Implicit Computation Geometry

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Overview

1. Introduction: Motivation for implicit computation
2. Skylines and convex hulls

Motivation

- Traditional focus in algorithm design: Running Time
- Here: Second core issue: Memory utilization.
- Historically: Space-Efficiency considered due to high memory prices.
- Nowadays: Space-Efficiency considered due to:
  - Larger datasets.
    - High-resolution survellaince data
    - Temporal and spatio-temporal data
  - Smaller computing devices.
    - Location based services for mobile communication networks
    - Data analysis and propagation in sensor networks
  - Limited (read/write)-memory

Sensor Networks

Car Navigation

Definition 1.1

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In-Place Algorithms

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Example:

- Heapsort is an in-place algorithm (uses in-place data structure).

Motivation: Dealing with Large Datasets

Algorithmic concepts for different scenarios:

- Small, fast working memory. Data resides on slow disks.
  - Cache-oblivious and I/O-efficient Algorithms: Minimize data (block) movement.
- Data is streamed and not constantly available.
  - Streaming algorithms: (Approximation of) data aggregates.
- (Almost) no memory to use additional to the given input.
- Implicit data structures, in-place algorithms.

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- In-place model "in between" I/O- and Streaming-Model . . .
- May provide insights in computational complexity of problems.

In-Place Sorting and Related Problems:

- Heapsort [Floyd, 1964].
- Linear-time merging/partitioning [Mannila & Ukkonen, 1984; Geffert et al., 2000; Katajainen & Pasanen, 1999]...
- Linear-time k-selection [Carlsson & Sundström, 1995; Geffert & Kollar, 2001; Bose et al., 2006].

In-Place, Cache-Oblivious(!) Dictionary:

- $\mathcal{O}(\log n)$ update/queries [Franceschini & Grossi, 2003].

Sapce-efficient Computational Geometry Results

In-Place Computational Geometry:

- Closest Pair etc. [Bose et al., 2006].
- Line-Segment Intersection [Bose et al., 2006; Vahrenhold, 2005].
- Convex Hull and Maxima problems etc. [Brönnimann et al., 2004b; Brönnimann & M.Chan, 2004; Blunck & Vahrenhold, 2006].

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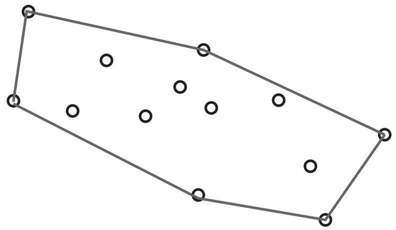
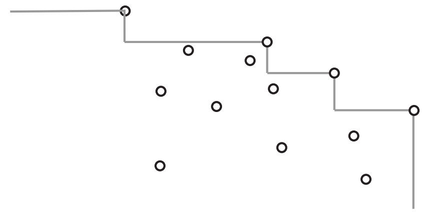
"Use-Polylog-Extra-Space-And-Time" Geometry Results:

- $3d$-convex hull and related [Brönnimann et al., 2004c].
- Multidimensional search sctructures [Brönnimann et al., 2004a].
- Klee's Measure Problem [Chen & M.Chan, 2005].

In this lecture . . .

In-place techniques and algorithms for:





In-place techniques and algorithms for:

- Convex hulls and sets of maxima ('skylines')

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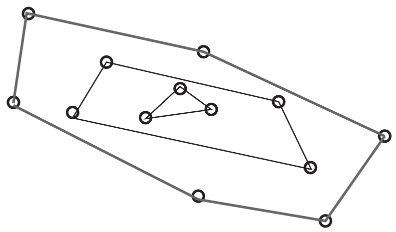

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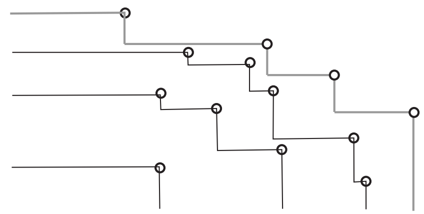
- Layers of convex hulls and maxima



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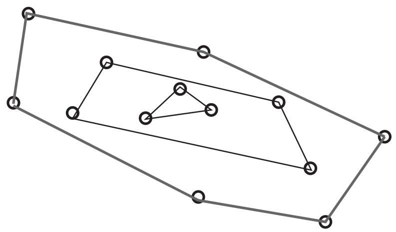

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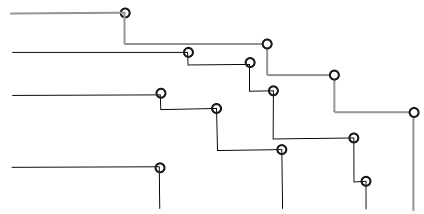
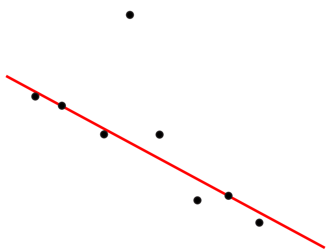


- Convex hulls and sets of maxima ('skylines')
- Layers of convex hulls and maxima
- Regression Analysis: Estimating linear correlations



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Computing the Skyline

Maximal Points:

- Given: Set $\mathcal{P}$ of n points in the plane.
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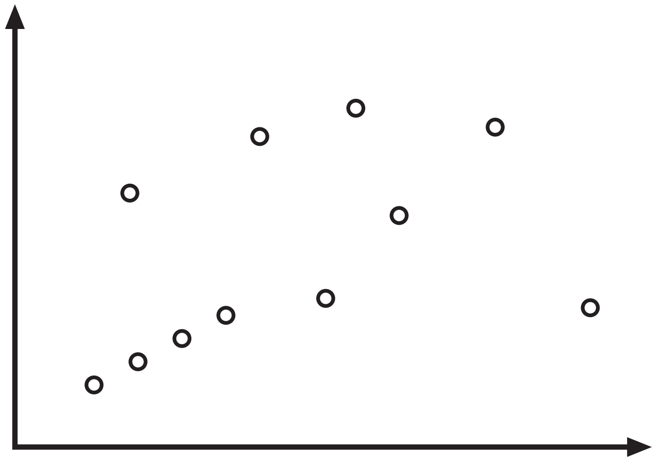

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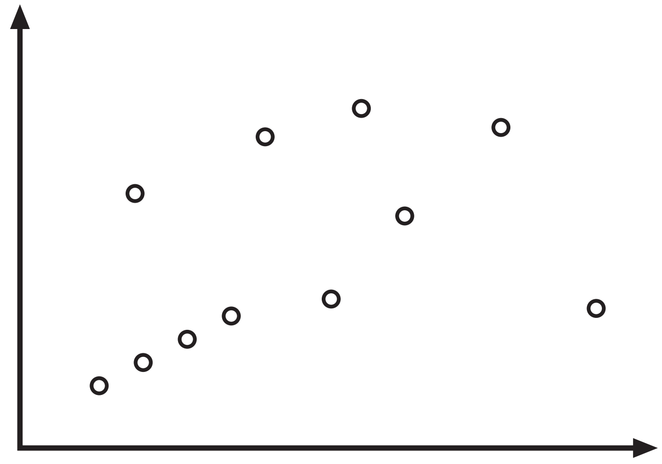

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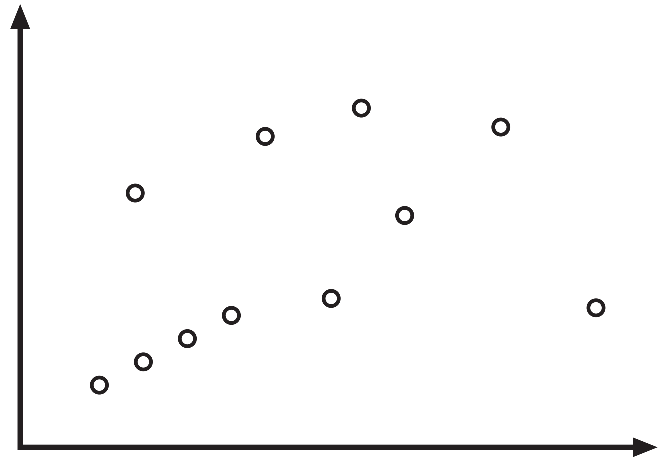

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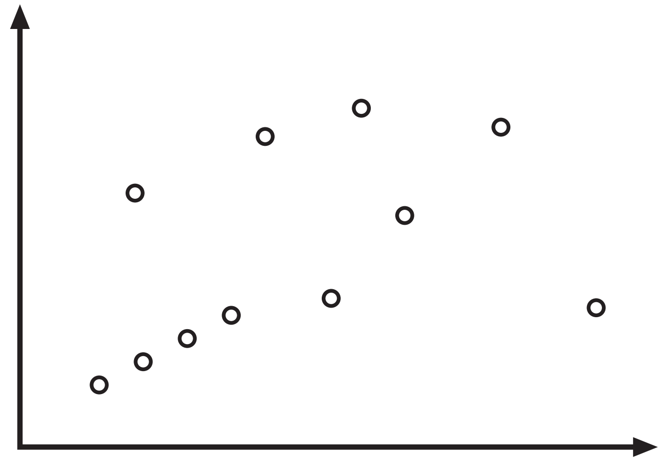

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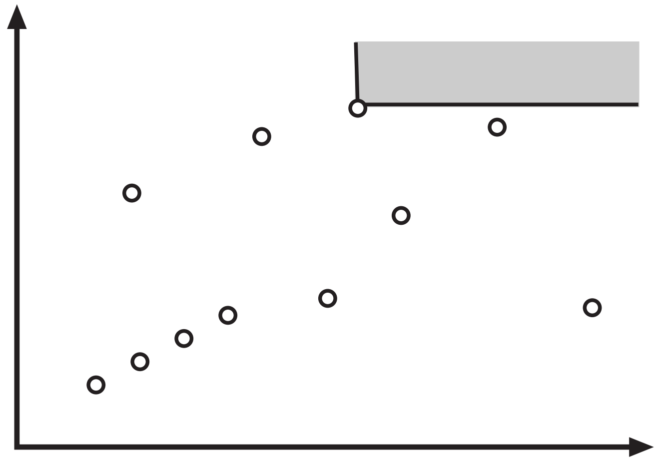

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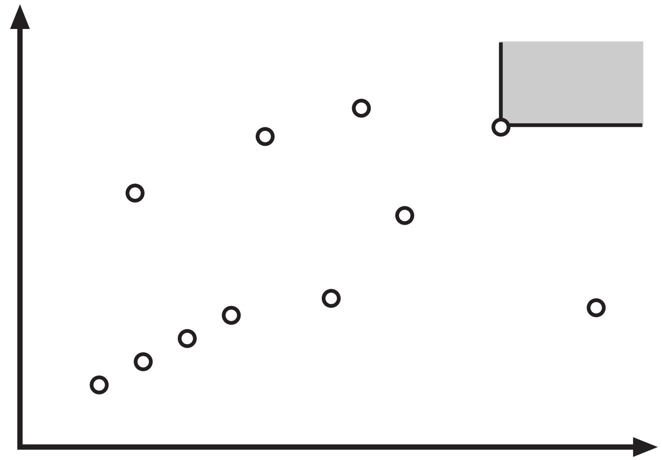

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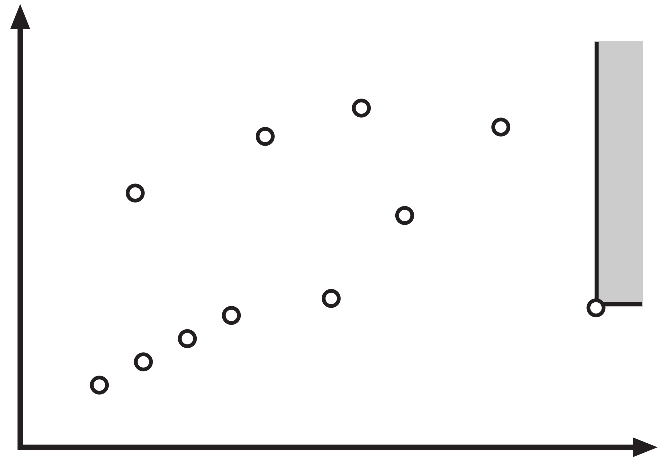

