Algoritmer og Datastrukturer 2

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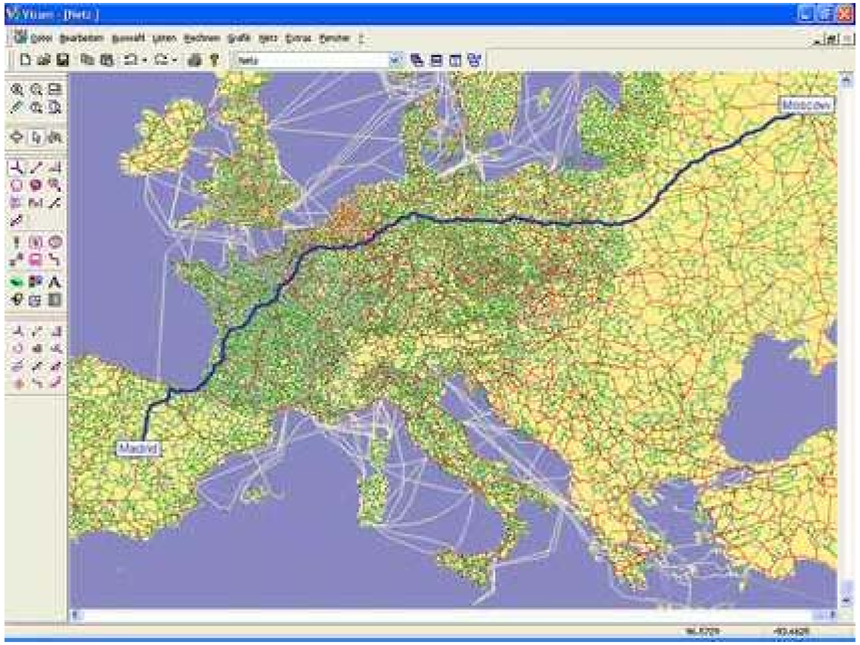
Korteste Veje
[CLRS, kapitel 24, 25.1-25.2]

