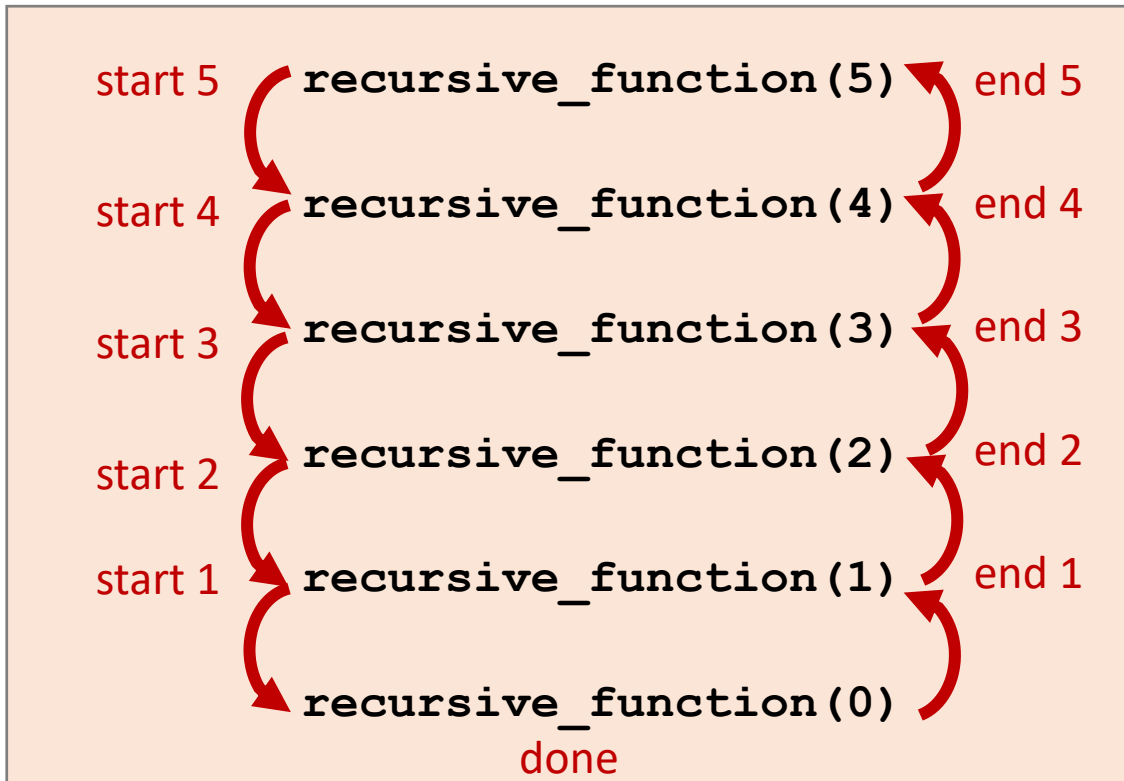


# Recursion

- symbol table
- stack frames

# Recursion

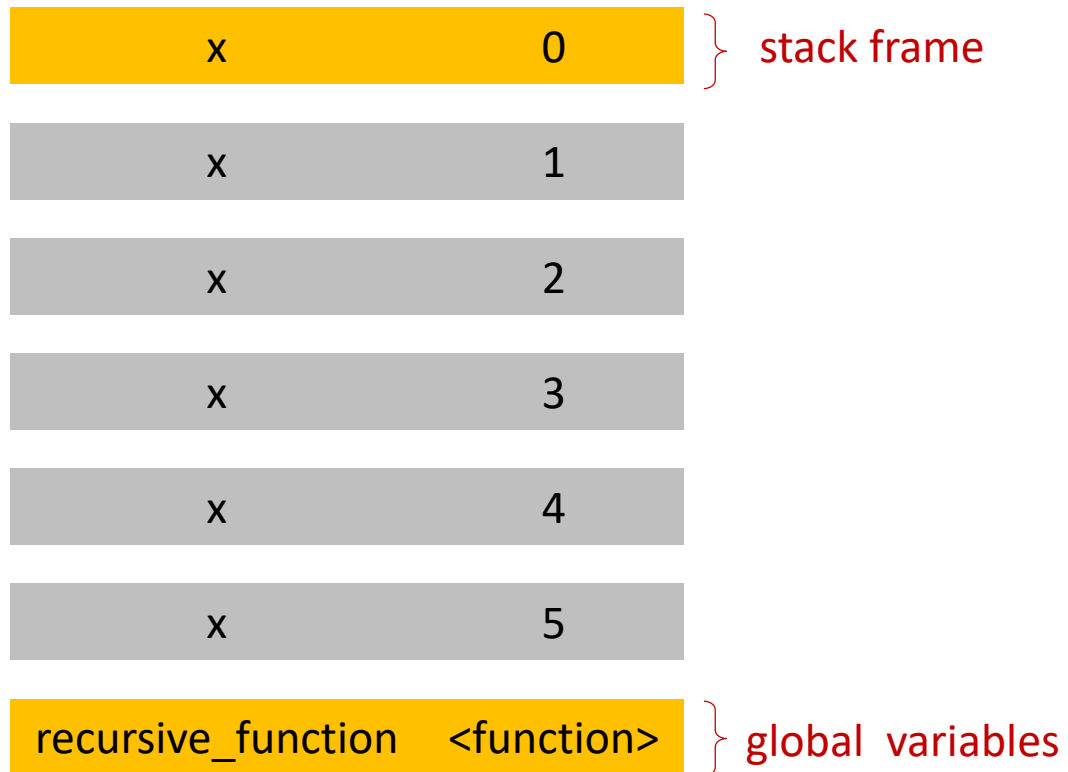
Recursive function  
≡  
"function that calls itself"



## Python shell

```
> def recursive_function(x):  
    if x > 0:  
        print("start", x)  
        recursive_function(x - 1)  
        print("end", x)  
    else:  
        print("done")  
  
> recursive_function(5)  
| start 5  
| start 4  
| start 3  
| start 2  
