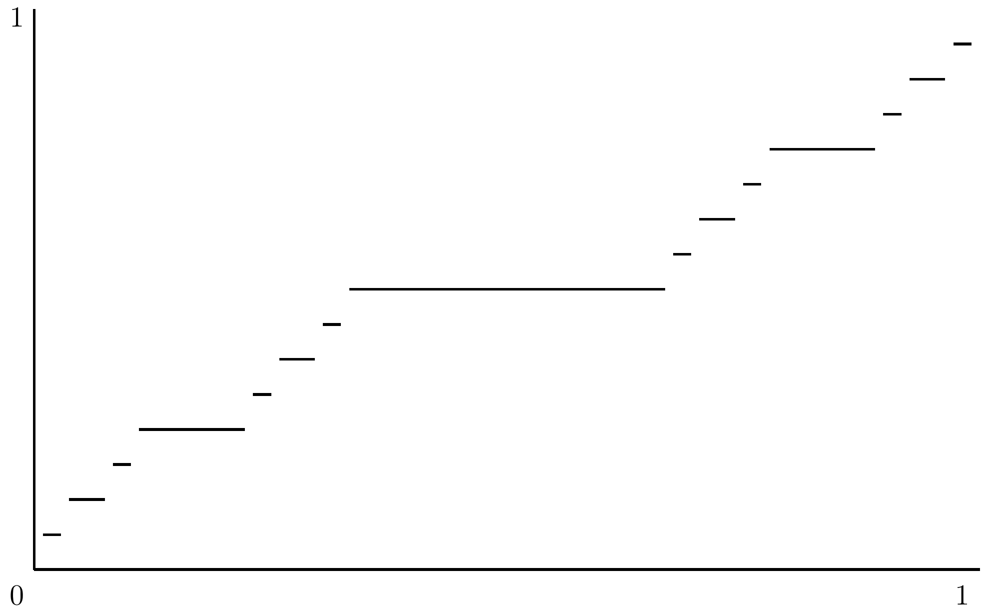

FIGURE 2.2. Cantor's function.

If this is not the case, but $f(z) > f(\gamma_0)$, define $\alpha_1 = \gamma_0$, $\beta_1 = \beta_0$ and $\gamma_1 = z$. Otherwise define $\alpha_1 = y$, $\beta_1 = z$ and $\gamma_1 = \gamma_0$.

Continuing in this way we obtain sequences $\alpha_0 \leq \alpha_1 \leq \ldots$, $\beta_0 \geq \beta_1 \geq \ldots$ and $\gamma_0, \gamma_1, \ldots$ all tending to the same limit, which we call γ. Because f is continuous and $f(a) < f(\gamma_0) \leq f(\gamma_1) \leq \ldots \leq f(\gamma)$ there is $\delta > 0$ such that $|\gamma - a| \geq \delta$. By a similar argument we see that $\gamma \# b$. Because for all n: $\alpha_n < \gamma_n < \beta_n$, $f(\alpha_n) < f(\gamma)$ and $f(\beta_n) < f(\gamma)$, it follows that $f'(\gamma) = 0$. $\square$

Let f be a function on $[a, b]$. For x and y in $[a, b]$, such that $x \# y$, define the difference quotient

$$\Delta_f(x, y) := \frac{f(x) - f(y)}{x - y}.$$

We omit the subscript when no confusion is possible.

Observe that for each z in $[a, b]$, such that $z \# x$ and $z \# y$

$$\begin{aligned} \Delta(x, y) &= \frac{(x - z)\Delta(x, z) + f(z) - f(y)}{x - y} \\ &= (\frac{x - z}{x - y})\Delta(x, z) + (1 - \frac{x - z}{x - y})\Delta(z, y). \end{aligned} \tag{2.1}$$

Theorem 2.6. [Mean Value Theorem] *Let f be differentiable on $[a, b]$. There is a countable set T, such that for all α and β, if $a \leq \alpha < \beta \leq b$ and $\Delta(\alpha, \beta)$ is apart from every t in T, then there is c in (α, β) such that $f'(c) = \Delta(\alpha, \beta)$.*

Proof. Define $T := \{\Delta(p,q) : p,q \in [a,b] \cap \mathbb{Q}, p \# q\}$. Suppose we have $\alpha, \beta \in [a,b]$, such that for all $t \in T$, $\Delta(\alpha,\beta) \# t$.
Define

$$g(x) := f(x) - f(\alpha) - \Delta_f(\alpha,\beta)(x-\alpha)$$

then $g(\alpha) = g(\beta) = 0$ and if $p,q \in [a,b] \cap \mathbb{Q}$ and $p \# q$, then

$$g(p) - g(q) = (p-q)(\Delta_f(p,q) - \Delta_f(\alpha,\beta)) \# 0.$$

By Rolle's Theorem there exists $c \in (a,b)$ such that $g'(c) = 0$, therefore $f'(c) = \Delta(\alpha,\beta)$. □

3. A constructive converse of the Mean Value Theorem

We will obtain a converse of the Mean Value Theorem in which we do not have to make exceptions as in the Theorems of Section 2. We need a few preparations.