AARHUS UNIVERSITET

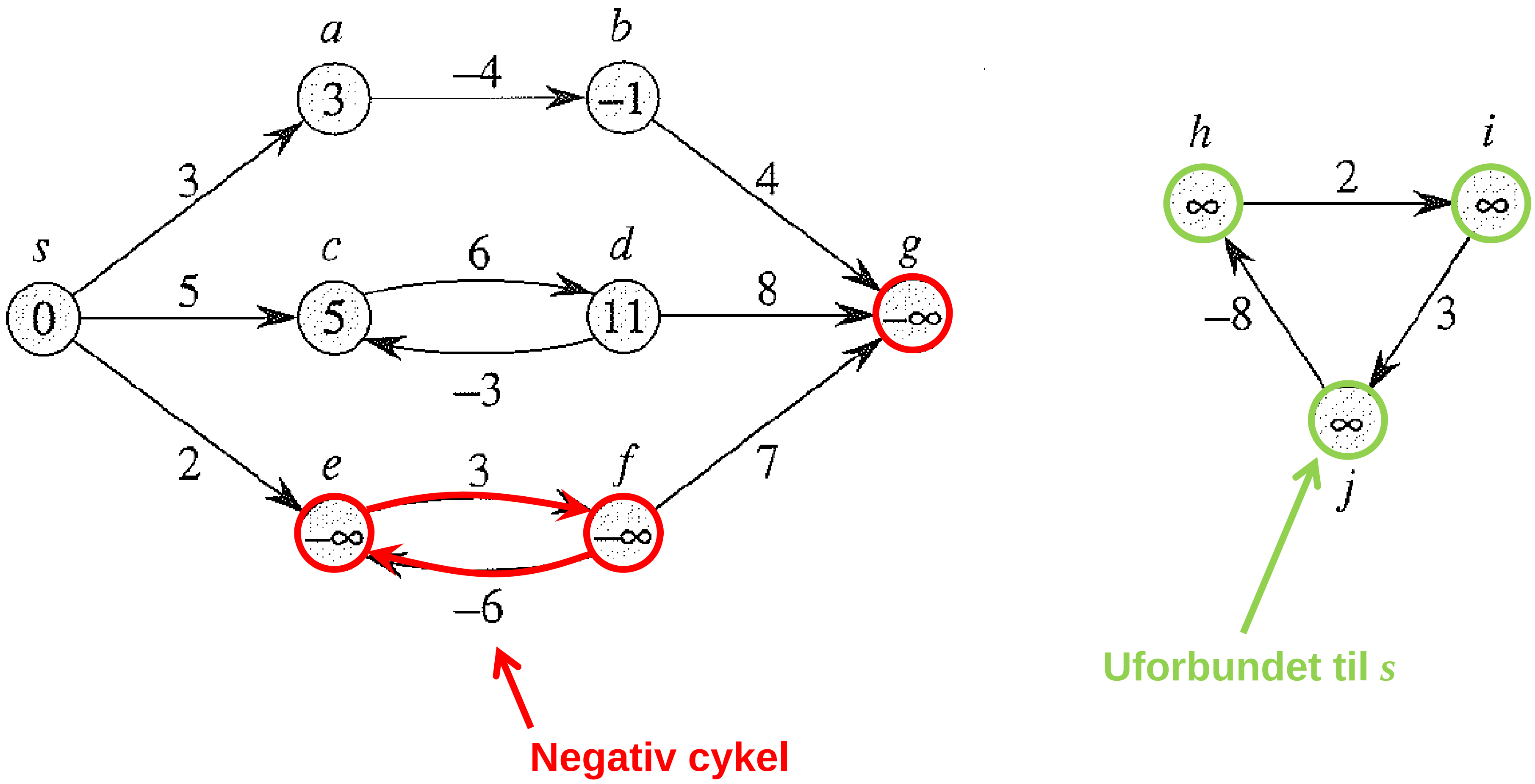
Kort over Vest-Europa

- **18.029.721 knuder**
- **42.199.587 orienterede kanter**

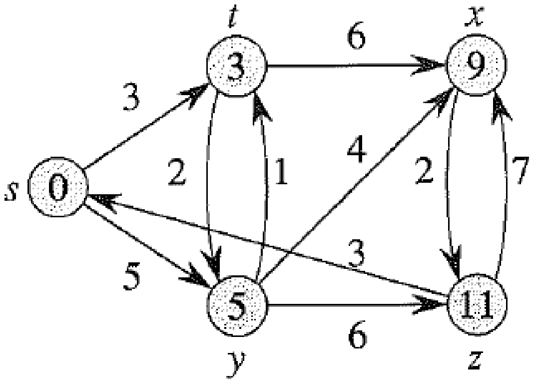
traffic mobility logistics.

