Modalities in HoTT

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Outline

Two generalizations of Sets

Groupoids:

To keep track of isomorphisms we generalize sets to
groupoids (proof relevant equivalence relations)
2-groupoids (add coherence conditions for associativity),
...
weak ∞-groupoids

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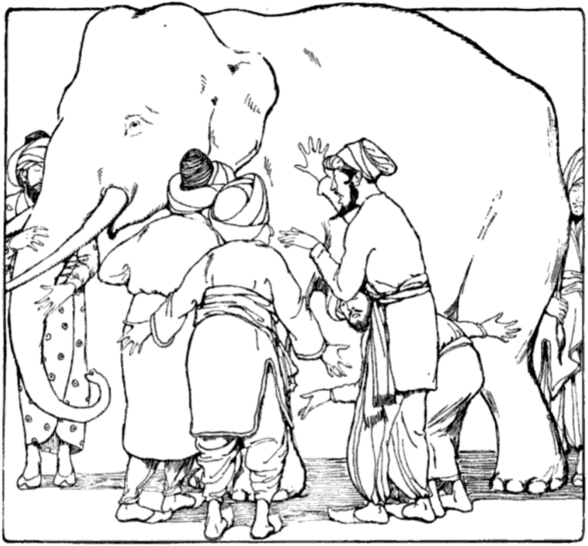
Weak ∞-groupoids are modeled by Kan simplicial sets.
(Grothendieck homotopy hypothesis)

Topos theory

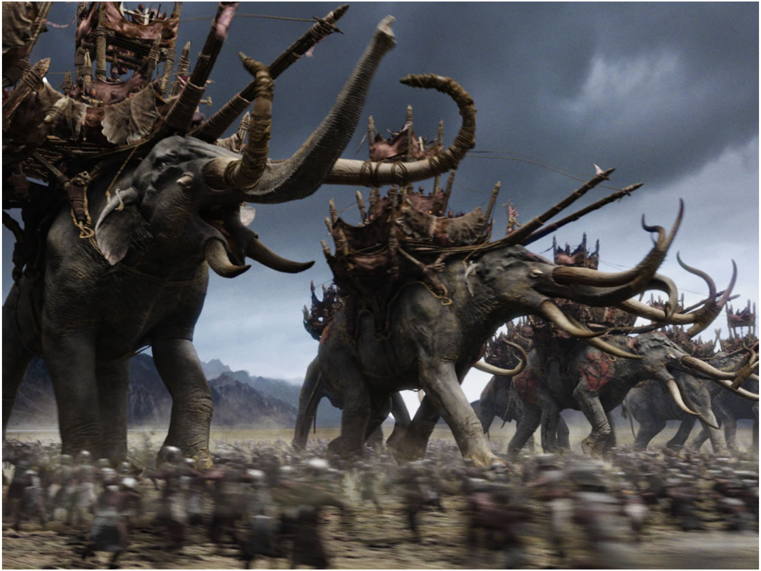


Topos theory

A topos is like:

- a semantics for intuitionistic formal systems
  model of intuitionistic higher order logic/type theory.
- a category of sheaves on a site (forcing)
- a category with finite limits and power-objects
- a generalized space

Higher topos theory



Higher topos theory

Combine these two generalizations of sets.

A higher topos is (represented by):
a model category which is Quillen equivalent to simplicial $Sh(C)_S$ for some model ∞-site (C, S)

Less precisely:

- a generalized space (presented by homotopy types)
- a place for abstract homotopy theory
- a place for abstract algebraic topology
- a semantics for Martin-Löf type theory with univalence (Shulman/Cisinski) and higher inductive types (Shulman/Lumsdaine). (current results are incomplete but promising)

Envisioned applications

Type theory with univalence and higher inductive types as the internal language for higher topos theory?