Lemma 3.1. *Let f be differentiable on $[a,b]$. If $\Delta(a,b) > t$, then there is $y \in [a,b]$, satisfying $f'(x) > t$. If $f'(z) < t$ for some z in $[a,b]$ and $f'(x) \leq t$ for all x in $[a,b]$, then $\Delta(a,b) < t$.*

Proof. Suppose $\Delta(a,b) > t$, say $\Delta(a,b) = t + \epsilon$. Because $f'(a) \leq t$ there is an a' such that $\Delta(a,a') < t + \frac{1}{2}\epsilon$. Now $(t + \frac{1}{2}\epsilon, t+\epsilon)$ is uncountable, so by the Intermediate Value Theorem (2.4) we construct $x \in (a', b)$ such that $\Delta(a,x) > t + \frac{1}{2}\epsilon$. Moreover we choose $\Delta(a,x)$ outside the set of exceptions of the Mean Value Theorem (2.6) in order to find y in (a,x) satisfying $f'(y) = \Delta(a,x) > t + \frac{1}{2}\epsilon$.

Assume $z \in (a,b)$ and $f'(z) < t$. Because $f'(z) = \lim_{y \to z} \Delta(z,y)$ there is $y_0 > z$ such that $\Delta(z,y_0) < t$. Now apply the argument above to $[a,z]$ and $[y_0,b]$ and conclude that $\Delta(a,z) \leq t$ and $\Delta(y_0,b) \leq t$. Applying Formula 2.1 twice we find first $\Delta(a,y_0) < t$, and then $\Delta(a,b) < t$. □

Lemma 3.2. *Let f be differentiable on $[a,b]$. If $t < f'(x)$ and $\delta > 0$, then for all $z < x$ there exists w in $(x-\delta, x)$ apart from z, such that $t < f'(w)$ and $\Delta(z,w) \# t$.*

Proof. Choose $y \in (x-\delta, x)$, such that $z < y$, $\Delta(x,y) > (f'(x)+t)/2$ and, by Lemma 3.1, $f'(y) > t$. Let $r := \frac{z-x}{z-y}$ and choose $\epsilon < \frac{(f'(x)-t)}{2}\frac{1-r}{r}$. Now either $|\Delta(z,x) - t| > \epsilon/2$ or $|\Delta(z,x) - t| < \epsilon$. In the former case use Lemma 3.1 to take w close enough to x. In the latter case:

$$\begin{aligned} \Delta(z,y) &= r\Delta(z,x) + (1-r)\Delta(x,y) && \text{by Formula 2.1} \\ &\geq r(t-\epsilon) + (1-r)(f'(x)+t)/2 \\ &\geq t + (1-r)(f'(x)-t)/2 - \epsilon r \\ &> t && \text{by choice of } \epsilon \end{aligned}$$

So in this case let $w := y$. □

Theorem 3.3. *Let f be differentiable on $[a,b]$ and $\epsilon > 0$. If $a < c_1 < c_2 < b$ and $f'(c_1) < t < f'(c_2)$, then there exist α and β such that $\alpha < \beta$ and $\Delta(\beta,\alpha) = t$ and $\alpha \in (c_1, c_1+\epsilon)$ or $\beta \in (c_2-\epsilon, c_2)$.*

The condition $f'(c_1) < t < f'(c_2)$ is necessary: consider the function $f : [-1,1] \to \mathbb{R}$ given by $f(x) = x^3$ and let $t := 0$.

Proof. Because $f'(c_1) = \lim_{y \to c_1} \Delta(c_1, y)$, there exists y_0 such that $\Delta(c_1, y_0) < t$ and $c_1 < y_0 < c_2$. Lemma 3.2 provides z_0 in (y_0, c_2) such that $t < f'(z_0)$ and $\Delta(c_1, z_0) \# t$. By taking z_0 close enough to c_2 we ensure $\Delta(c_2, z_0) > t$. Now there are two possibilities: 1. $\Delta(c_1, z_0) > t$ or 2. $\Delta(c_1, z_0) < t$. We first consider case 1.

In classical mathematics one could simply define $\alpha := c_1$ and then use successive bisection in order to find β such that $\Delta(\alpha, \beta) = t$. In constructive mathematics we have to construct both α and β, but we may ensure that α is not too far away from c_1.