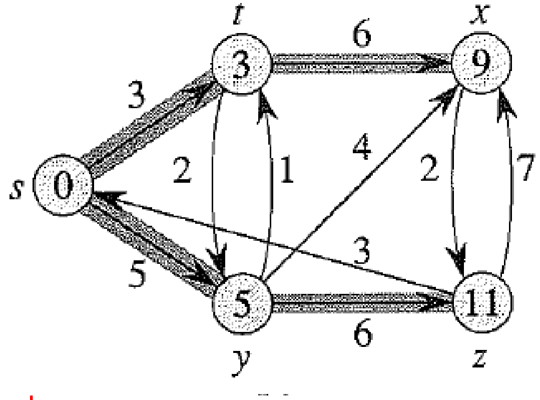


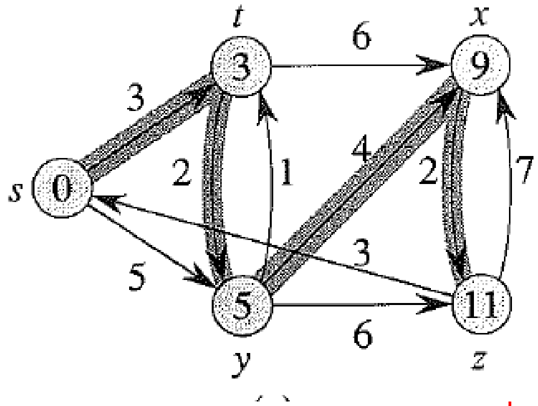
Eksempel: Korteste veje fra *s*



Eksempel: Korteste veje træer



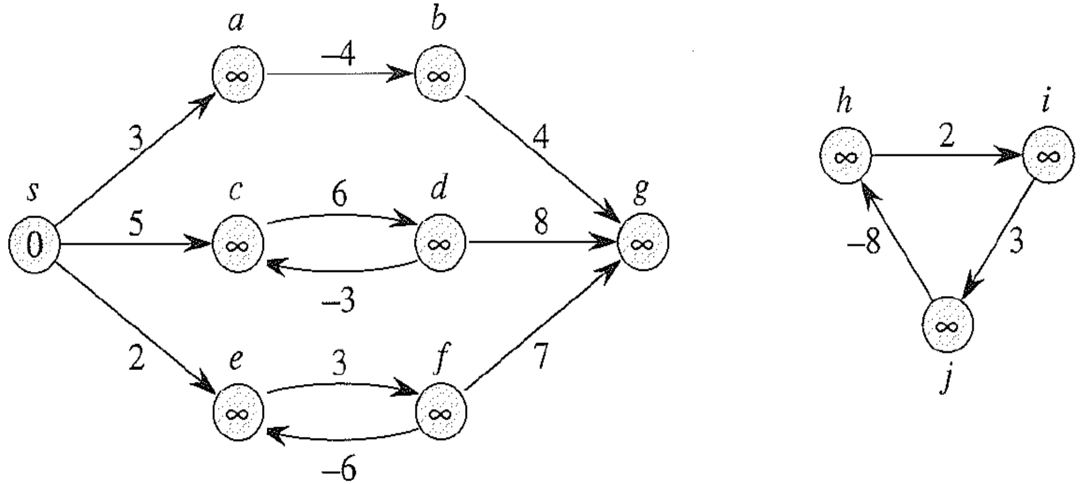




2 forskellige korteste veje træer der repræsenterer stier fra s med samme længde

Korteste Veje Estimater : Initialisering

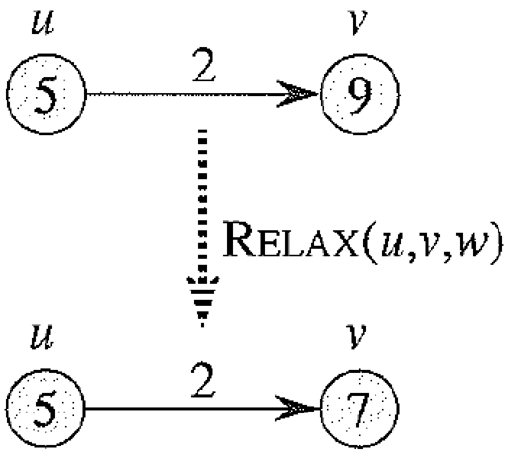
```
INITIALIZE-SINGLE-SOURCE(G, s)
for each vertex v ∈ V[G]
    do d[v] ← ∞
       π[v] ← NIL
d[s] ← 0
```