- higher categorical foundation of mathematics
- framework for large scale formalization of mathematics
- foundation for constructive mathematics
  e.g. type theory with the fan rule
- expressive programming language with a clear semantics (e.g. `cubical`)

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Towards elementary ∞-topos theory.
Effective ∞-topos?, glueing (Shulman),. . .
Coq formalization

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Towards elementary ∞-topos theory.
Effective ∞-topos?, glueing (Shulman),...
Coq formalization[1]

[1] https://github.com/HoTT/HoTT/

Type theory

Type theory is another elephant

- a foundation for constructive mathematics
  an abstract set theory ($\Pi\Sigma$).
- a calculus for proofs
- an abstract programming language
- a system for developing computer proofs

topos axioms

HoTT+UF gives:

- functional extensionality
- propositional extensionality
- quotient types

In fact, hSets forms a predicative topos (Rijke/Spitters)
as we also have a large subobject classifier

Large subobject classifier

The subobject classifier lives in a higher universe.

$$\begin{array}{ccc} B & \xrightarrow{\;!\;} & 1 \\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle \mathsf{True}} \\ A & \xrightarrow{\;P\;} & \mathsf{hProp_i} \end{array}$$

With propositional univalence, hProp classifies monos into A.

$$A, B : U_i \qquad hProp_i := \Sigma_{B:U_i} isprop(B) \quad hProp_i : U_{i+1}$$

Equivalence between predicates and subsets.
Use universe polymorphism (Coq). Check that there is some way to satisfy the constraints.
This correspondence is the crucial property of a topos.

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Sanity check: epis are surjective (by universe polymorphism).

higher toposes

Definition

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2. Locally cartesian closed
3. Has a subobject classifier (a “universe of truth values”)

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Definition (Rezk, Lurie, . . .)

A **higher topos** is an $(\infty, 1)$-category which is

1. Locally presentable (cocomplete and "small-generated")
2. Locally cartesian closed (has right adjoints to pullback)
3. Has **object** classifiers ("universes")

Object classifier

$Fam(A) := \{(B, \alpha) \mid B : Type, \alpha : B \to A\}$ (slice cat)
$Fam(A) \cong A \to Type$
(Grothendieck construction, using univalence)

$$\begin{array}{ccc} B & \xrightarrow{i} & \mathrm{Type}_\bullet \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\pi_1} \\ A & \xrightarrow{P} & \mathrm{Type} \end{array}$$

$\mathrm{Type}_\bullet = \{(B, x) \mid B : \mathrm{Type}, x : B\}$
Classifies all maps into A + group action of isomorphisms.
Crucial construction in ∞-toposes.
Grothendieck universes from set theory by universal property

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Grothendieck universes from set theory by universal property
Accident: $hProp_\bullet \equiv 1$?

Object classifier

Theorem (Rijke/Spitters)

In type theory, assuming pushouts, TFAE

1. *Univalence*
2. *Object classifier*
3. *Descent: Homotopy colimits (over graphs) defined by higher inductive types behave well.*

In category theory, $2, 3$ are equivalent characterizing properties of a higher topos (Rezk/Lurie).
Shows that univalence is natural.

Examples of toposes I

Example

The $(\infty, 1)$-category of ∞-groupoids is an ∞-topos. The object classifier $\mathscr{U}$ is the ∞-groupoid of (small) ∞-groupoids.

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Example

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Example

If $\mathcal{E}$ is an ∞-topos and $\mathcal{F} \subseteq \mathcal{E}$ is reflective with accessible left-exact reflector, then $\mathcal{F}$ is an ∞-topos: a sub-∞-topos of $\mathcal{E}$.

Every ∞-topos arises by combining these.

Examples of toposes II

Example

X a topological space; the $(\infty, 1)$-category $\mathsf{Sh}(X)$ of sheaves of ∞-groupoids on X is an ∞-topos.

For nice spaces X, Y,

- Continous maps $X \to Y$ are equivalent to ∞-topos maps $\mathsf{Sh}(X) \to \mathsf{Sh}(Y)$.
- Every subspace $Z \subseteq X$ induces a sub-∞-topos $\mathsf{Sh}(Z) \subseteq \mathsf{Sh}(X)$.

Outline

Topos-general mathematics

Idea

- We can "do mathematics" to apply generally in any ∞-topos.
- A single theorem yields results about many different models.

Example

The topos-general theory of "abelian groups" yields:

- In ∞-Gpd, classical abelian groups
- In $\mathsf{Sh}(X)$, sheaves of abelian groups
- In ∞-Gpd$/X$, local systems on X
- In presheaves on $\mathcal{O}(G)$, equivariant coefficient systems

Topos-general mathematics

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Example

The topos-general theory of "spectra" yields:

- In ∞-Gpd, classical stable homotopy theory
- In $\mathrm{Sh}(X)$, sheaves of spectra
- In ∞-Gpd$/X$, parametrized stable homotopy theory
- In presheaves on $\mathcal{O}(G)$, equivariant stable homotopy theory*

Topos-general mathematics

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Example

The topos-general construction of "Eilenberg–MacLane objects"

$$\text{abelian groups} \to \text{spectra}$$

can be done once and applied in all cases.