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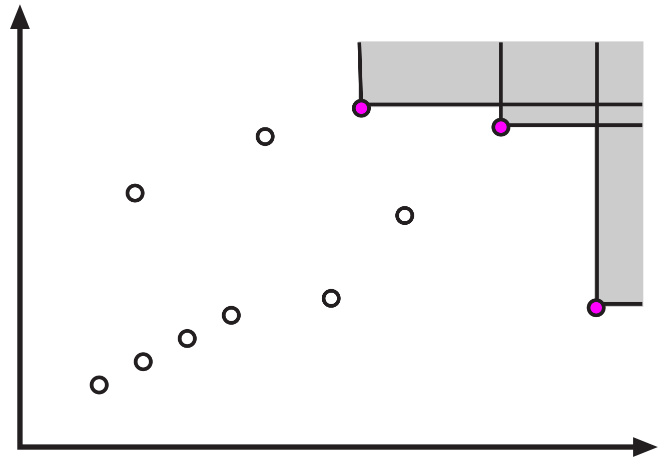

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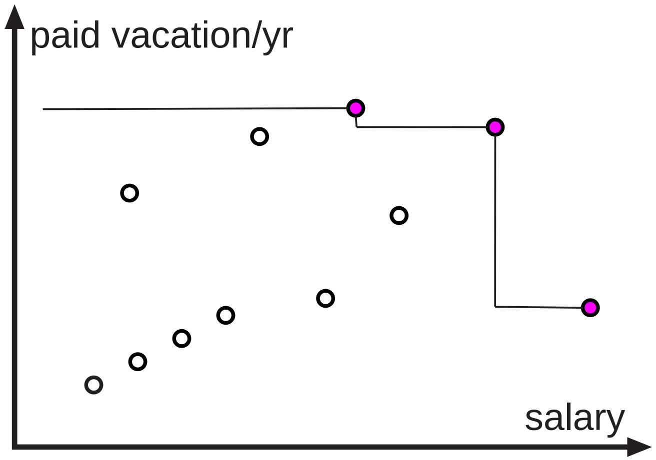

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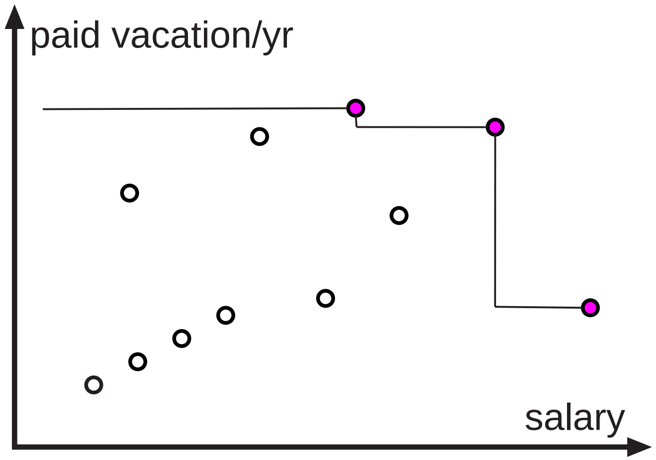

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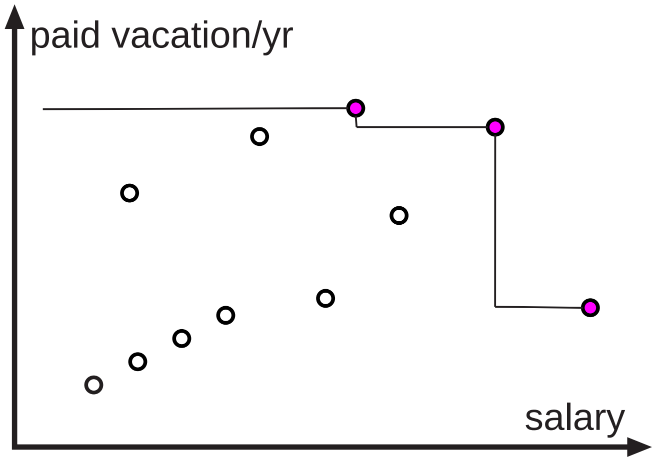

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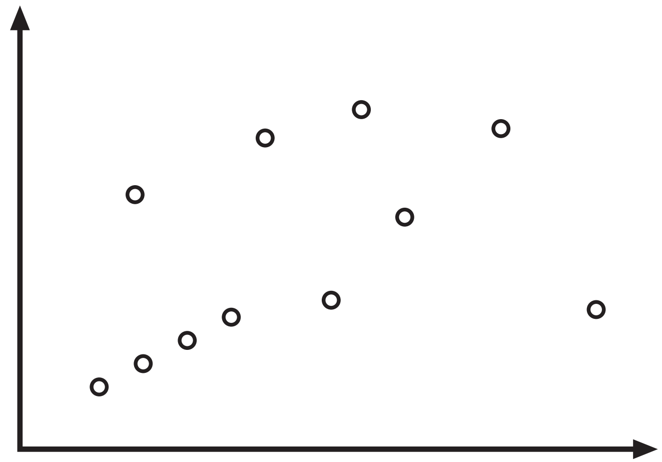
Generalizations:

- Definition generalizes to:
  - Arbitrary dimensions d.
  - 'Maxima' w.r.t. d arbitrary chosen coordinate axes.

Computing all skylines in-place

Layers of Maxima:



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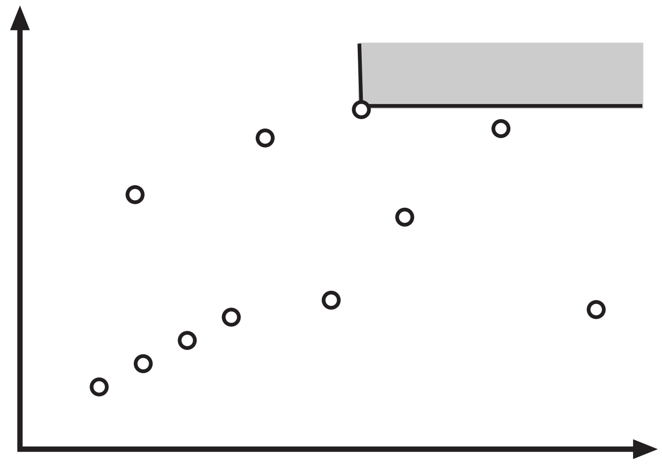

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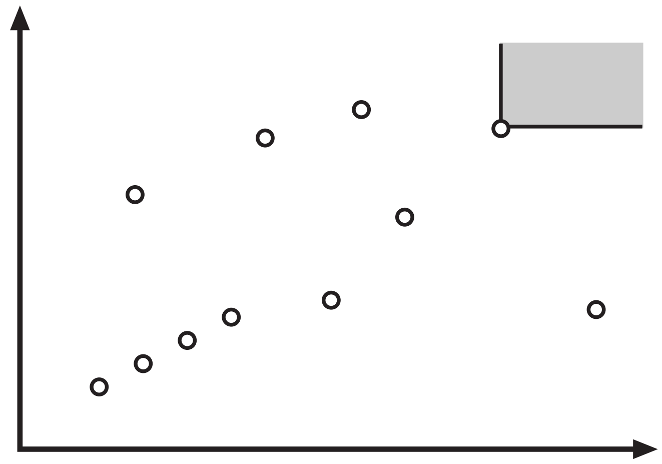

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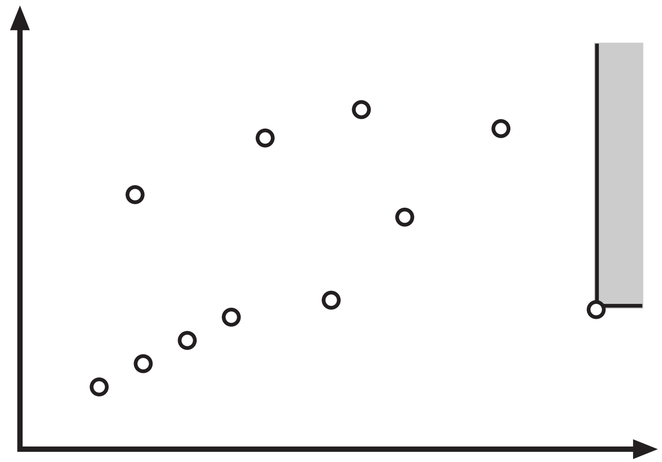

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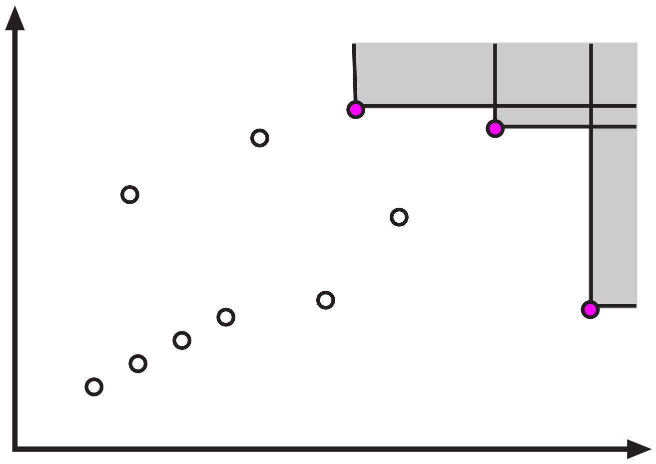

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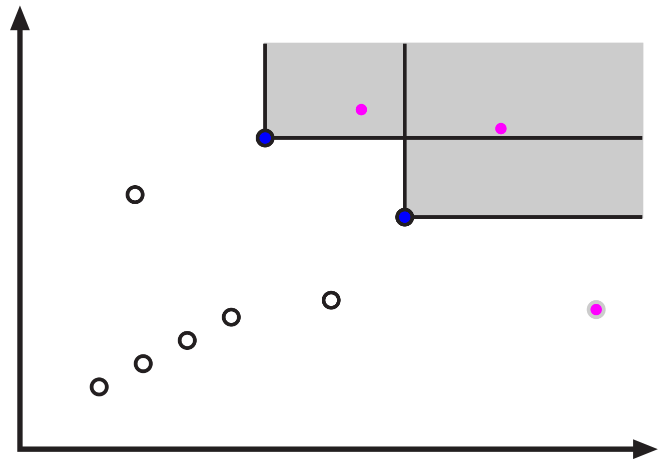

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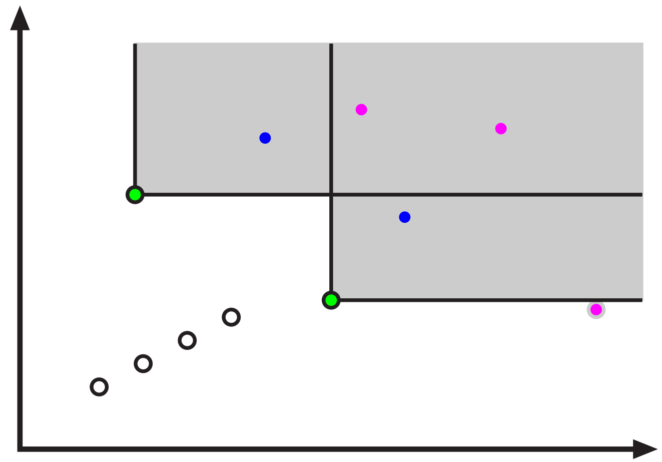