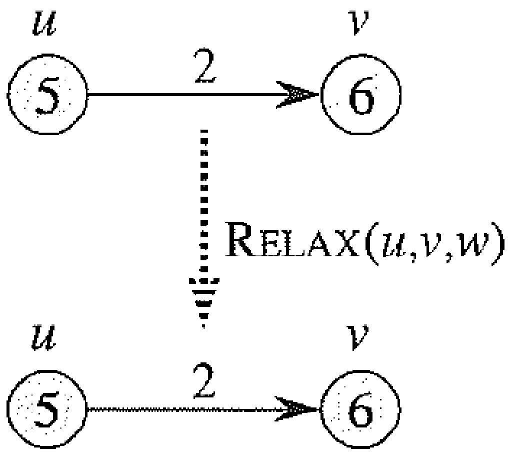
Korteste Veje Estimater : Relax

RELAX(u, v, w)

if $d[v] > d[u] + w(u, v)$
    then $d[v] \leftarrow d[u] + w(u, v)$
        $\pi[v] \leftarrow u$



Kortere afstand til *v* fundet



Forbedrer ikke afstanden til *v*

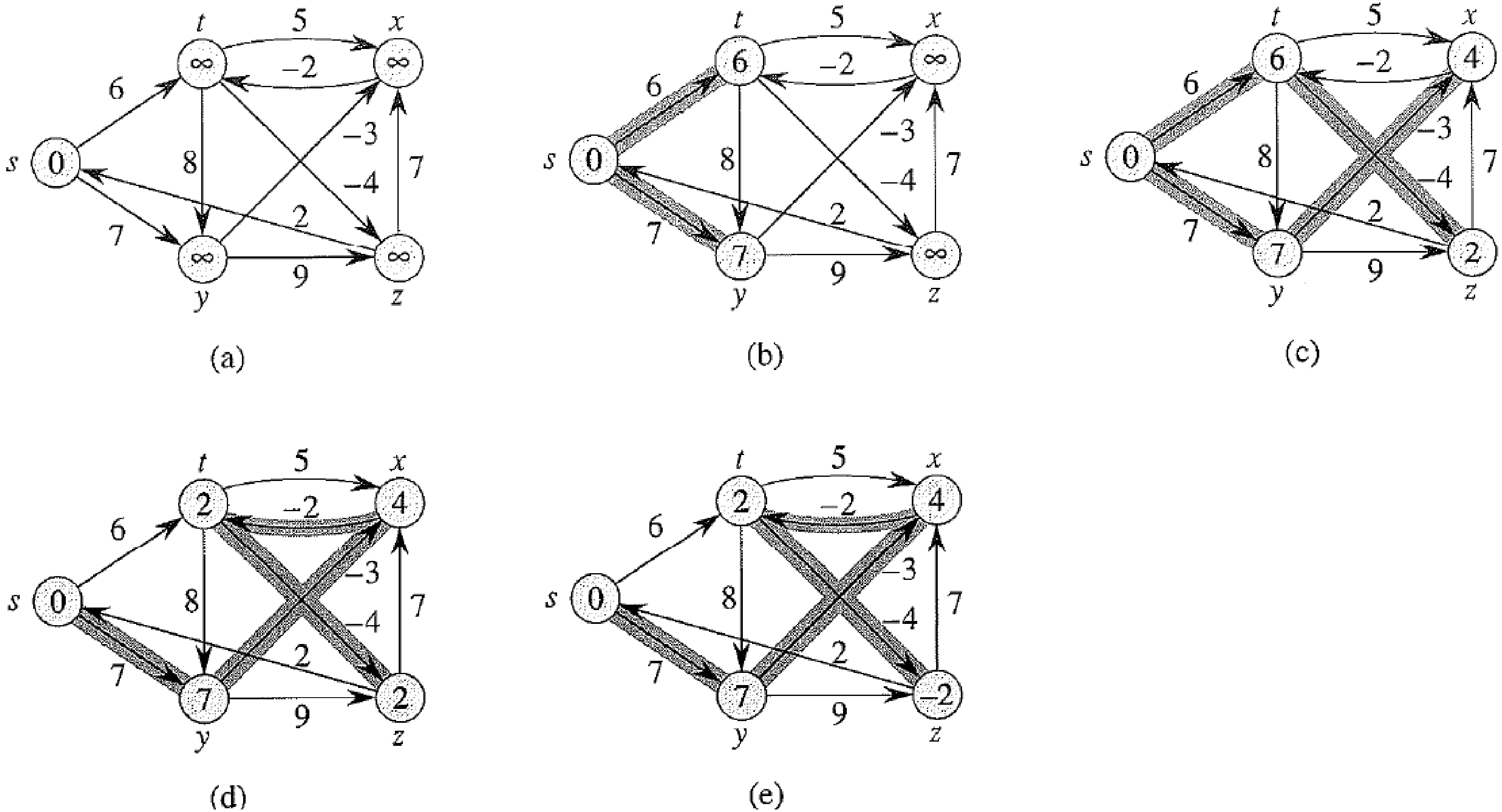
Bellman-Ford:
Korteste Veje i Grafer med Negative Vægte

```
BELLMAN-FORD(G, w, s)
INITIALIZE-SINGLE-SOURCE(G, s)
for i ← 1 to |V[G]| − 1
        do for each edge (u, v) ∈ E[G]
                do RELAX(u, v, w)
for each edge (u, v) ∈ E[G]
        do if d[v] > d[u] + w(u, v)
                then return FALSE
return TRUE
```