| start 1  
| done  
| end 1  
| end 2  
| end 3  
| end 4  
| end 5
```

# Recursion



Recursions stack when x = 0 is reached

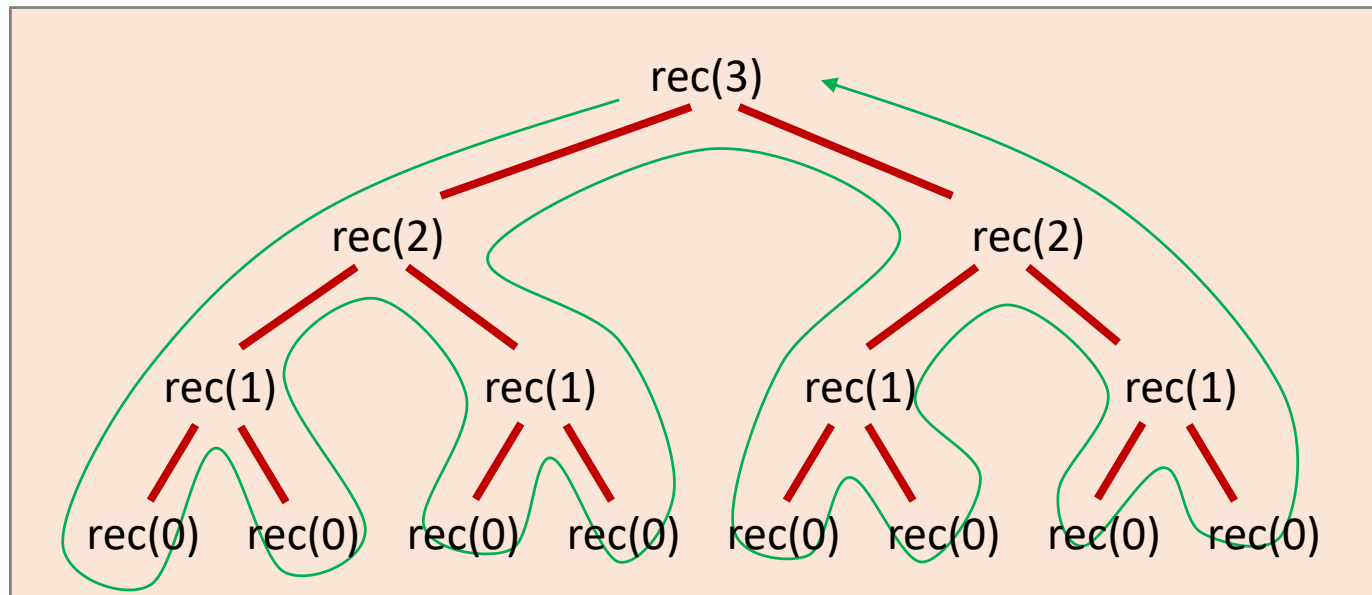
## Python shell

```
> def recursive_function(x):
    if x > 0:
        print("start", x)
        recursive_function(x - 1)
        print("end", x)
    else:
        print("done")

> recursive_function(5)
| start 5
| start 4
| start 3
| start 2
| start 1
| done
| end 1
| end 2
| end 3
| end 4
| end 5
```

