CONSTRUCTIVE ALGEBRAIC INTEGRATION THEORY

BAS SPITTERS

Abstract. For a long time people have been trying to develop probability theory starting from 'finite' events rather than collections of infinite events. In this way one can find natural replacements for measurable sets and integrable functions, but measurable functions seemed to be more difficult. We present a solution. Moreover, our results are constructive (in the sense of Bishop).

1. Introduction

In modern, set-theoretic, probability theory we represent the event that the start of a sequence of coin tosses is 'head' by the set of all infinite sequences of 0s and 1s beginning with a 0. From a conceptual point of view it would be more satisfying to model it by just one event, which one can take to be a primitive notion. In fact, this is the way probability theory started: the finite events were taken as primitive and infinite events were derived notions. This problem has been stressed ever since the introduction of set-theoretic methods in probability theory, see for instance [7, 4].

The focus on finite events, instead of infinite ones, is also characteristic for formal topology [9]. Ideas from formal topology have been used by Coquand and Palmgren [5] to develop the foundations of constructive algebraic probability theory. Their theory does not contain a theory of integrable or measurable functions. In this article we will develop a theory of these functions.

Another reason for developing measure theory in this way is that we feel that Bishop's integration theory is not entirely satisfactory. It uses the set of *all* partial functions. Not only is the construction of this set impredicative, but it also seems to be unlikely that Bishop's approach is useful when viewing Bishop-style mathematics as a high-level programming language [1, 8].

The present theory can be can also be seen as an alternative for Bishop and Bridges' approach [2] to measurable functions. In fact, this is the way in which it was presented in [11]. We feel that the theory presented here is both technically and conceptually simpler than the theory in [2].

In this article we reason constructively, but we will not assume any axioms that are classically false, so all our results are acceptable in Bishop-style mathematics [2].

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2. Preliminaries

The set of natural numbers will be denoted by $\mathbf{N}$. The set of real numbers will be denoted by $\mathbf{R}$. Apartness is denoted by $\neq$. Finally, if $A \subset \mathbf{R}$, then $\mathbf{R} \sim A$ denotes the set $\{x \in \mathbf{R} \,|\, \forall a \in A\, [a \neq x]\}$.

For other definitions we refer to Bishop and Bridges [2].

3. Measurable sets

We will now briefly sketch the Boolean ring approach to measurable sets taken by Coquand and Palmgren.

A commutative ring[1] A is *Boolean* if $x^2 = x$, for each x in A. For a Boolean ring we define for all $x, y \in A$, $x \wedge y := xy$, $x \vee y := x + y + xy$ and $x \leq y$ as $x = x \wedge y$. A binary relation $\neq$ on A is a *strong apartness relation* if the following hold for all $x, y, z \in A$:

(1) $\neg x \neq x$;
(2) $x \neq y \Rightarrow y \neq x$;
(3) $\neg x \neq y \Rightarrow x = y$;
(4) $x \neq y \Rightarrow x + z \neq y + z$;
(5) $xy \neq 0 \Rightarrow x \neq 0$ or $y \neq 0$;
(6) $x + y \neq 0 \Rightarrow x \neq 0$ or $y \neq 0$.

A *measure* on a Boolean ring A with a strong apartness relation $\neq$ is a function $\mu : A \to [0, \infty)$ such that for all x, y in A:

(1) $\mu(x \vee y) = \mu(x) + \mu(y) - \mu(x \wedge y)$,
(2) $\mu(x) > 0 \Rightarrow x \neq 0$.
The measure is *positive* if for all $x \in A$:
(3) $x \neq 0 \Rightarrow \mu(x) > 0$.

A simple, but important, observation is that when μ is positive the function $\rho : A \times A \to [0, \infty)$, defined by $\rho(x, y) := \mu(x + y)$ is a metric on A and that the operations $\cdot$ and $+$ are uniformly continuous with respect to this metric.