Check for negativ cykel

Tid $O(nm)$

Bellman-Ford: Eksempel

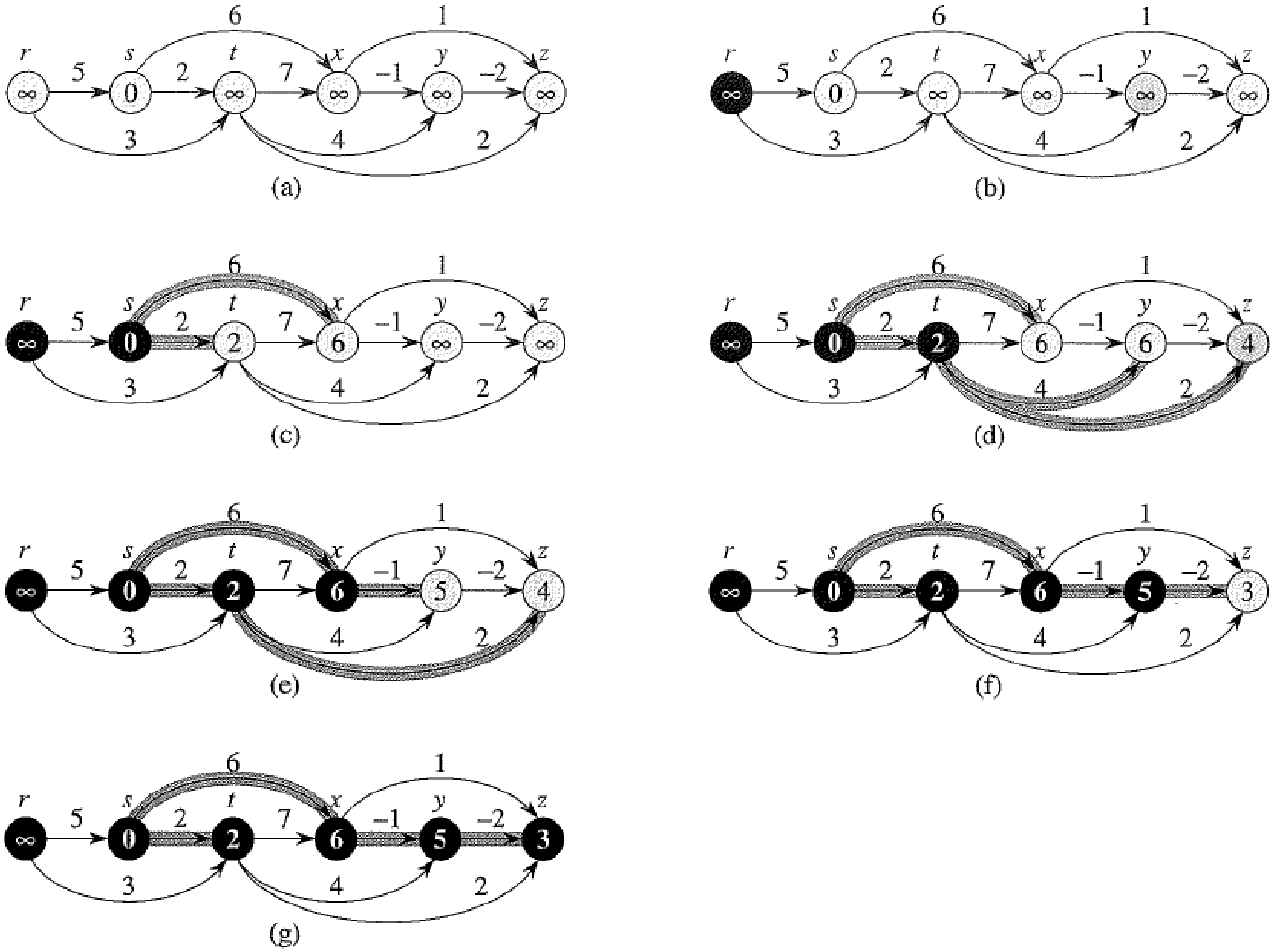


Korteste Veje i Acycliske Grafer

```
DAG-SHORTEST-PATHS(G, w, s)
topologically sort the vertices of G
INITIALIZE-SINGLE-SOURCE(G, s)
for each vertex u, taken in topologically sorted order
      do for each vertex v ∈ Adj[u]
            do RELAX(u, v, w)
```