## Python shell

```
> def rec(x):  
    if x > 0:  
        print("start", x)  
        rec(x - 1)  
        rec(x - 1)  
        print("end", x)  
    else:  
        print("done")
```



## Python shell

```
> rec(3)  
| start 3  
| start 2  
| start 1  
| done  
| done  
| end 1  
| start 1  
| done  
| done  
| end 1  
| end 2  
| start 2  
| start 1  
| done  
| done  
| end 1  
| done  
| done  
| end 1  
| end 2  
| end 3
```

Question – How many times does `rec(5)` print "done"?

```
Python shell
> def rec(x):
    if x > 0:
        print("start", x)
        rec(x - 1)
        rec(x - 1)
        rec(x - 1)
        print("end", x)
    else:
        print("done")
```

a) 3

b) 5

c) 15

d) 81

e) 125



f) 243

g) Don't know

# Factorial

$$n! = n \cdot \underbrace{(n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1}_{(n-1)!}$$

**Observation**  
(recursive definition)

$$1! = 1$$

$$n! = n \cdot (n-1)!$$

`factorial.py`

```
def factorial(n):  
    if n <= 1:  
        return 1  
    return n * factorial(n - 1)
```

`factorial_iterative.py`

```
def factorial(n):  
    result = 1  
    for i in range(2, n + 1):  
        result *= i  
    return result
```

# Binomial coefficient $\binom{n}{k}$

- $\binom{n}{k}$  = number of ways to pick  $k$  elements from a set of size  $n$

- $$\binom{n}{k} = \begin{cases} 1 & \text{if } k = 0 \text{ or } k = n \\ \binom{n-1}{k} + \binom{n-1}{k-1} & \text{otherwise} \end{cases}$$

```
binomial_recursive.py
```

```
def binomial(n, k):  
    if k == 0 or k == n:  
        return 1  
    return binomial(n - 1, k) + binomial(n - 1, k - 1)
```

- Unfolding computation shows  $\binom{n}{k}$  1's are added → **slow**

# Binomial coefficient $\binom{n}{k}$

Observation  $\binom{n}{k} = \frac{n!}{(n-k)! \cdot k!}$

```
binomial_factorial.py
```

```
def binomial(n, k):  
    return factorial(n) // factorial(k) // factorial(n - k)
```

- Unfolding computation shows  $2n - 2$  multiplications and 2 divisions → **fast**
- Intermediate value  $n!$  can have significantly more digits than result (**bad**)



# Binomial coefficient $\binom{n}{k}$

Observation  $\binom{n}{k} = \frac{n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)}{k \cdot (k-1) \cdot (k-2) \cdots 1} = \binom{n-1}{k-1} \cdot \frac{n}{k}$


```
binomial_recursive_product.py
```

```
def binomial(n, k):  
    if k == 0:  
        return 1  
    else:  
        return binomial(n - 1, k - 1) * n // k
```