One can prove that the completion $\overline{A}$ of A with respect to this metric is again a measure ring. The completion $\overline{A}$ is σ-complete and μ is σ-additive in the following sense: if $(a_k)_{k \in \mathbf{N}}$ is a sequence in $\overline{A}$ and $a := \lim_{n \to \infty} \bigvee_{k=1}^{n} a_k$ exists with respect to the metric, then a is the smallest upper bound for the sequence $(a_k)_{k \in \mathbf{N}}$ and $\mu(a - \bigvee_{k=1}^{n} a_k) \to 0$ when $n \to \infty$.

One can also prove that a measure space in the sense of Bishop and Bridges [2] is a Boolean ring and that its completion is complete as a metric space.

4. Integrable functions

In this section we define the space of integrable functions as the completion of the metric space of "simple functions".

[1]We do not assume that a ring has a unit.

Let A be a fixed measure ring and μ a positive measure on A. We define the set $S(A)$ of *simple functions* as follows. An element of $S(A)$ consists of a finite lists of pairs $((c_i, a_i))_{i=1}^n$, where for all $i \in \{1, \dots, n\}$, $c_i \in \mathbf{R}$ and $a_i \in A$ and for all $i, j \in \{1, \dots, n\}$, $a_i \wedge a_j = 0$ when $i \neq j$. We will suggestively denote $((c_i, a_i))_{i=1}^n$ as $\sum_{i=1}^n c_i \chi_{a_i}$. One may think of a finite linear combination of characteristic functions. Let $*$ be any of the operations $\{+, -, \cdot, \wedge, \vee\}$ on $\mathbf{R}$. Define the function $*$ on $S(A)$ as follows:

$$(\sum_{i=1}^n c_i \chi_{a_i}) * (\sum_{j=1}^m d_j \chi_{b_j}) := \sum_{i=1}^n \sum_{j=1}^m (c_i * d_j) \chi_{a_i \wedge b_j}.$$

One may check that the usual relations between the operations hold. Let f be any continuous function on $\mathbf{R}$, define the function f on $S(A)$ by $f(\sum_{i=1}^n c_i \chi_{a_i}) = \sum_{i=1}^n f(c_i) \chi_{a_i}$. In particular, we have defined the absolute value function on $S(A)$ in this way. Define the linear functional I on $S(A)$ such that $I(\chi_a) = \mu(a)$. Moreover, define a semi-norm on $S(A)$ by $\|f\| = I(|f|)$. We make $S(A)$ into a normed space by identifying elements which have distance 0. From now on we assume that $S(A)$ is a normed space. The functions $+, -, \wedge, \vee$ and $|\cdot|$ are uniformly continuous with respect to the norm and $\cdot$ is continuous. Define $L_1(A)$ to be the the completion of $S(A)$ with respect to the norm. Elements of $L_1(A)$ are called *integrable functions*.

It is convenient to identify $\overline{A}$ with the subset $\{\chi_a : a \in \overline{A}\}$ of $L_1(A)$. The set $\overline{A}$ coincides with the set $\{g \in L_1(A) : g = g^2\}$ of idempotents in $L_1(A)$.

Using the theory of profiles [2] one can prove the following theorem.

Theorem 4.1. *If $f \in L_1(A)$, then there is a countable set $T \subset \mathbf{R}$ and a partial function $t \mapsto g_t \in \overline{A}$ defined on $\mathbf{R} \sim T$ such that for all $t \in \sim T$, $f g_t \geq t g_t$. Moreover, for each admissible $t > 0$ and each $\epsilon > 0$, there exists $\delta > 0$ such that*

$$|\mu(g_t - g_s)| < \epsilon$$

whenever $s > 0$ is admissible and $|t - s| < \delta$.

We will suggestively denote g_t by $[f \geq t]$.

Proof. The proof is an adaptation of the proof of Theorem 6.4.11 in [2]. Here we indicate the key steps.

Define the functions $h(x_0, x_1, .) : \{x \in \mathbf{R} : x \geq 0\} \to [0, 1]$ by

$$h(x_0, x_1, x) := (x_1 - x_0)^{-1} (\min(x, x_1) - \min(x, x_0)) \qquad (0 < x_0 < x_1).$$

As explained above one can define $h(b)$ first for all simple functions b and then extend it too all integrable functions b.