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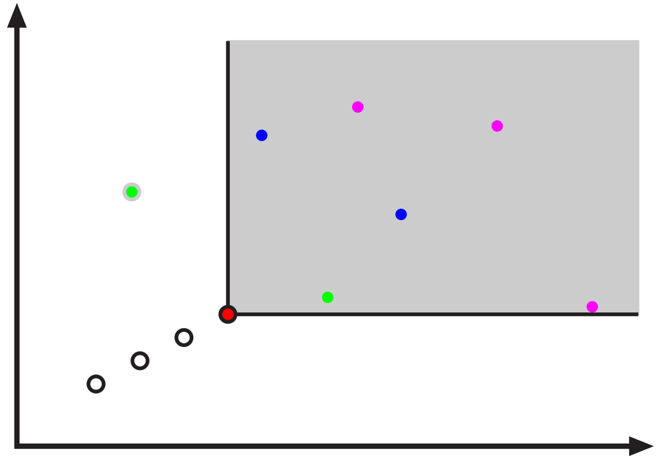

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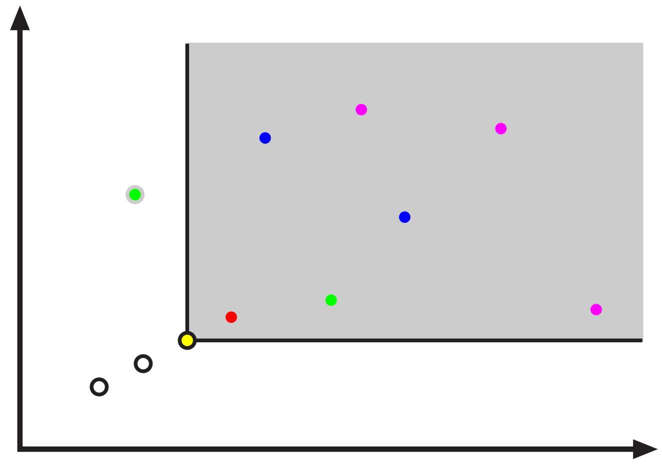

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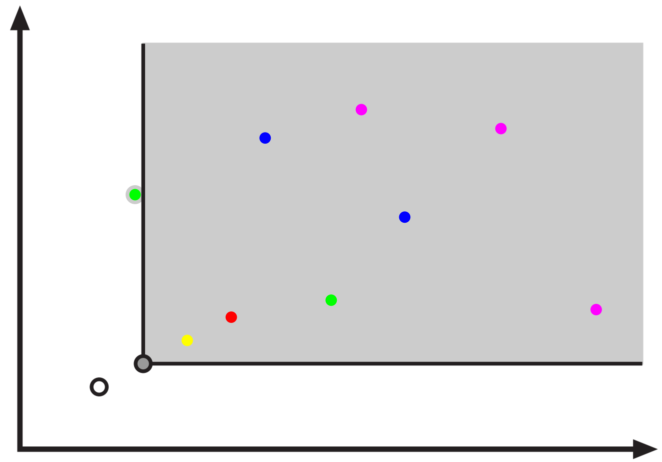

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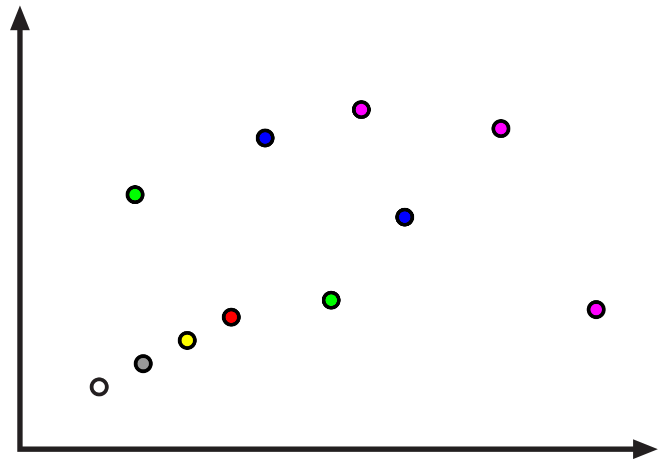

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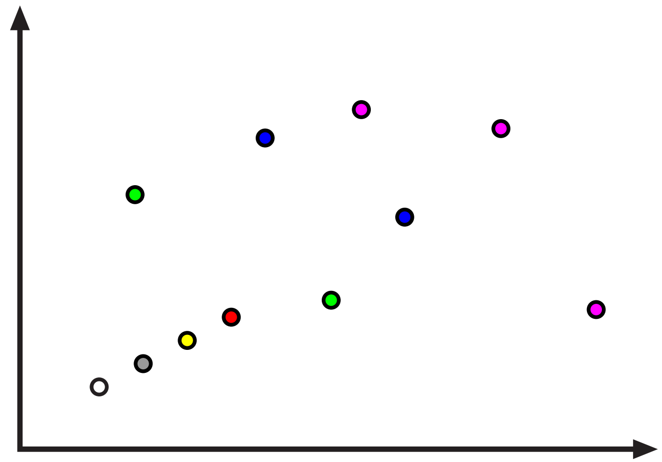

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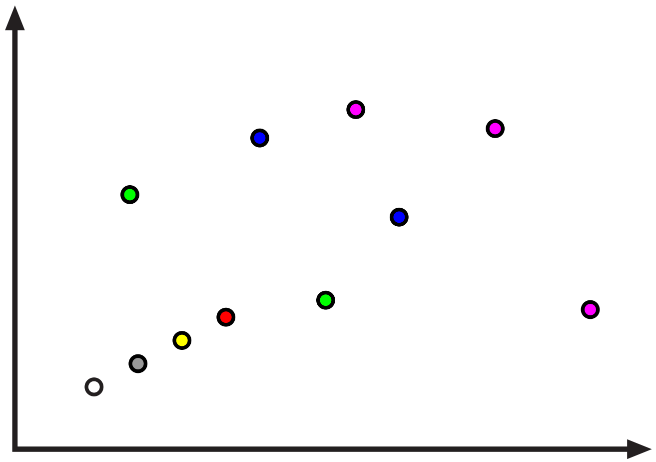

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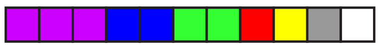
In-place setting:

- Group points by layer.
- In each layer: points sorted (by x).



Computing the Skyline: Selecting Maximal Points in 2D

- Presort points by y-coordinate using, e.g., heapsort.

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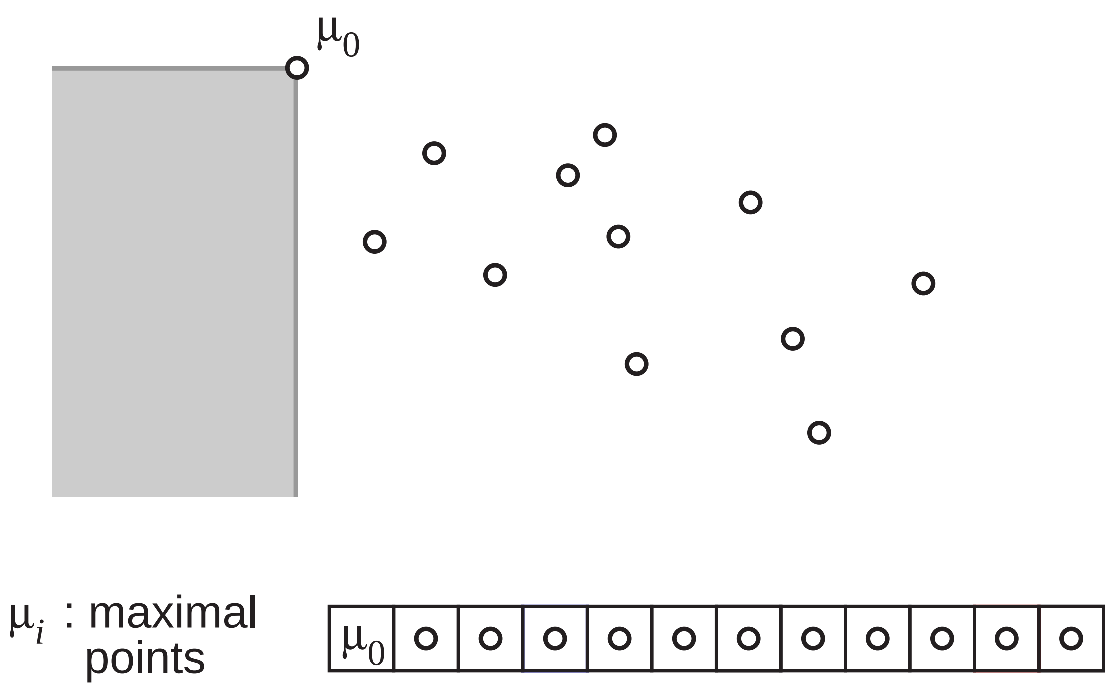
- Sweep top-down, maintain lowest maximal point (='tail') seen so far.
  - Invariant: Next point p: is maximum $\Rightarrow$ p right of tail.



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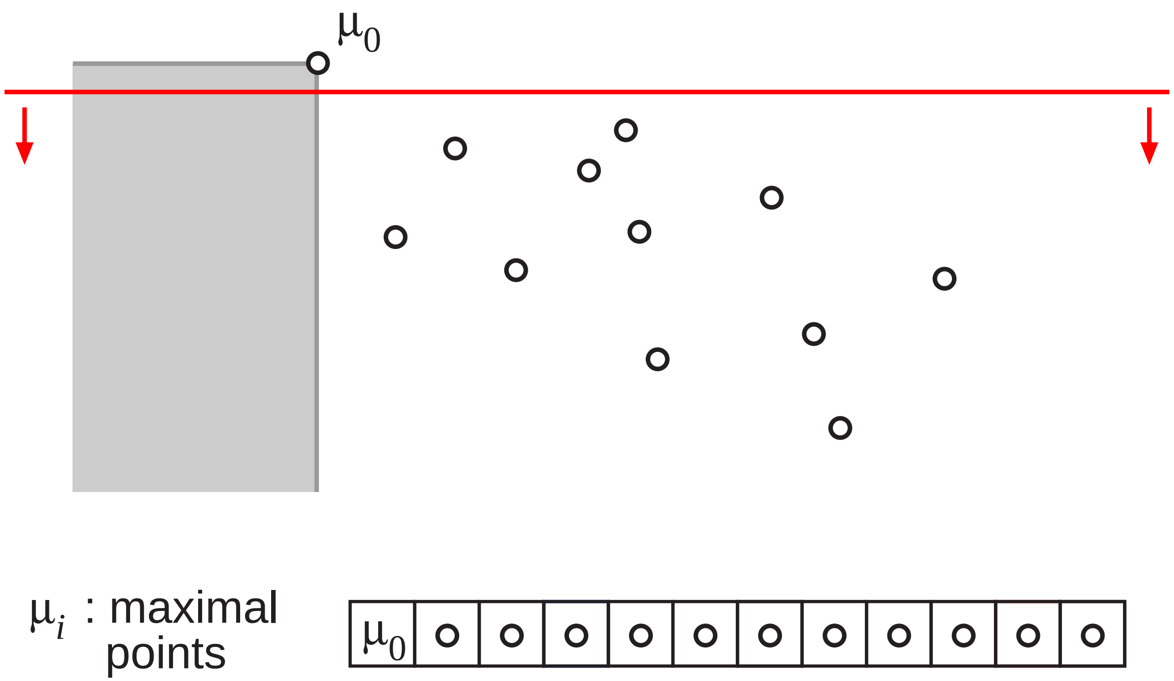