Tid $O(n+m)$

Acykliske Grafer : Eksempel



Dijkstra:

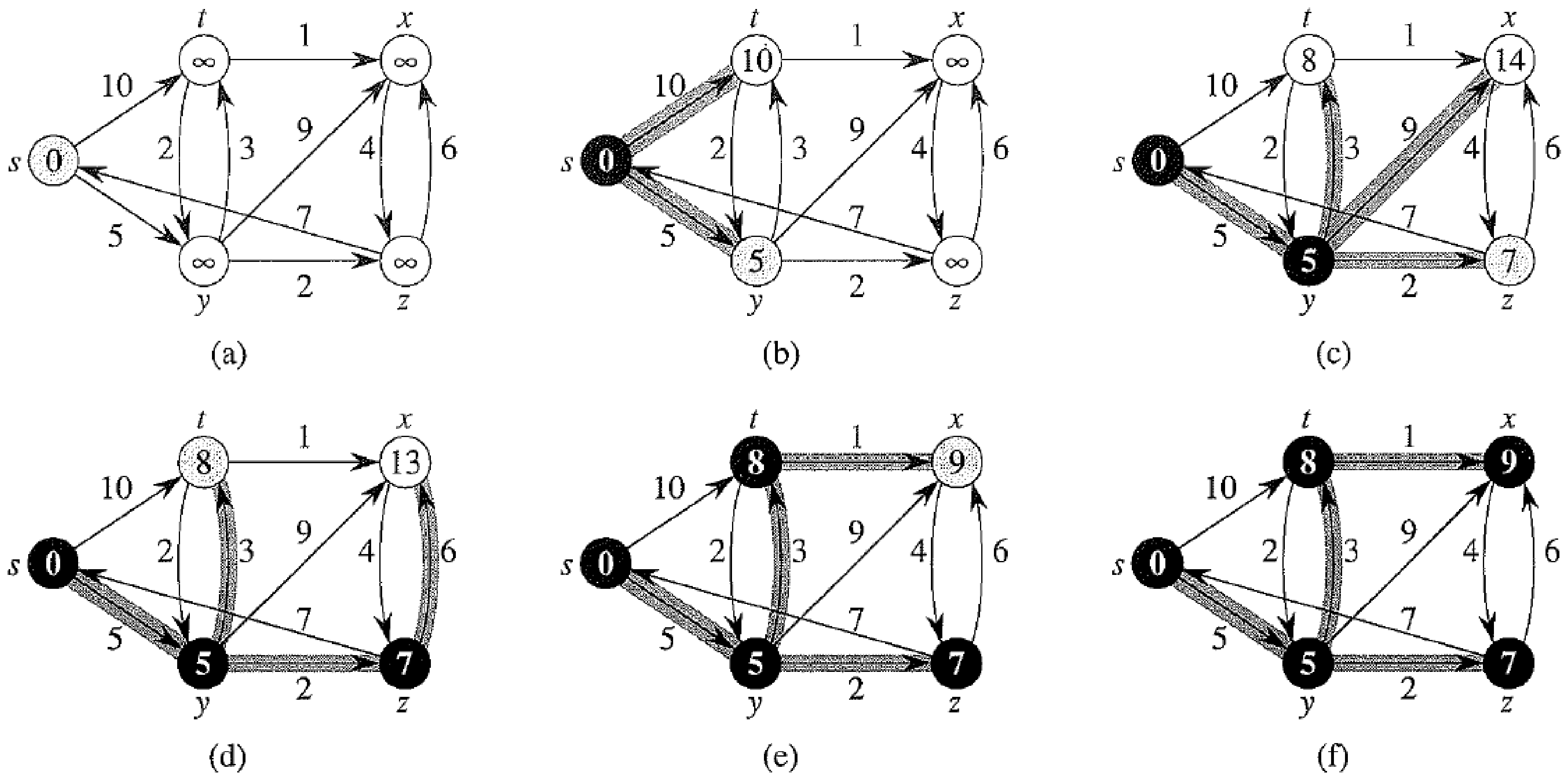
Korteste Veje i Grafer uden Negative Vægte

```
DIJKSTRA(G, w, s)
    INITIALIZE-SINGLE-SOURCE(G, s)
    S ← Ø
    Q ← V[G]
    while Q ≠ Ø
        do u ← EXTRACT-MIN(Q)
            S ← S ∪ {u}
            for each vertex v ∈ Adj[u]
                do RELAX(u, v, w)
```