We may assume that $f \geq 0$. There is a sequence x_n of positive numbers such that if $t > 0$ and $t \neq x_n$ for all n, then $\{t\}$ is smooth relative to the (I, f)-profile. Consider such admissible $t > 0$. Choose N such that $t > 2^{-N}$. For each integer $n \geq N$ define

$$\phi_n = h(t - 2^{-n}, t - 2^{-n-1}, f)$$

and

$$\psi_n := h(t + 2^{-n}, t + 2^{-n-1}, f).$$

It is not difficult to show that $\phi := \lim \phi_n$ and $\psi := \lim \psi_n$ are well-defined. Observe that $\phi = \psi$, because $0 \leq \psi \leq \phi$ and $I(\psi) = I(\phi)$. Moreover $\phi_n \geq \phi_n^2 \geq \psi_n$ for all n, so $\phi \geq \phi^2 \geq \psi = \phi$, that is $\phi = \phi^2$. The rest of the proof is similar to the proof of Theorem 6.4.11 in [2]. The result is obtained by taking g_t equal to ϕ. □

5. Uniform spaces

We briefly discuss uniform spaces, a convenient generalization of metric spaces.

Definition 5.1. Let X be a set. A *pseudometric* ρ on X is a map from $X \times X$ to $\mathbf{R}$ such that for all $x, y, z \in X$, $\rho(x, y) \geq 0$, $\rho(x, y) = \rho(y, x)$ and $\rho(x, z) \leq \rho(x, y) + \rho(y, x)$. A *uniform space* (X, M) consists of a set X and a set M of pseudometrics on X, such that the relation $\neq$ defined on $X \times X$ by

$$x \neq y \Leftrightarrow \exists \rho \in M\, [\rho(x, y) > 0]$$

is a tight[2] apartness relation on X. A function $f : X \to Y$ from a uniform space (X, M) to a uniform space (Y, N) is *uniformly continuous* if for each d in N and each $\varepsilon > 0$ there are $\rho_1, \ldots, \rho_m$ in M and $\delta > 0$ such that for all $x, y \in X$,

$$\forall i \leq m\, [\rho_i(x, y) < \delta] \to d(f(x), f(y)) < \varepsilon.$$

If f is a uniformly continuous function from (X, M) to (Y, N) with an inverse which is also uniformly continuous, then f is called a *metric equivalence* and X and Y are *metrically equivalent*.

The family of subsets

$$U_{x,\varepsilon,\rho_1,\ldots,\rho_k} := \{y \in X : \forall i \leq k[\rho_i(x, y) < \varepsilon]\}$$

with $\varepsilon > 0$, $x \in X$ and $\rho_1, \ldots, \rho_k$ a finite sequence in M forms a neighborhood structure on X as defined in [2, Section 3.3]. The words open, closed and dense refer to this neighborhood structure.

Bishop's construction of the completion of a general uniform space [2, p.124] seems to use quantification over the class of all subsets of a set. Since we feel this class can not be surveyed as a whole, and since this construction is not possible in certain formal systems for constructive mathematics we prefer not to use it. We follow Bishop's advice '[...] to avoid pseudo-generality. (Separability hypotheses are freely employed.)' [2, p.3].

Definition 5.2. Let (X, M) be a uniform space. The uniform space (X, M) is *complete* if (X, M) is metrically equivalent to a uniform space $(Y, \{d\})$ and the metric space (Y, d) is complete. A uniform space (Y, N) is a *completion* of (X, M) if (X, M) is dense in (Y, N) and (Y, N) is complete.

[2]that is $\neg a \neq b$ implies $a = b$.

Any two completions of a uniform space are metrically equivalent, so we can talk about *the* completion of a uniform space.

6. Measurable functions

For a positive measure on a locally compact space, the space of integrable functions is the metric completion of the space of test functions, see [2, Cor. 6.2.17]. In this section this idea is extended to measurable functions. It is possible to define a metric on a σ-finite measure space so that the set of measurable functions is the completion of the set of test functions with respect to this metric. Instead of a metric we prefer to use a uniform structure.