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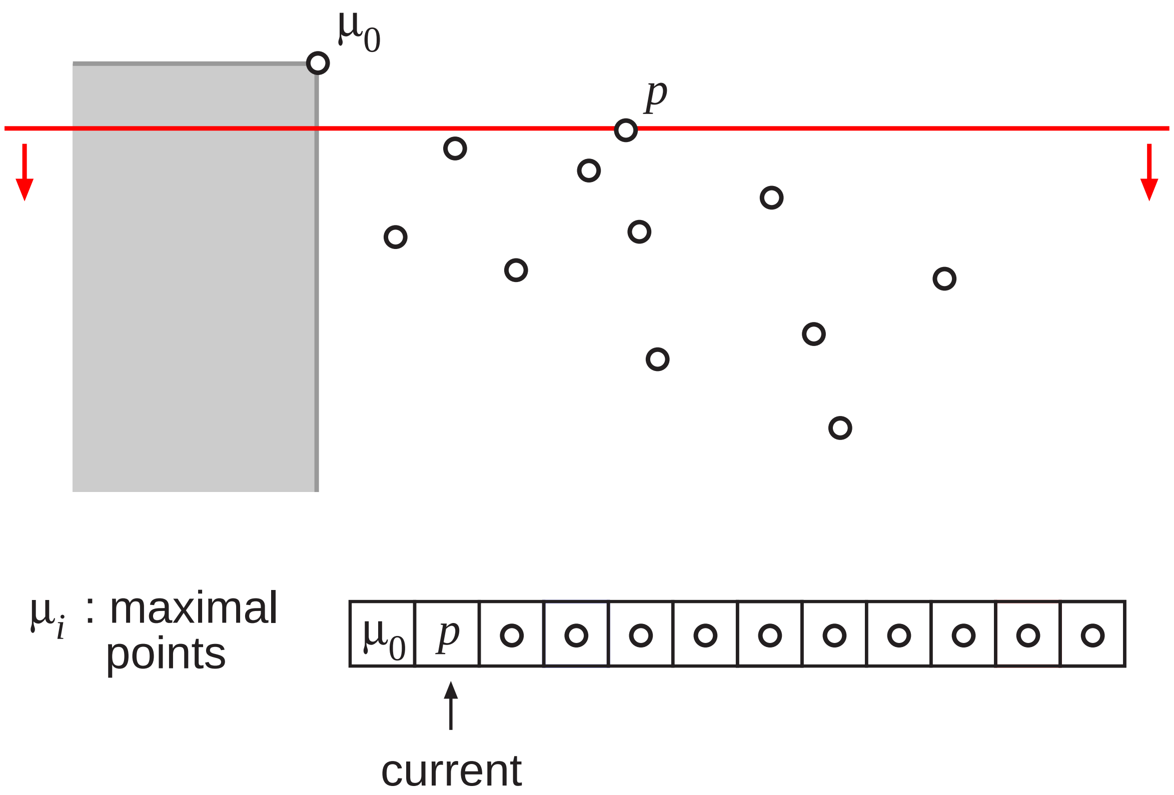

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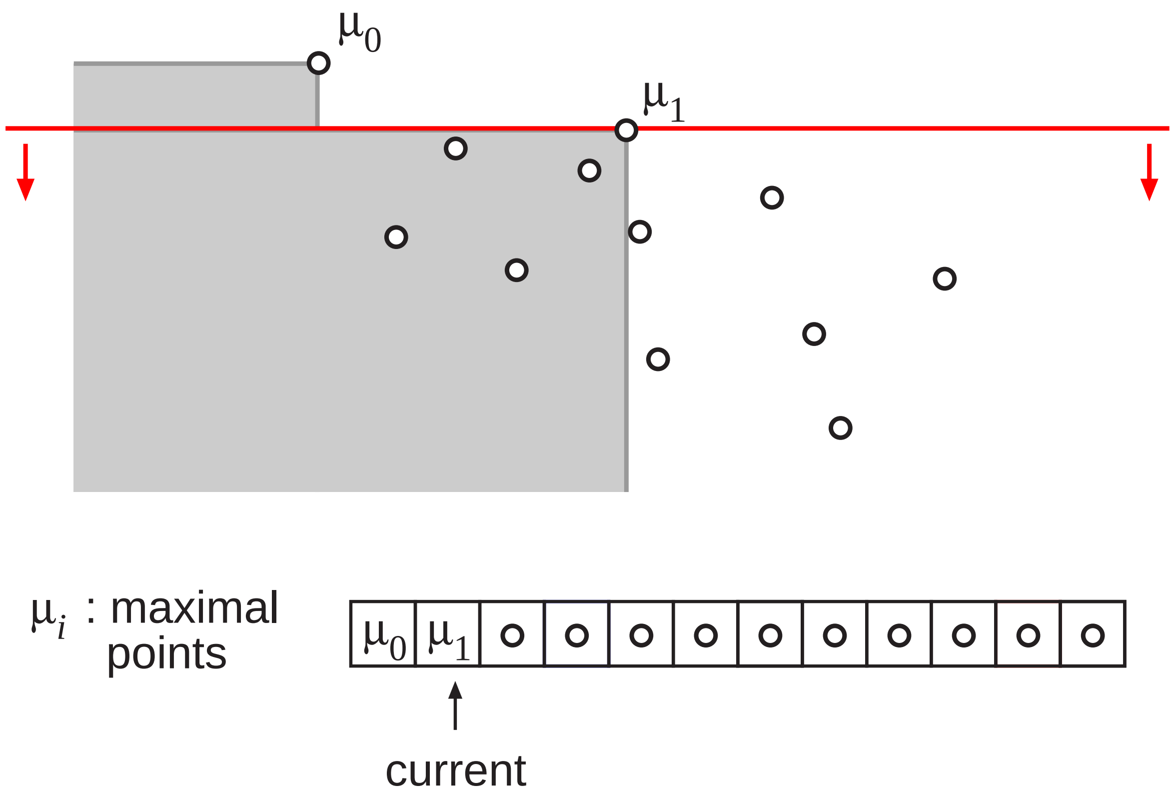

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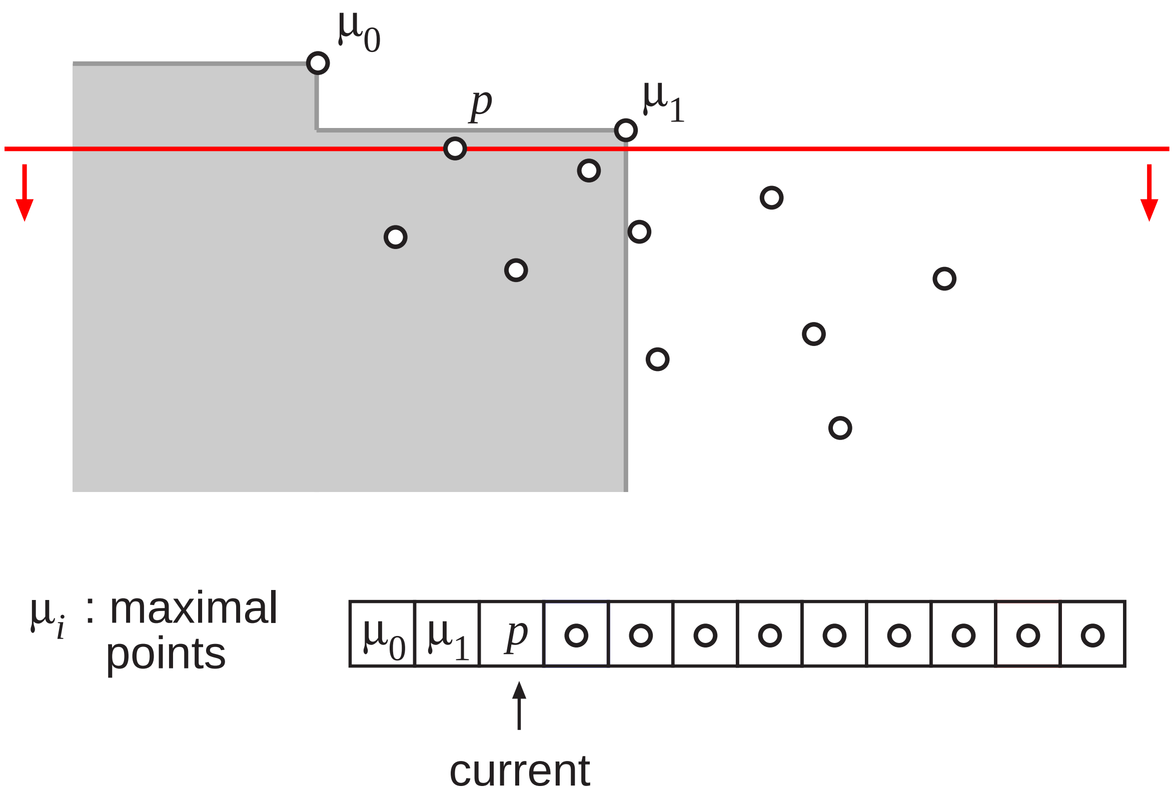

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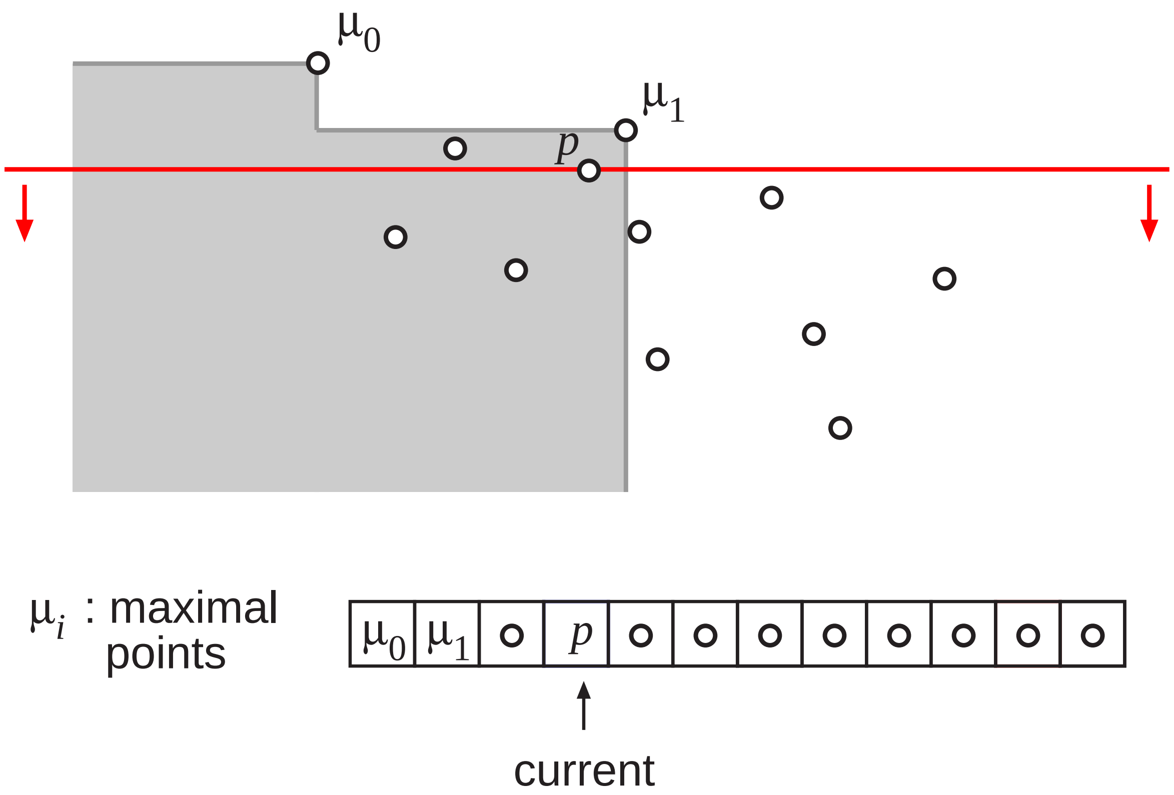