Eilenberg-MacLane object: For any abelian group G and positive integer n, there is an n-type $K(G,n)$ such that $\pi_n(K(G,n)) = G$, and $\pi_k(K(G,n)) = 0$ for $k \neq n$.

Internalization

Idea

We can "do mathematics" to apply generally in any ∞-topos.

There are two ways to do this:

1. Write mathematics in a "point-free" category-theoretic style, in terms of objects and morphisms.
2. Give a procedure that "compiles" point-ful mathematics to make sense in any ∞-topos — the internal logic / type theory.

Internalization – first style

Example

A group object in a category is

- an object G,
- a morphism $m : G \times G \to G$,
- the square

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{m \times 1} & G \times G \\ {\scriptstyle 1 \times m}\downarrow & & \downarrow{\scriptstyle m} \\ G \times G & \xrightarrow[m]{} & G \end{array}$$

  commutes.
- more stuff . . .

Internalization – second style

Example

A group is

- A set G,
- For each $x, y \in G$, an element $x \cdot y \in G$
- For each $x, y, z \in G$, we have $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,
- more stuff ...

Internalization – second style

Example

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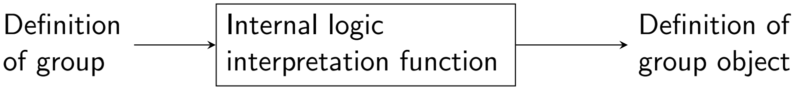

Outline

1. Higher toposes
2. Internal logic
3. Modalities
4. Sub-∞-toposes
5. Formalization

Modalities in Logic

In traditional logic:

- A "modality" is a unary operation on propositions like "it is possible that P" (denoted $\diamond P$) or "it is necessary that P" (denoted $\Box P$).
- Lawvere-Tierney topologies j: 'P holds locally'.
- j is an idempotent monad on the poset of propositions, while $\Box$ is a comonad.

Our "modalities" $\bigcirc$ are higher modalities, which act on all types, not just subterminals.
Idempotent monads on Type

Modalities

Two classes of examples of modalities:

- n-truncations
- Lawvere-Tierney j-operators (closure operators) on hProp.
  - $\neg\neg$
  - For u : hProp
    open modality $p \mapsto (u \Rightarrow p)$
    closed modality $p \mapsto (u \star p)$

Reflective subuniverses

Definition (in HoTT)

A reflective subuniverse consists of

- A predicate $\mathrm{in}_{\bigcirc} : \mathscr{U} \to \Omega$.
- A reflector $\bigcirc : \mathscr{U} \to \mathscr{U}$ with units $\eta_A : A \to \bigcirc A$.
- For all A we have $\mathrm{in}_{\bigcirc}(\bigcirc A)$.
- If $\mathrm{in}_{\bigcirc}(B)$, then $(- \circ \eta_A) : B^{\bigcirc A} \to B^A$ is an equivalence.

Examples: truncated types, $\neg\neg$-stable types

Lex Modalities

Definition (in HoTT)

A reflective subuniverse is a lex modality if $\bigcirc$ preserves pullbacks.

Lex=left exact, preserves finite limits

Modalities

Theorem (in HoTT)

A reflective subuniverse $\bigcirc$ is a modality if:
If $\mathsf{in}_\bigcirc(A)$ *and* $\forall(x : A)\ \mathsf{in}_\bigcirc(B(x))$, *then* $\mathsf{in}_\bigcirc(\sum_{x:A} B(x))$.

It is a lex modality if:
If $\bigcirc A = *$ *then* $\bigcirc(x = y) = *$ *for all* $x, y : A$.

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The types and type families that are $\mathsf{in}_\bigcirc$ are called modal.

Example

Every Lawvere-Tierney topology on Prop lifts to a lex modality.
The n-truncation τ_n, for any $n > -2$, is a non-lex modality.