To show how our approach works we will rewrite Sections 6.7 and 6.8 of Bishop and Bridges [2]. New definitions for 'measurable function' and 'convergence in measure' will be used. These new definitions will be shown to be equivalent to the ones used in [2].

Let (A, μ) be a fixed separable[3] measure ring and denote $L_1(A)$ by L_1. We will sometimes write $\int$ for I. When V is vector space with an order $\geq$ and zero vector 0, define $V^+ := \{v \in V : v \geq 0\}$.

For all $h \in L_1^+$ we define a pseudo-metric by

$$d_h(f, g) := \int |f - g| \wedge h \qquad f, g \in L_1.$$

If $h_1, h_2 \in L_1^+$, then for all $f, g \in L_1$,

$$|d_{h_1}(f, g) - d_{h_2}(f, g)| \leq \|h_1 - h_2\|_1.$$

So the uniform space $(L_1, \{d_h : h \in L_1^+\})$ is metrically equivalent to the uniform space $(L_1, \{d_h : h \in K\})$ when K is a dense subset of L_1^+. Let $\{h_1, h_2, \ldots\}$ be a dense set in L_1^+. Define the metric d by $d(f, g) := \sum_{n=1}^{\infty} 2^{-n}(d_{h_n}(f, g) \wedge 1)$, for all $f, g \in L_1$; then $(L_1, \{d_h : h \in L_1^+\})$ is equivalent to $(L_1, \{d\})$.

One may wonder why we use a uniform space instead of a metric space. There is in general no *canonical* metric space equivalent to $(L_1, \{d_h : h \in L_1^+\})$, so we prefer not to pick one. This makes the development of the theory is a little smoother. In the case that (A, μ) is *finite*, that is the ring A contains a unit 1, $\int |f - g| \wedge 1$ is a canonical metric, so the theory can be slightly simplified.

Convergence in the uniform space $(L_1, \{d_h : h \in L_1^+\})$ is called *convergence in measure*. So a sequence $(f_n)_{n \in \mathbf{N}}$ in L_1 converges to f in L_1 in measure if and only if for all finite sequences $h_1, \ldots, h_m \in L_1^+$ and each $\varepsilon > 0$, there exists N such that

$$d_{h_i}(f_n, f) < \varepsilon \qquad \text{(for all } n \geq N \text{ and } i \in \{1, \ldots, m\}).$$

[3]Separability is assumed to avoid complications with the completion of a uniform space, as indicated in section 5.

Because $\bigvee_{i=1}^m h_i \leq \sum_{i=1}^m h_i \leq m \bigvee_{i=1}^m h_i$, in order to show that a sequence $(f_n)_{n\in\mathbf{N}}$ in L_1 converges to f in L_1, it is enough to show that for all $h \in L_1^+$, and each $\varepsilon > 0$, there is N such that

$$d_h(f_n, f) < \varepsilon \qquad \text{(for all } n \geq N).$$

A sequence $(f_n)_{n\in\mathbf{N}}$ in $(L_1, \{d_h : h \in L_1^+\})$ is *Cauchy in measure* if for all $h \in L_1^+$ and each $\varepsilon > 0$ there is $N \in \mathbf{N}$ such that $d_h(f_n, f_m) < \varepsilon$, for all $n, m \geq N$.

Definition 6.1. An element in the completion of the uniform space $(L_1, \{d_h : h \in L_1^+\})$ is called a *measurable function*. The collection of measurable functions will be denoted by L_0.

We allow ourselves the poetic license to call an element of L_0 a function, but in Section 7 we show that elements of L_0 may be identified with a.e. defined functions.