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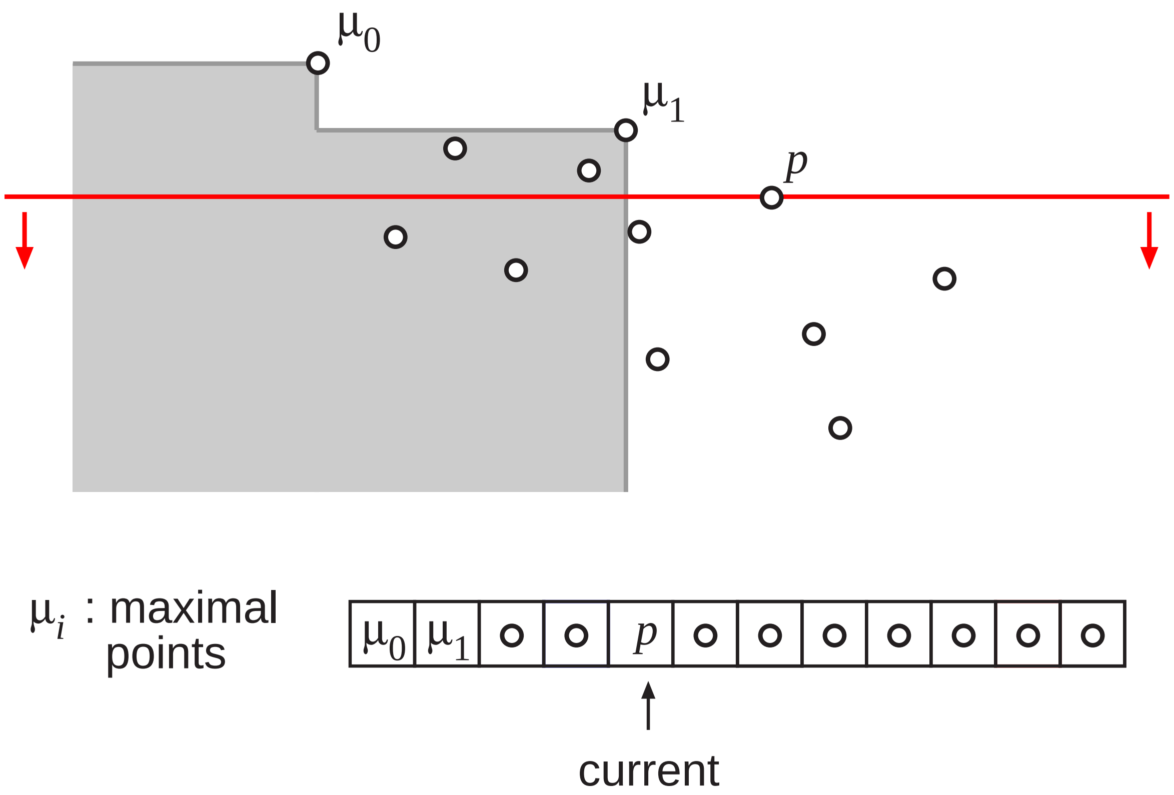

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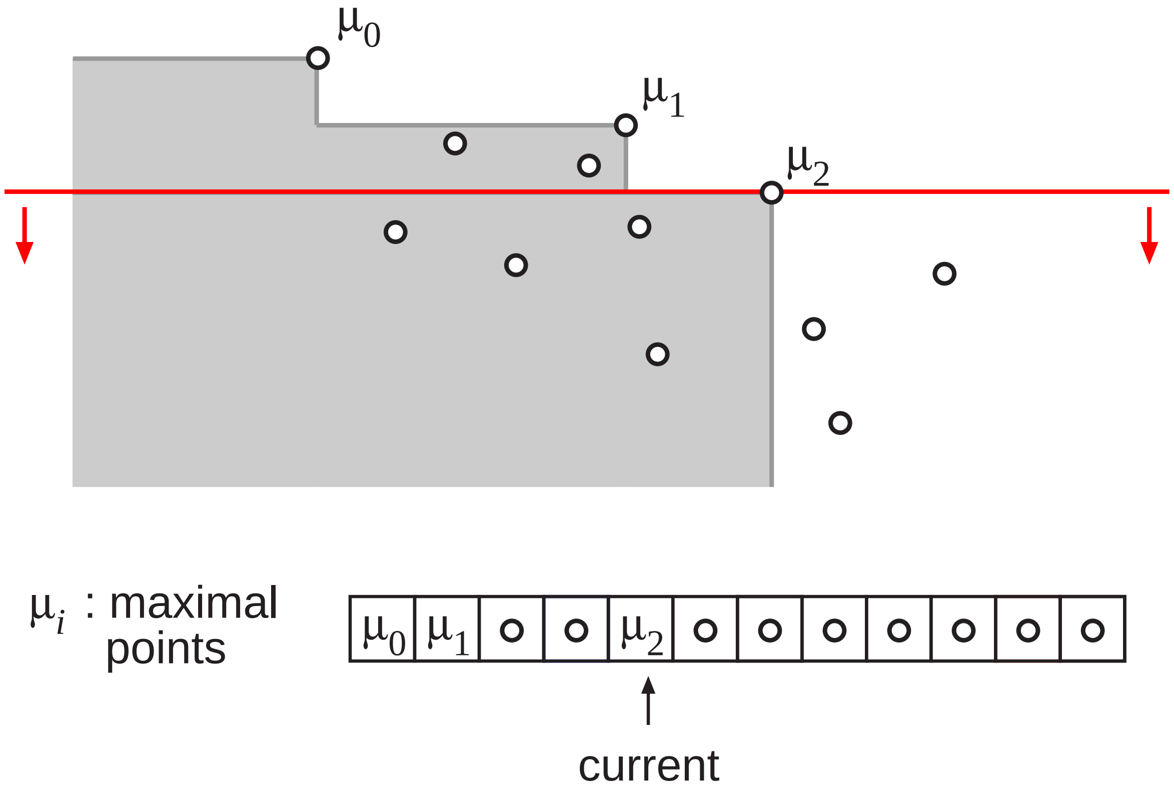

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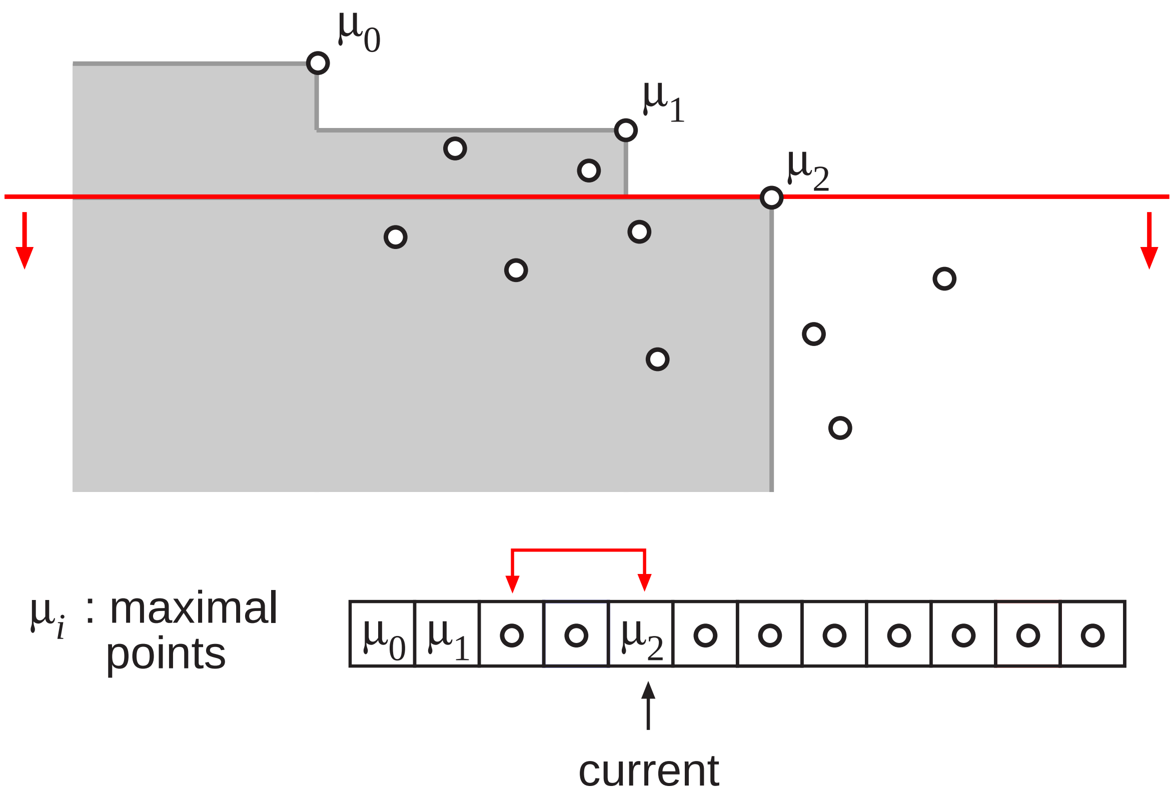

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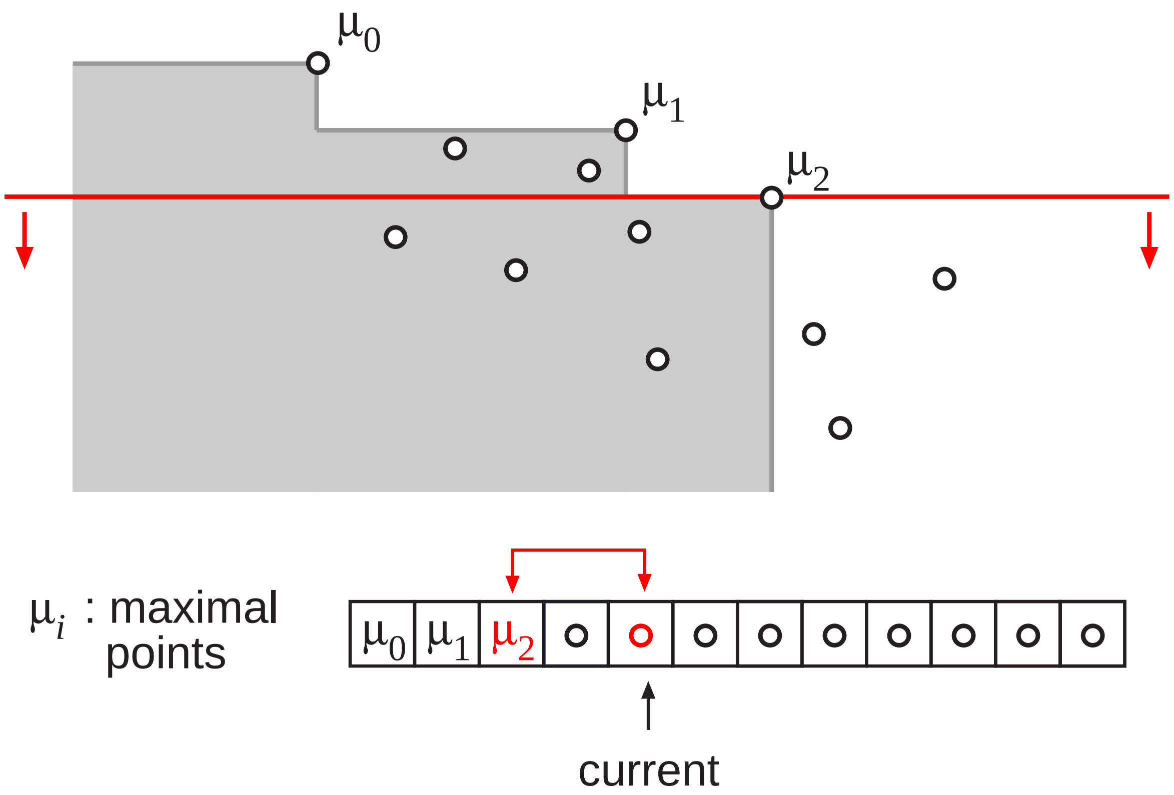