***Q* = prioritets kø (besøger knuderne efter stigende afstand fra *s*)**

Tid $O((n+m) \cdot \log n)$
eller $O(n^2+m)$

Dijkstra : Eksempel



Opsummering

		SSSP En-til-alle korteste veje
Acykliske grafer (positive og negative vægte)		$O(n+m)$
Generelle grafer	Kun positive vægte	**Dijkstra** $O((n+m)\cdot\log n)$ $O(n^2+m)$
	Positive og negative vægte	**Bellman-Ford** $O(m\cdot n)$

Relaxer hver kant præcis én gang

Korteste Veje mellem alle Par af Knude

```
PRINT-ALL-PAIRS-SHORTEST-PATH(Π, i, j)
if i = j
    then print i
    else if π_ij = NIL
            then print "no path from" i "to" j "exists"
            else PRINT-ALL-PAIRS-SHORTEST-PATH(Π, i, π_ij)
                 print j
```

EXTEND-SHORTEST-PATHS(L, W)

$n \leftarrow rows[L]$
let $L' = (l'_{ij})$ be an $n \times n$ matrix
for $i \leftarrow 1$ **to** n
  do for $j \leftarrow 1$ **to** n
    do $l'_{ij} \leftarrow \infty$
      for $k \leftarrow 1$ **to** n
        do $l'_{ij} \leftarrow \min(l'_{ij}, l_{ik} + w_{kj})$
return L'

Tid $O(n^3)$

MATRIX-MULTIPLY(A, B)