Because for all $h \in L_1^+$: $d_h(|f|, |g|) \leq d_h(f, g)$ on $S(A)$, we can extend the operation $f \mapsto |f|$ from $S(A)$ to L_0. We then define the operations f^+, f^-, $\wedge$, $\vee$ and the relation $\leq$ on L_0, using $|\cdot|$. They extend the already defined operations and relations on L and the usual relations hold. For instance, to see that $|f + g| \leq |f| + |g|$, we have to show that $|f| + |g| - |f + g| \geq 0$, i.e. $||f| + |g| - |f + g|| = |f| + |g| - |f + g|$. But this holds on $S(A)$ and therefore on L_0.

Theorem 6.2. [Dominated convergence] *Let f be a measurable function, let $(f_n)_{n\in\mathbf{N}}$ be a sequence in L_1, and let g be an element of L_1 such that for all $n \in \mathbf{N}$, $|f_n| \leq g$. Suppose that $f_n \to f$ in measure. Then $f_n \to f$ in norm.*

Proof. Because $\int |f_m - f_n| = \int |f_m - f_n| \wedge 2g = d_{2g}(f_m, f_n) \to 0$ when $m, n \to \infty$, we see that there exists f' in L_1 such that $f_n \to f'$ in norm and hence in measure, so $f' = f$ and $f_n \to f$ in norm. □

We have not assumed that $f \in L_1$ as Bishop and Bridges did.

Theorem 6.3. *Let f be a measurable function and let g be an integrable function. If $|f| \leq g$, then $f \in L_1$.*

Proof. Let $h \in L_1^+$. If $f \leq g$, then for all $f' \in L_1$, $d_h(f' \wedge g, f) \leq d_h(f', f)$. Indeed,

$$\begin{aligned} |f' - f| \wedge h &= (f' - f)^+ \wedge h + (f' - f)^- \wedge h \\ &\geq (f' \wedge g - f)^+ \wedge h + (f' \wedge g - f)^- \wedge h \\ &= |f' \wedge g - f| \wedge h. \end{aligned}$$

If $(f_n)_{n\in\mathbf{N}}$ is a sequence in L_1 and $f_n \to_{d_g} f$, then $f_n \wedge g \to f$ in measure. Now apply Theorem 6.2 and use the completeness of L_1. □

Lemma 6.4. *The uniform space $(L_1, \{d_h : h \in L_1^+\})$ is metrically equivalent to the uniform space $(L_1, \{d_a : a \in A\})$.*

Proof. We prove that the injection map from $(L_1, \{d_h : h \in L_1^+\})$ to $(L_1, \{d_a : a \in A\})$ is uniformly continuous. In order to do so fix $\varepsilon > 0$ and $h \in L_1^+$. Because the simple functions are dense we can pick an $a \in A$ and $c \in \mathbf{R}$ such that $\int (h - ca)^+ < \varepsilon/2$. If f, g are elements of L_1 such that $d_a(f, g) < \varepsilon/2c$, then

$$d_h(f,g) = \int |f-g| \wedge h \leq c \int |f-g| \wedge a + \int (h-ca)^+ < \varepsilon. \qquad \square$$

Theorem 6.5. *Let $(f_n)_{n\in\mathbf{N}}$ be an increasing sequence of integrable functions. Then the sequence $(f_n)_{n\in\mathbf{N}}$ converges in measure to an integrable function if and only if $\lim_{n\to\infty} I(f_n)$ exists.*

Proof. Suppose that $l := \lim_{n\to\infty} I(f_n)$ exists. Then $\int (f_n - f_m) \to l - l = 0$ when $n, m \to \infty$. So the sequence $(f_n)_{n\in\mathbf{N}}$ is Cauchy in norm and hence converges to some $f \in L_1$.

Conversely, suppose that the sequence $(f_n)_{n\in\mathbf{N}}$ converges in measure to an integrable function $f \in L_1$. We may assume that $f_n \geq 0$ for all $n \in \mathbf{N}$. For all $m, n \in \mathbf{N}$, such that $m \geq n$,

$$\int f_m - f_n = \int |f_m - f_n| \wedge f,$$

which converges to 0 when $m, n \to \infty$. So the sequence $(\int f_m)_{m\in\mathbf{N}}$ converges. $\square$

Definition 6.6. A measure ring (A, μ) is *σ-finite* if there is a sequence $(a_n)_{n\in\mathbf{N}}$ such that $a := \lim_{n\to\infty} \bigvee_{k=1}^{n} a_k$ exists in L_0 and a is a unit with respect to the multiplication in L_0.