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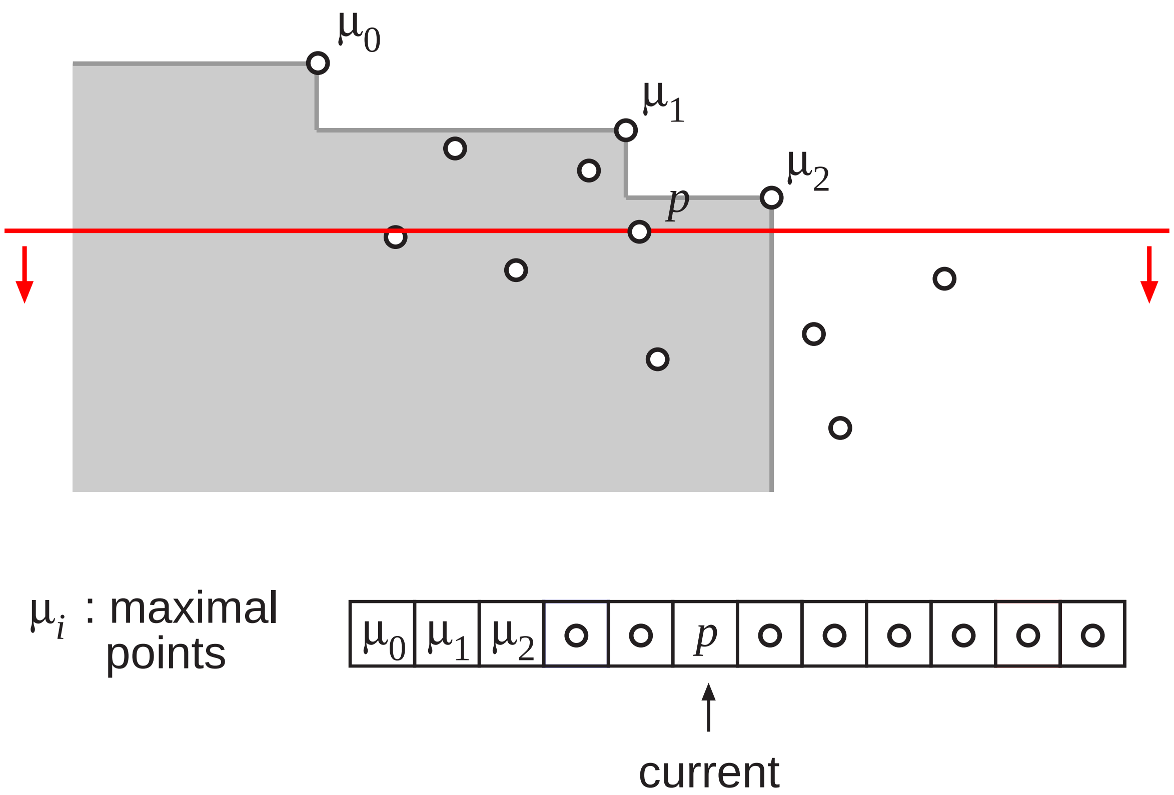

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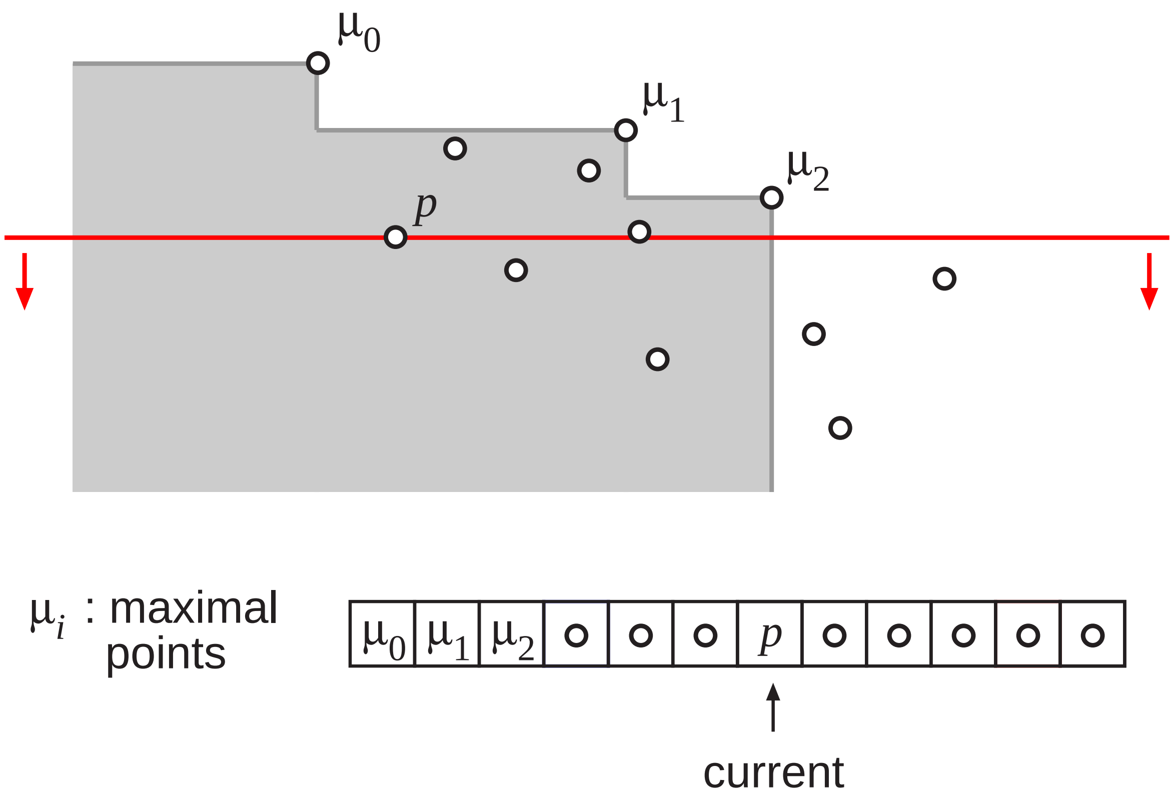

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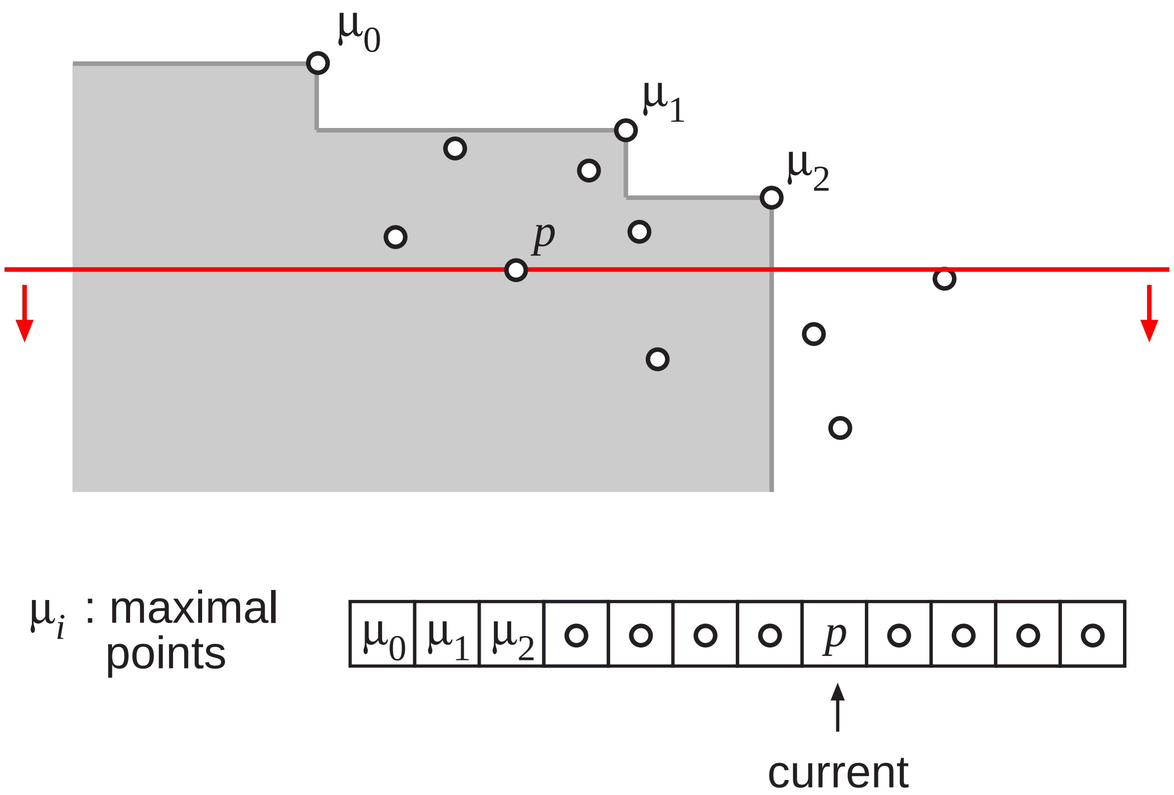

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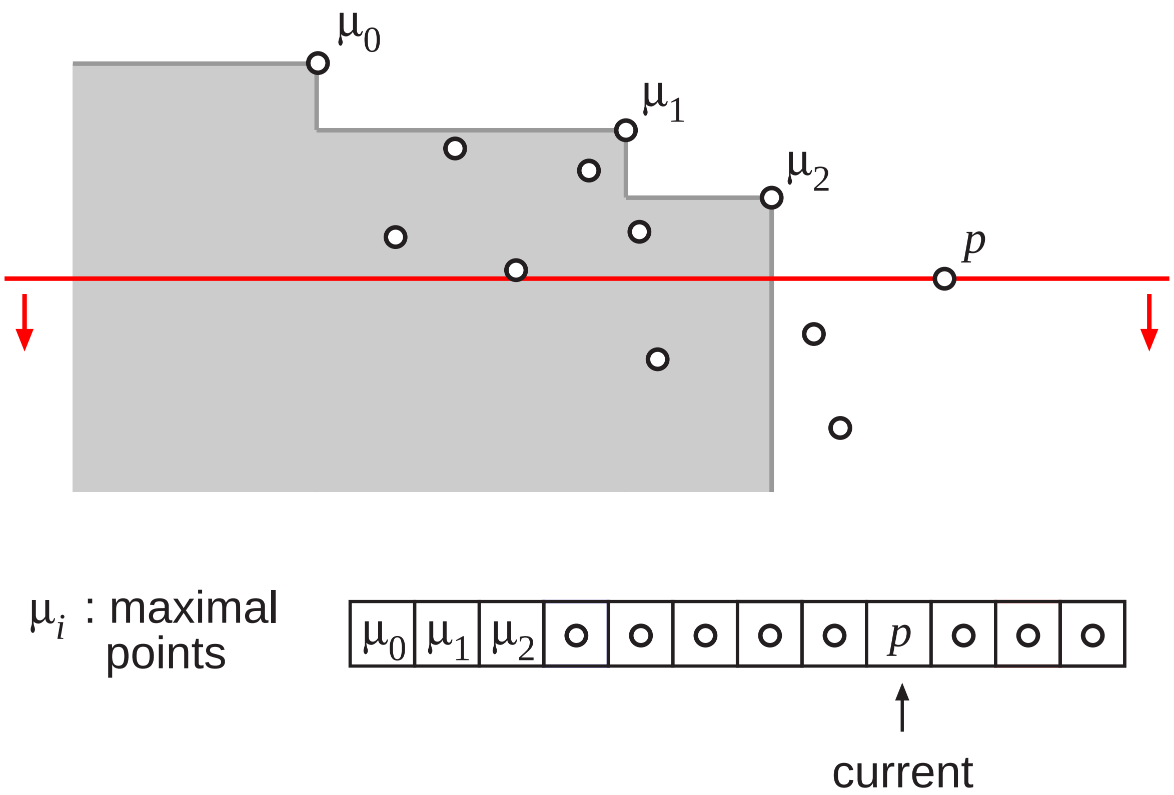