Factorization systems

In an ∞-topos, a modality corresponds to a pullback-stable orthogonal factorization system $(\mathcal{L}, \mathcal{R})$:

- $\mathcal{R}$ = the maps $E \to B$ which are modal in $\mathcal{E}/B$.
- the factorization = the local reflection $A \to \bigcirc_B A \to B$.

Can be internalized in HoTT.

Example

For the n-truncation τ_n, we have the (n-connected, n-truncated) factorization system.
$n = -1$ epi-mono factorization

Accessibility in ∞-toposes

Definition

For a family $\{f_i : S_i \to T_i\}_{i \in I}$ of maps in $\mathcal{E}$, an object X is externally f-local if

$$\mathcal{E}(T_i, X) \xrightarrow{-\circ f_i} \mathcal{E}(S_i, X)$$

is an equivalence for all i.

Since $\mathcal{E}$ is locally presentable, if f is small then the externally f-local types are reflective.

Definition

A reflective subcategory is accessible if it consists of the externally f-local types for some (small) family $\{f_i\}$.

Accessibility in HoTT

Definition (in HoTT)

Given type families $S, T : I \to \mathscr{U}$ and a family of maps $f : \prod_{i:I}(S_i \to T_i)$, a type X is internally f-local if

$$X^{T_i} \xrightarrow{-\circ f_i} X^{S_i}$$

is an equivalence for all i.

With higher inductive types, the internally f-local types form a reflective subuniverse.

Definition

A reflective subuniverse is accessible if it consists of the internally f-local types for some family f.

Accessible modalities

Theorem (in HoTT)

An accessible reflective subuniverse is a modality iff it is generated by some $f : \prod_{i:I}(S_i \to *)$ ('nullification').

- Such an f is completely determined by a type family $S : I \to \mathcal{U}$, hence by a map $p : \sum_{i:I} S_i \to I$.
- internally f-local $\iff$ externally local for all pullbacks of p.

Example

The n-truncation τ_n is generated by $S^n \to *$ (with $I = *$).

Outline

The modal universe

- In HTT, the universe of a sub-∞-topos is constructed by an inexplicit local-presentability argument.
- In HoTT, we can be very explicit about it:

Theorem

For an accessible lex modality, the universe of modal types

$$\mathscr{U}_{\bigcirc} := \sum_{X:\mathscr{U}} \mathsf{in}_{\bigcirc}(X)$$

is again modal. Thus, it is an object classifier for the sub-∞-topos of modal types.

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Conversely

If $\bigcirc$ is a modality and $\mathscr{U}_{\bigcirc}$ is modal, then $\bigcirc$ is lex.

"A quasitopos with a (sub)object classifier is a topos."

Topological localizations

- In HTT, a topological localization is a left exact localization generated by monomorphisms.
- For internal localizations in HoTT:

Theorem (in HoTT)

If $S : I \to \Omega$ is a family of truth values, then its localization modality is lex.

Example

Hypercompletion is lex, but not topological.

The propositional fracture theorem, a.k.a. Artin gluing

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Gluing allows us to 'reconstruct' the topos
from the open and the closed modalities.

Example: Freyd cover

Scones, Logical Relations, and Parametricity

Outline

Formalization

All of this theory has been formalized (by Shulman) in the HoTT-library for Coq.
HoTT-library Bauer, Gross, Lumsdaine, Shulman, Sozeau, Spitters

Interesting use of module system:
A modality is an operator $\bigcirc$ which acts on types and satisfies a universal property that quantifies over all types. We need to express that $\bigcirc$ at level i has the universal propety with respect to every level j, not only i. We needed a construct like record types, but allowing each field to be individually universe-polymorphic. Modules do the job.

Perhaps, Set in agda?

Applications

- Coquand: stack models for independence of
- Program/proof transformations (judgemental variant/Coq plugin by Tabareau et al)
- New mathematics:
  generalized Blakers-Massey (Anel, Biedermann, Finster, Joyal)
- physics by cohesive higher toposes (Schreiber, Shulman)

Conclusion

- Modal type theory internalizes subtoposes from higher toposes
- Joint generalization of n-truncations and Lawvere-Tierney topologies
- three classes:
  - reflective universes, orthogonal factorization systems
  - modalities
  - lex modalities
- semantics in higher toposes

Basic theory of modalities (83pp) 1706.07526
formalization in the HoTT library