Because we assumed A to be separable, it follows that A is σ-finite.

Theorem 6.7. *Let (A, μ) be a σ-finite measure ring. Let f be a measurable function. Then there exist a countable set $T \subset \mathbf{R}$ and a partial function $t \mapsto g_t$ defined on $\mathbf{R} \sim T$ such that for all $t \in \sim T$, g_t is a measurable function such that $g_t = g_t^2$, $fg_t \leq tg_t$ and $f(1 - g_t) \geq t(1 - g_t)$.*

Moreover, for each admissible $t > 0$ and each $\epsilon > 0$ and $h \in L_1^+$, there exists $\delta > 0$ such that

$$d_h(g_t, g_s) < \epsilon$$

whenever $s > 0$ is admissible and $|t - s| < \delta$. Finally, $g_t \to 1$ in measure as $t \to \infty$.

Intuitively, for all $t \in \sim T$, g_t may be identified with $[f \leq t]$ or $[f < t]$.

Proof. We first assume that there is $N \in \mathbf{N}$ such that $0 \leq f \leq N = N \cdot 1$. Let a be in A. Then fa is integrable by Theorem 6.3. Theorem 4.1 supplies a countable set T such that for all $t \in \mathbf{R}^+ \sim T$, $[fa \leq t]$ is integrable.

We may drop the assumption that f is bounded, by noting that if $N \in \mathbf{N}$ and $t \in [-N/2, N/2]$ is admissible for fa, then $[fa \leq t] = [(fa \wedge N/2) + N/2 \leq t + N/2]$ and $(fa \wedge N/2) + N/2$ is bounded by N. We see that for all $a \in A$, there is a countable set T_a of exceptions.

Observe that for every pair of integrable sets a, b and $h \in L_1^+$ and every t which is admissible for fa and fb,

$$d_h([fa \leq ta], [fb \leq tb]) \leq d_h(a, b).$$

It follows that if $(a_m)_{m\in\mathbf{N}}$ is a sequence in A satisfying the conditions of Definition 6.6, then

$$g_t := \lim_{N\to\infty} [(\bigvee_{n=1}^{N} a_m) f \leq t]$$

exists for all $t \in \mathbf{R} \sim \bigcup_{n=0}^{\infty} T_{a_n}$. It is clear that g_t has the properties mentioned in the theorem. $\square$

Fix $m \in \mathbf{R}^+$. Define $S(A)_m := \{f \in S(A) : |f| \leq m\}$ and $L_{0,m} := \{f \in L_0 : |f| \leq m\}$.

Lemma 6.8. *Consider $S(A)$ with the uniform structure inherited from L_0. Multiplication from $S(A)_m \times S(A)_m$ to $S(A)$ is uniformly continuous. Since $S(A)_m$ is dense in $L_{0,m}$, the multiplication can be uniquely extended to a uniformly continuous function from $L_{0,m} \times L_{0,m}$ to L_0.*

Proof. We prove that the map $f \mapsto f^2$ is uniformly continuous from $S(A)_m$ to $S(A)$. The lemma then follows from the observation that $fg = \frac{1}{2}((f+g)^2 - f^2 - g^2)$.

By Lemma 6.4 we may consider the uniform space $(S(A), \{d_a : a \in A\})$. Let $f, g \in S(A)_m$, $\epsilon > 0$ and suppose that $d_a(f, g) \leq \epsilon$. Then

$$\begin{aligned} |f^2 - g^2| \wedge a &= |f-g||f+g| \wedge a \\ &\leq 2m(|f-g| \wedge a), \end{aligned}$$

so $d_a(f^2, g^2) \leq 2m\epsilon$. $\square$

Let p be a polynomial on $\mathbf{R}$ and f a measurable function. We define $p \circ f$ using Lemma 6.8.