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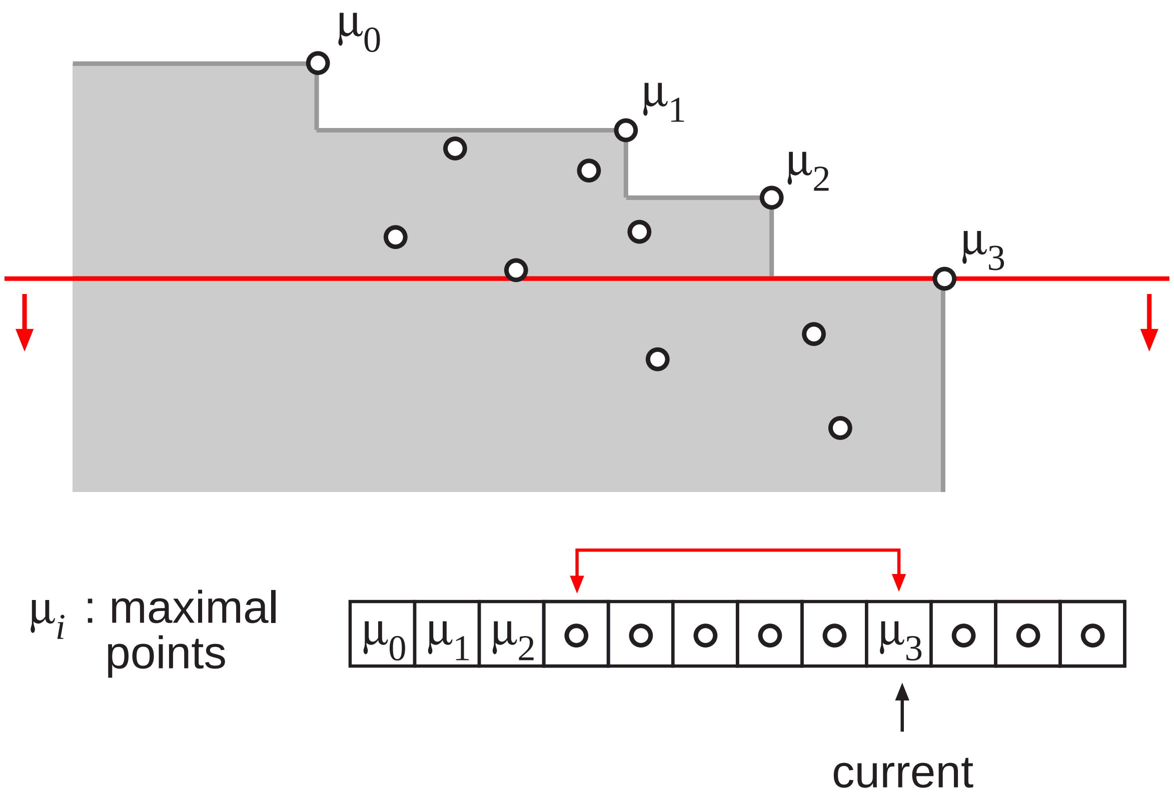

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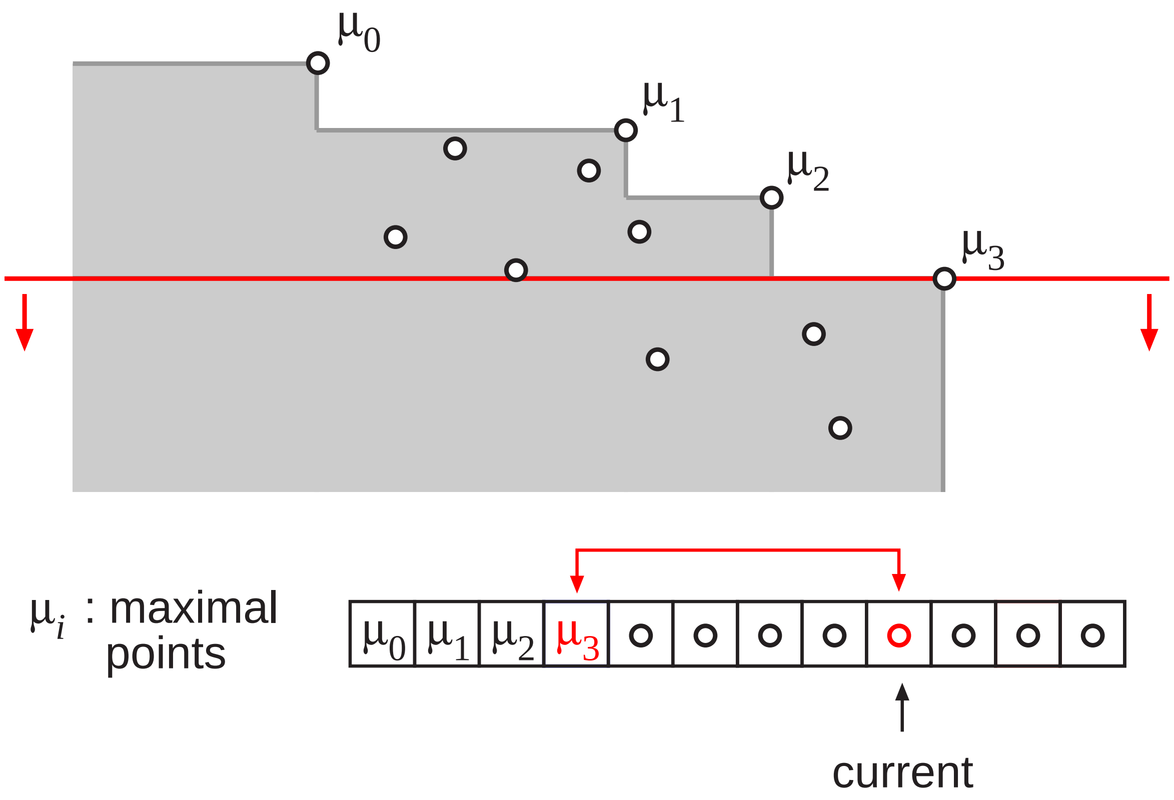

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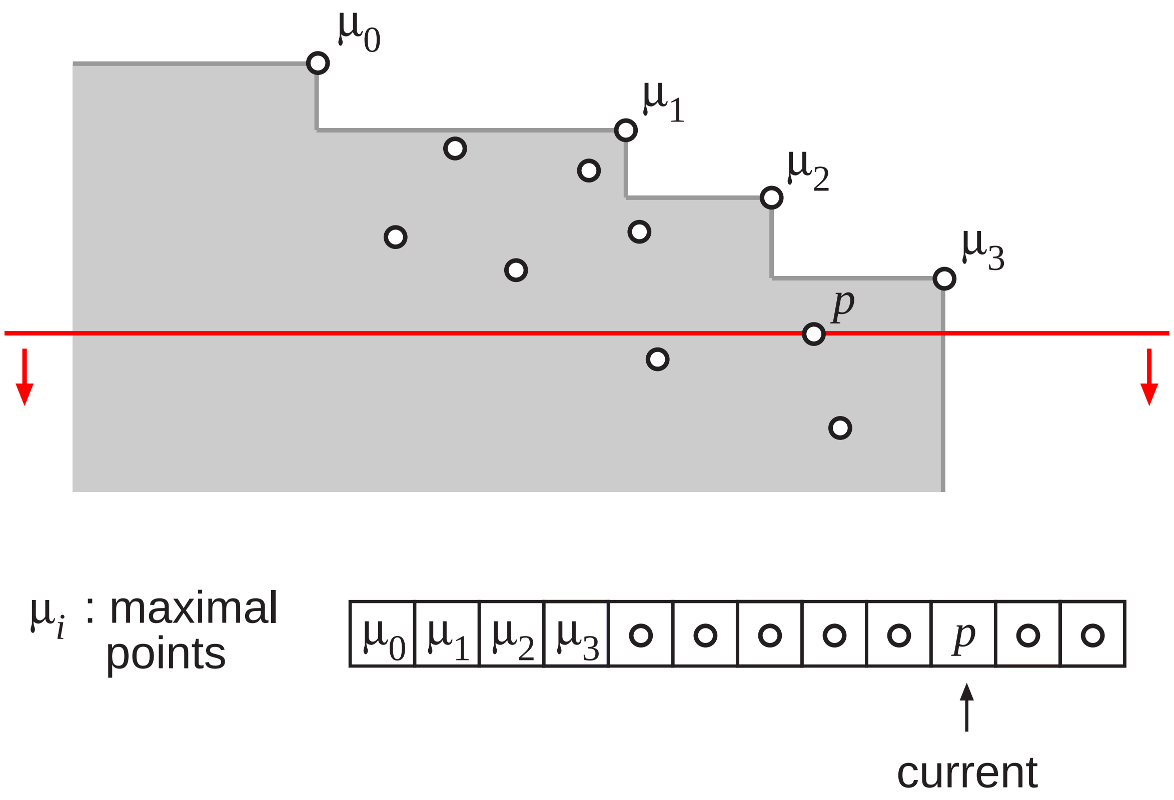

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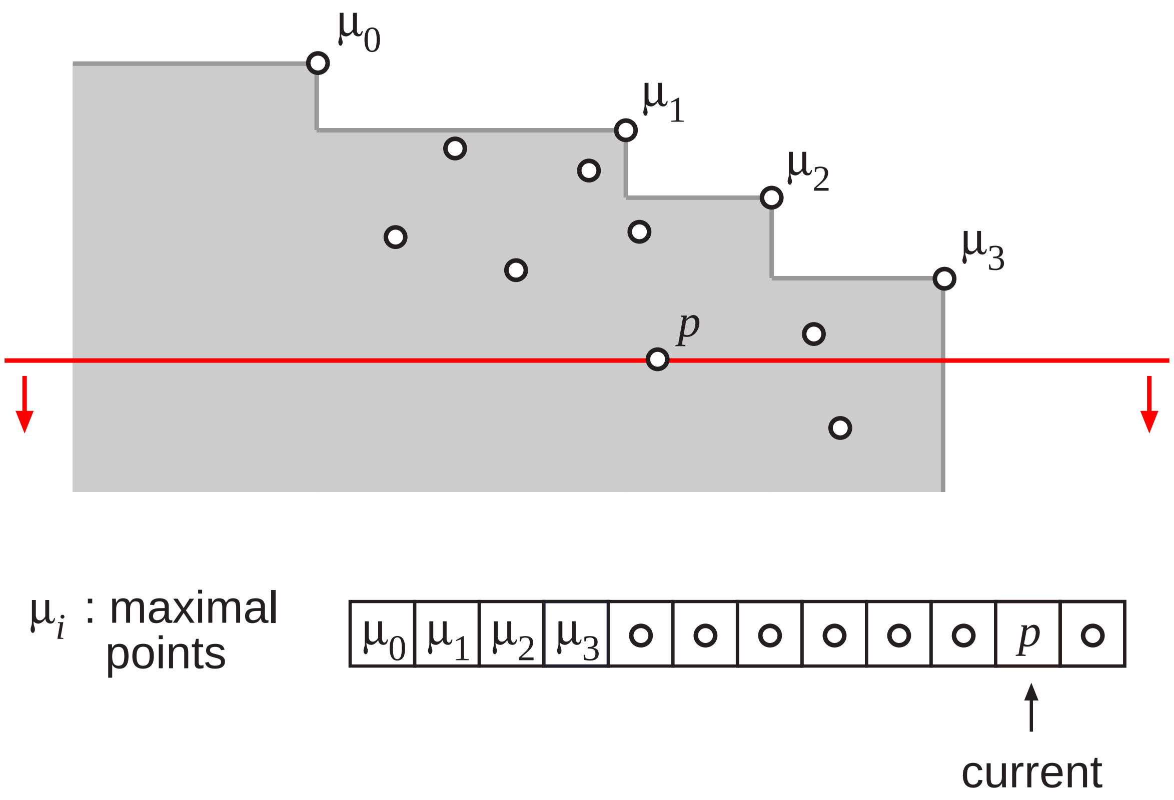