Let $\alpha_0 := c_1$. Now $\Delta(\alpha_0, z_0) > t$ and $\Delta(\alpha_0, y_0) < t$. Since Δ is continuous, there exists an open interval I containing α_0 such that for all x in I: $\Delta(x, z_0) > t$ and $\Delta(x, y_0) < t$. Let $y := \frac{z_0+y_0}{2}$. Lemma 3.2 applied to $-f$ provides $\alpha_1 < y_0$ in $I \cap (\alpha_0 - \epsilon/2, \alpha_0 + \epsilon/2)$ satisfying $f'(\alpha_1) < t$ and $\Delta(y, \alpha_1) \# t$. If $\Delta(\alpha_1, y) > t$ let $y_1 := y_0$ and $z_1 := y$, if $\Delta(\alpha_1, y) < t$ let $y_1 := y$ and $z_1 := z_0$. Thus $\alpha_1 \in (\alpha_0 - \epsilon/2, \alpha_0 + \epsilon/2)$ and $|z_1 - y_1| < \frac{|z_0 - y_0|}{2}$.

By repeating the above construction, we obtain sequences $(\alpha_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ such that for all n: $\Delta(\alpha_n, y_n) < t$ and $\Delta(\alpha_n, z_n) > t$. Let $\alpha := \lim_{n\to\infty} \alpha_n$ and $\beta := \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n$, and observe $\Delta(\alpha, \beta) = t$ and $|\alpha - c_1| < \epsilon$.
In case 2 we follow a similar construction and obtain the conclusion $|\beta - c_2| < \epsilon$. $\square$

By taking $t = f'(c)$ in the previous theorem one obtains a constructive weak converse of the Mean Value Theorem, in which α and β are found such that $f'(c) = \Delta(\beta, \alpha)$, possibly not satisfying $\alpha < c < \beta$. For the stronger conclusion we need two more lemmas.

Lemma 3.4. *If f is differentiable on $[a,b]$, then f' is strongly extensional, i.e. if $c, d \in [a,b]$ and $f'(c) \# f'(d)$, then $c \# d$.*

Proof. Observe that either $c \# d$ or, as we will assume, c and d are close enough to each other to find x, such that $\Delta(c,x) \# \Delta(d,x)$, i.e. $\frac{f(c)-f(x)}{c-x} \# \frac{f(d)-f(x)}{d-x}$. So $f(c) - f(x) \# f(d) - f(x)$ or $c - x \# d - x$. In the latter case the proof is complete. In the former case we remake that f is continuous and therefore $c \# d$. $\square$

Lemma 3.5. [Darboux]*Let f be differentiable on $[a,b]$. If $f'(d) < t < f'(e)$ and f' is densely apart from t, then there is c between d and e such that $f'(c) = t$.*

Proof. Define $g(x) := f(x) - tx$. Then $g'(d) < 0 < g'(e)$ and g' is densely apart from 0. So g is locally nonconstant.

We assume $d < e$. Because $g'(d) < 0 < g'(e)$, there is y between d and e, such that $g(y) < \inf\{g(d), g(e)\}$. So by the Intermediate Value Lemma (2.3) there are $z_1 \in [d,y]$ and $z_2 \in [y,e]$ such that $g(z_1) = g(z_2)$. By Rolle's Theorem (2.5) there exists c such that $g'(c) = 0$, therefore $f'(c) = t$. □

We now prove the promised constructive strong converse of the Mean Value Theorem. The conditions in this theorem are classically equivalent to: f' does not have a local extremum in c and c is not an accumulation point of $A_c := \{x \in (a,b) : f'(x) = f'(c)\}$. Tong and Braza [3] showed the statement is not true without these conditions: it suffices to consider the continuous function $g : [-1/2, 1/2] \to \mathbb{R}$ satisfying $g(x) := x^3 \sin(1/x) + x|x|/2$ for $x \# 0$, and $g(0) := 0$.

Theorem 3.6. *Let f be differentiable on $[a,b]$ and $\delta > 0$ such that for all x in $(c-\delta, c+\delta)$ apart from c: $f'(c) \# f'(x)$. If for all $\epsilon > 0$ there exist c_1 and c_2 in $(c-\epsilon, c+\epsilon)$, satisfying $f'(c_1) < f'(c) < f'(c_2)$, then there are α and β in (a,b) such that $\alpha < c < \beta$ and $\Delta(\alpha,\beta) = f'(c)$.*

Proof. Take c_1 and c_2 in $(c-\delta, c+\delta)$ satisfying $f'(c_1) < f'(c) < f'(c_2)$. Lemma 3.4 assures that $c_1 \# c$, say $c_1 < c$. Suppose that $c_2 < c$, then the Lemma 3.5 provides y, satisfying $c_1 < y < c$ and $f'(y) = f'(c)$, which contradicts the assumptions. Hence $c_2 \geq c$, moreover because $f'(c_2) \# f'(c)$ it follows that $c_2 > c$.
Theorem 3.3 provides α and β, such that $\alpha < \beta$ and $\Delta(\alpha,\beta) = f'(c)$. We may decide $\alpha < c$ or $c < \beta$. We only consider the case $\alpha < c$. By Lemma 3.1 we have $\Delta(\alpha, x) < f'(c)$ for $x \in [\alpha, c]$. So $\alpha < c < \beta$. □

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