Theorem 6.9. *Let P be the set of polynomials on $\mathbf{R}$. Let C be the uniform space of continuous functions with the sequence of pseudometrics $(\rho_n)_{n\in\mathbf{N}}$ defined by $\rho_n(f) := \sup_{[-n,n]} |f|$ for all $n \in \mathbf{N}$ and $f \in C$. The map $\circ : P \times L_{0,m} \to L_0$ defined by $\circ(p, f) := p \circ f$ is uniformly continuous and can therefore be uniquely extended to a uniformly continuous map from $C \times L_{0,m}$ to L_0. Moreover, for each $f \in L_0$ and each test-function ψ with support included in $[-n, n]$ we define $\psi \circ f := \psi \circ (f \wedge n \vee -n)$. The map $\circ_f : C_0 \to L_0$ defined by $\circ_f(\psi) = \psi \circ f$ is uniformly continuous and can therefore be extended uniquely to a uniformly continuous map from C to L_0. Here C_0 denotes the set of test-functions.*

Proof. Let $m \in \mathbf{R}^+$ and fix $\varepsilon > 0$ and $a \in A$. Suppose that $p \in P$ and $|p| \leq \varepsilon$ on $[-m, m]$. Then for all $f, g \in L_{0,m}$, $\int_a |p(f) - p(g)| \wedge 1 \leq 2\varepsilon\mu(a)$, because this holds for all simple functions f, g bounded by m. We see that the map $\circ$ is uniformly continuous on $P \times L_{0,m}$.

Fix a measurable function f. We assume that it is positive. Let ψ be a test-function. We use the notation of Theorem 6.7. For each $t \in \mathbf{R}$, which is admissible for f, fg_t is bounded by t. We write $\psi(fg_t)$ for $\psi \circ (fg_t)$. Let s be admissible for f and such that $s < t$. Then $\psi(fg_t)g_s = \psi(fg_s)$. So

$$\psi(fg_t) - \psi(fg_s) = \psi(fg_t)(g_t - g_s) + \psi(fg_t)g_s - \psi(fg_s) = \psi(fg_t)(g_t - g_s).$$

It follows that for every integrable set a, $d_a(\psi(fg_t), \psi(fg_s)) \leq d_a(g_t, g_s)$. Finally, remark that $g_t \to 1$ as $t \to \infty$. Consequently, for each a, we can fix $t \in \mathbf{R}^+$ such that for all $t', t'' \geq t$, $d_a(g_{t'}, g_{t''}) \leq \epsilon$. Consequently, for all test functions ψ, ψ' if $\rho_t(\psi - \psi') \leq \epsilon$, then $d_a(\psi(f), \psi'(f)) \leq \epsilon + 2\epsilon\mu(a)$. □

7. Equivalence with the other definitions.

Coquand and Palmgren showed that the algebra of integrable sets of a complete integration space is a metrically complete measure ring. It follows that L_1 in the sense of Bishop and Bridges is isomorphic as an ordered normed space with our L_1. We prove a similar result for measurable functions.

Proposition 7.1. *A sequence $(f_n)_{n\in\mathbf{N}}$ of integrable functions that is Cauchy in measure in the sense of* [2] *is d_h-Cauchy for every $h \in L_1^+$.*

Proof. By Theorem 6.4 we may restrict ourselves to the case $h = a$, where $a \in A$. Suppose that the sequence $(f_n)_{n\in\mathbf{N}}$ is Cauchy in the sense of [2]. Let $\varepsilon > 0$. Choose an integrable set $b \subset a$ and $N \in \mathbf{N}$ such that for all $n, m > N$: $|f_m - f_n| < \varepsilon/\mu(A)$ on b and $\mu(a - b) < \varepsilon$. Now

$$\begin{aligned} \int_a |f_m - f_n| \wedge 1 &= \int_{a-b} |f_m - f_n| \wedge 1 + \int_b |f_m - f_n| \wedge 1 \\ &\leq 2\varepsilon. \end{aligned}$$

□

Proposition 7.2. *Let $(f_n)_{n\in\mathbf{N}}$ be sequence of integrable functions that is Cauchy in the uniform space $(L_1, \{d_h : h \in L_1^+\})$. Then $(f_n)_{n\in\mathbf{N}}$ is Cauchy in measure in the sense of* [2].