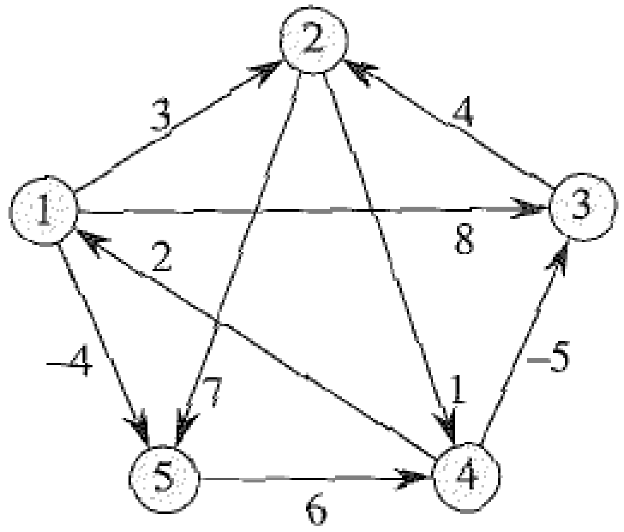
```
n ← rows[A]
let C be an n × n matrix
for i ← 1 to n
        do for j ← 1 to n
                do c_ij ← 0
                    for k ← 1 to n
                        do c_ij ← c_ij + a_ik · b_kj
return C
```

Tid $O(n^3)$

SLOW-ALL-PAIRS-SHORTEST-PATHS(W)

$n \leftarrow rows[W]$
$L^{(1)} \leftarrow W$
for $m \leftarrow 2$ **to** $n - 1$
  do $L^{(m)} \leftarrow$ EXTEND-SHORTEST-PATHS$(L^{(m-1)}, W)$
return $L^{(n-1)}$

Tid $O(n^4)$



$$L^{(1)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \quad L^{(2)} = \begin{pmatrix} 0 & 3 & 8 & 2 & -4 \\ 3 & 0 & -4 & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & \infty & 1 & 6 & 0 \end{pmatrix}$$

$$L^{(3)} = \begin{pmatrix} 0 & 3 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix} \quad L^{(4)} = \begin{pmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix}$$

FASTER-ALL-PAIRS-SHORTEST-PATHS(W)

$n \leftarrow rows[W]$
$L^{(1)} \leftarrow W$
$m \leftarrow 1$
while $m < n - 1$
    do $L^{(2m)} \leftarrow$ EXTEND-SHORTEST-PATHS$(L^{(m)}, L^{(m)})$
        $m \leftarrow 2m$
return $L^{(m)}$

Tid $O(n^3 \cdot \log n)$

Floyd-Warshall

FLOYD-WARSHALL(W)

$n \leftarrow rows[W]$
$D^{(0)} \leftarrow W$
for $k \leftarrow 1$ **to** n
    do for $i \leftarrow 1$ **to** n
        do for $j \leftarrow 1$ **to** n
            do $d_{ij}^{(k)} \leftarrow \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$
return $D^{(n)}$

Tid $O(n^3)$

Transitive Lukning

TRANSITIVE-CLOSURE(G)

$n \leftarrow |V[G]|$
for $i \leftarrow 1$ **to** n
    do for $j \leftarrow 1$ **to** n
        do if $i = j$ or $(i, j) \in E[G]$
            then $t_{ij}^{(0)} \leftarrow 1$
            else $t_{ij}^{(0)} \leftarrow 0$
for $k \leftarrow 1$ **to** n
    do for $i \leftarrow 1$ **to** n
        do for $j \leftarrow 1$ **to** n
            do $t_{ij}^{(k)} \leftarrow t_{ij}^{(k-1)} \vee \left(t_{ik}^{(k-1)} \wedge t_{kj}^{(k-1)}\right)$
return $T^{(n)}$

Tid $O(n^3)$

Opsummering

		SSSP En-til-alle korteste veje	APSP Alle-til-alle korteste veje
Acykliske grafer (positive og negative vægte)		$O(n+m)$	$O(n \cdot (n+m))$
Generelle grafer	Kun positive vægte	**Dijkstra** $O((n+m) \cdot \log n)$	**Floyd-Warshall** $O(n^3)$
	Positive og negative vægte	**Bellman-Ford** $O(m \cdot n)$	