- Unfolding computation shows  $2k - 2$  multiplications and  $k$  divisions  $\rightarrow$  **fast**
- Multiplication with fractions  $\geq 1 \rightarrow$  intermediate numbers limited size

# Questions – Which correctly computes $\binom{n}{k}$ ?

Observation  $\binom{n}{k} = \frac{n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)}{k \cdot (k-1) \cdot (k-2) \cdots 1}$

- a) binomial\_A
-  b) binomial\_B
- c) both
- d) none
- e) Don't know

## binomial\_iterative.py

```
def binomial_A(n, k):  
    result = 1  
    for i in range(k):  
        result = result * (n - i) // (k - i)  
    return result  
  
def binomial_B(n, k):  
    result = 1  
    for i in range(k) [::-1]:  
        result = result * (n - i) // (k - i)  
    return result
```

# Recursively print all leaves of a tree

- Assume a recursively nested tuple represents a tree with strings as leaves

## Python shell

```
> def print_leaves(tree):  
    if isinstance(tree, str):  
        print("Leaf:", tree)  
    else:  
        for child in tree:  
            print_leaves(child)  
  
> print_leaves(('a', ('b', 'c')))  
| Leaf: a  
| Leaf: b  
| Leaf: c
```

Question – How many times is `print_leaves` function called in the example?

Python shell

```
> def print_leaves(tree):  
    if isinstance(tree, str):  
        print("Leaf:", tree)  
    else:  
        for child in tree:  
            print_leaves(child)  
  
> print_leaves(('a', ('b', 'c')))  
| Leaf: a  
| Leaf: b  
| Leaf: c
```

a) 3

b) 4

 c) 5

d) 6

e) Don't know

# Collect all leaves of a tree in a set

## Python shell

```
> def collect_leaves_slow(tree):
    leaves = set()
    if isinstance(tree, str):
        leaves.add(tree)
    else:
        for child in tree:
            leaves |= collect_leaves_slow(child)
    return leaves
```

copies all labels  
from child from one  
set to another set

```
> collect_leaves_slow(('a', ('b', 'c')))
| {'a', 'c', 'b'}
```

## Python shell

```
> def collect_leaves_wrong(tree, leaves = set()):  
    if isinstance(tree, str):  
        leaves.add(tree)  
    else:  
        for child in tree:  
            collect_leaves_wrong(child, leaves)  
    return leaves
```



```
> def collect_leaves_right(tree, leaves = None):  
    if leaves == None:  
        leaves = set()  
    if isinstance(tree, str):  
        leaves.add(tree)  
    else:  
        for child in tree:  
            collect_leaves_right(child, leaves)  
    return leaves
```

```
> collect_leaves_wrong(('a', ('b', 'c')))  
| {'a', 'c', 'b'}  
> collect_leaves_wrong(('d', ('e', 'f')))  
| {'b', 'e', 'a', 'f', 'c', 'd'}
```

```
> collect_leaves_right(('a', ('b', 'c')))  
| {'b', 'a', 'c'}  
> collect_leaves_right(('d', ('e', 'f')))  
| {'f', 'd', 'e'}
```

## Python shell

```
> def collect_leaves(tree):
    leaves = set()

    def traverse(tree):
        nonlocal leaves # can be omitted
        if isinstance(tree, str):
            leaves.add(tree)
        else:
            for child in tree:
                traverse(child)

    traverse(tree)
    return leaves
```

```
> collect_leaves(('a', ('b', 'c')))
| {'b', 'a', 'c'}
```

```
> collect_leaves(('d', ('e', 'f')))
| {'f', 'd', 'e'}
```

# Maximum recursion depth ?

- Python's maximum allowed recursion depth can be increased by

```
import sys
sys.setrecursionlimit(1500)
```

## Python shell

```
> def f(x):
    print("#", x)
    f(x + 1)

> f(1)
| # 1
| # 2
| # 3
| ...
| # 975
| # 976
| # 977
| # 978
| RecursionError: maximum
| recursion depth exceeded
| while pickling an object
```