Proof. Let a be an integrable set. Suppose that $f, g \in L_1$ and $d_a(f, g) < \alpha^2$ and α is admissible for $|f - g|$. Define $b := [|f - g| \leq \alpha]$ and observe that $\mu(a - b) \leq \alpha^2/\alpha = \alpha$.

Let $\varepsilon > 0$ be admissible for all integrable functions $|f_n - f_m|$, where n and m range over $\mathbf{N}$. Choose N such that for all $n, m \geq N$, $d_a(f_n, f_m) < \varepsilon^2$. Then for all $n, m \geq N$ there is an integrable set b such that $\mu(a - b) < \varepsilon$ and $|f_n - f_m|\chi_b \leq \varepsilon$. □

8. Convergence almost everywhere

As we have already seen convergence in norm and convergence in measure are topological properties which can be treated conveniently in a point-free way. One would hope that the same is true for convergence almost everywhere, that is, that there is a collection of open sets such that a sequence $(f_n)_{n\in\mathbf{N}}$ of measurable functions converges almost everywhere to a measurable function f if and only if for all open U, if $f \in U$, then for all sufficiently large n, $f_n \in U$. This is NOT the case as the following adaptation of a classical example shows. Our example uses the extended fan-theorem, an intuitionistic principle which is inconsistent with classical mathematics, but can be added consistently to Bishop-style mathematics. See for instance [12] or [3] for more on this principle.

Identify a finite sequence a in $\{0,1\}$ with the set $\{\alpha \in \{0,1\}^{\mathbf{N}} : \alpha \text{ starts with } a\}$ and define the measure μ on $\{0,1\}^{\mathbf{N}}$, by $\mu(\chi_a) := 2^{-n}$, where n is the length of a. For all $\alpha \in \{0,1\}^{\mathbf{N}}$, the sequence $n \mapsto \chi_{\bar{\alpha}n}$ converges to the 0-function almost everywhere. Fix an open neighborhood $U \subset L_1$ of the 0-function. Then for all α, there is an $N \in \mathbf{N}$ such that for all $n \geq N$, $\chi_{\bar{\alpha}n} \in U$. By the extended fan theorem there is N_U such that for all α and for all $n \geq N_U$, $\chi_{\bar{\alpha}n} \in U$. Hence U contains all functions associated with finite sequences that are sufficiently long.

Define the sequence $f_1, f_2, \ldots$ by $\chi_0, \chi_1, \chi_{00}, \chi_{01}, \chi_{10}, \chi_{11}, \chi_{000}, \ldots$ As we just showed, for all open U containing 0 there is $N \in \mathbf{N}$ such that for all $n \geq N$, $f_n \in U$, but the sequence $(f_n)_{n\in\mathbf{N}}$ does not converge almost everywhere to the 0-function; in fact it does not converge at any point. It follows that convergence in measure is not a topological property. This example also shows that almost uniform convergence is not a topological property.

We have been deliberately vague about what a topology is. But the argument above should work in any framework for constructive topology, for instance the one in [13].

9. Final remarks

We presented an algebraic treatment of integration theory including a theory of measurable functions. We mentioned a problem of this approach: it does not include convergence a.e. We conjecture that this can be avoided in a number of cases by considering convergence in measure instead of convergence a.e.

Treating integration theory algebraically has a been advocated by a number of researchers: A. Weil [14], Kolmogorov [7], Caratheodory [4], Segal [10], Fremlin [6]. The present approach seems to be new. It was well-known that the topology of convergence in measure is metrizable, but I have been unable to find a treatment of the set of measurable functions as a completion, even in classical mathematics.

One difference between our approach and the ones mentioned above is that we studied the measure space together with a measure, rather then the measure space on its own.

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