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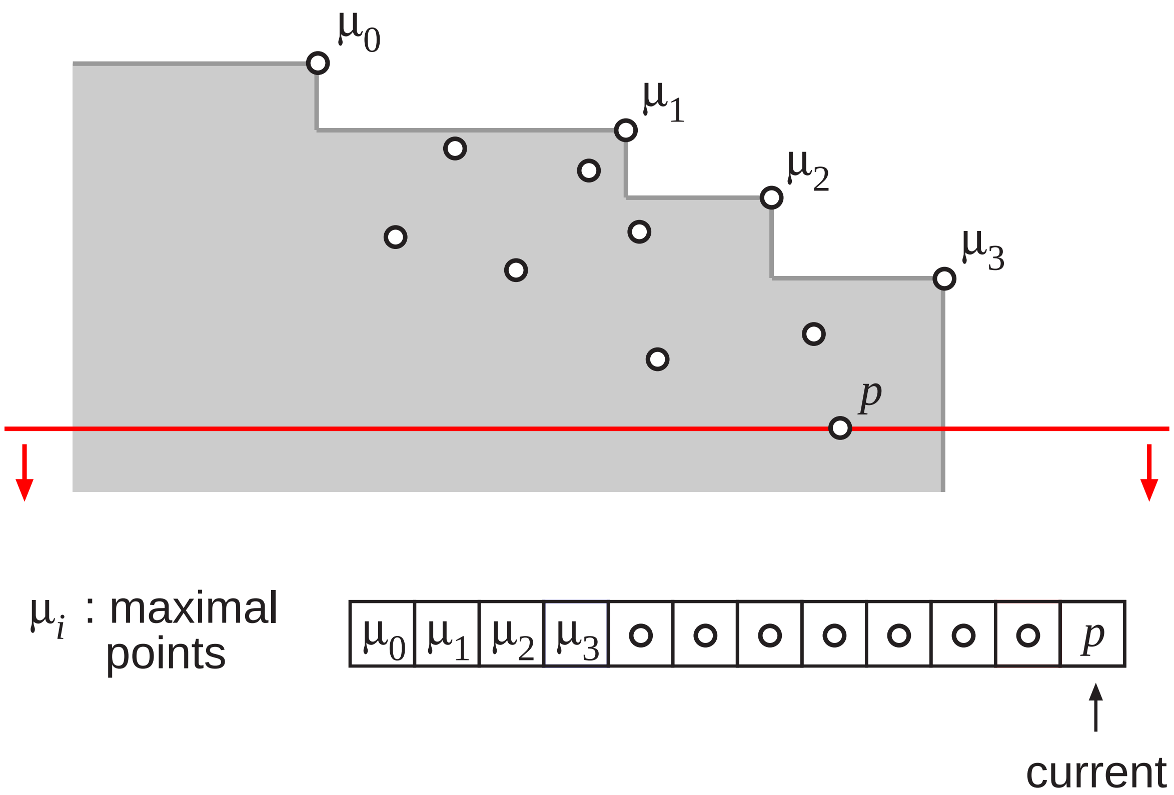

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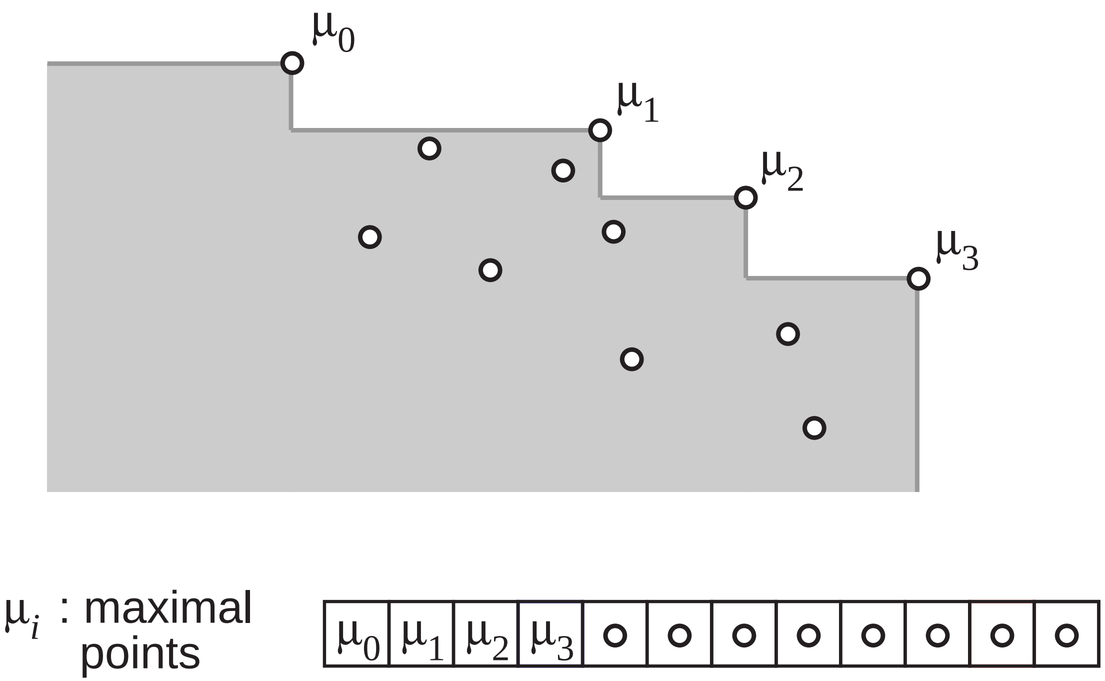

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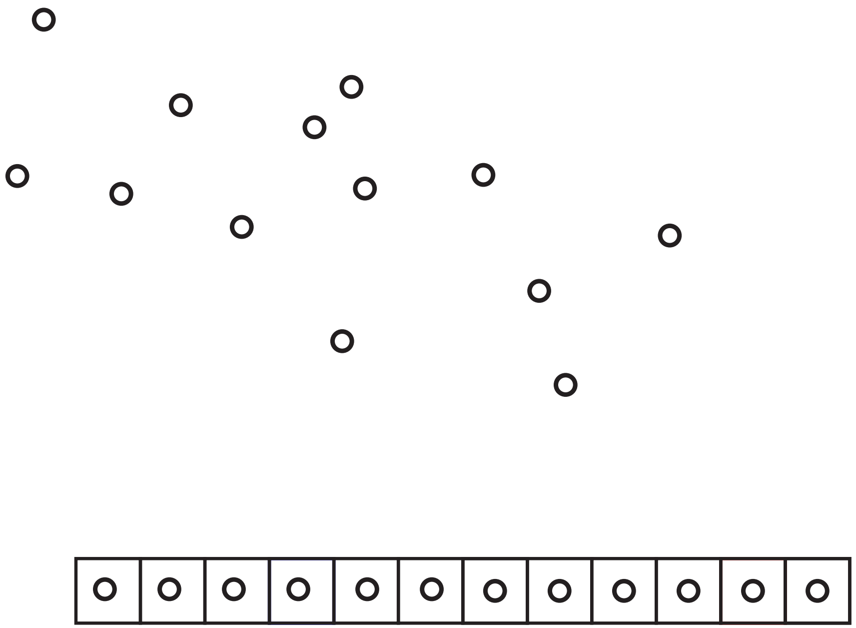

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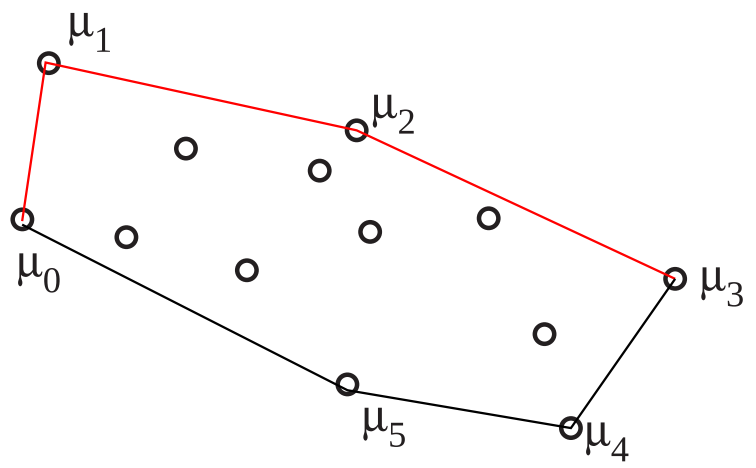

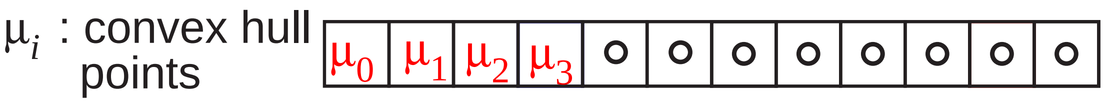

Computing the Convex Hull in 2D: Graham's Scan

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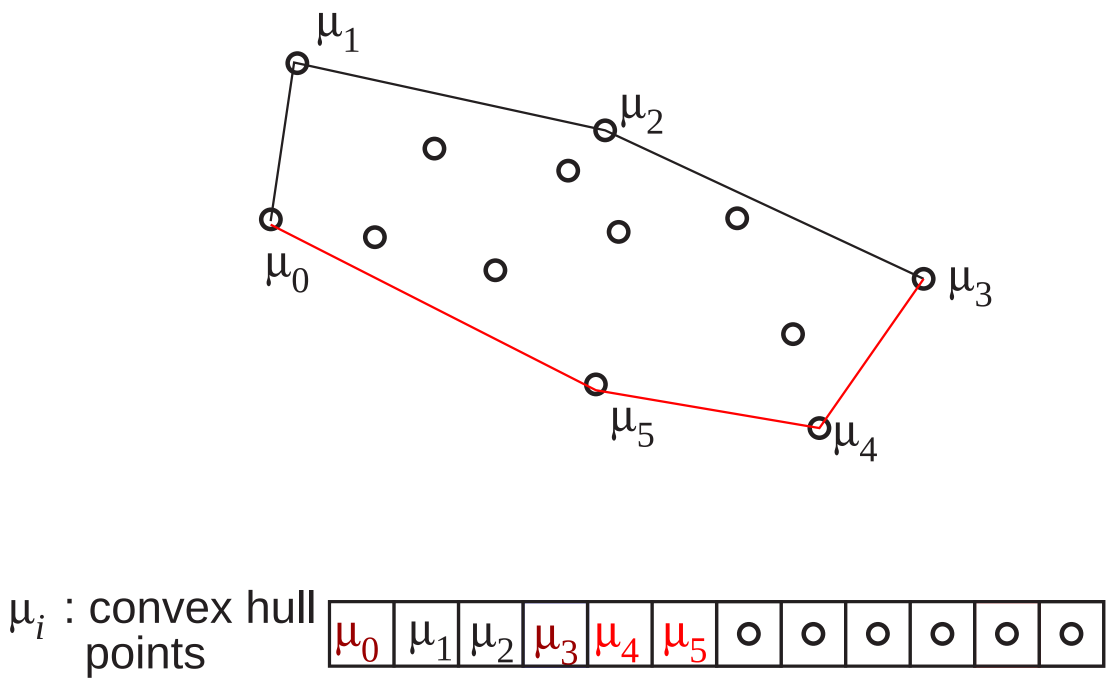

- Convex hull: In-place, $\mathcal{O}(n \log n)$ (sort & scan) [Graham, 1972].

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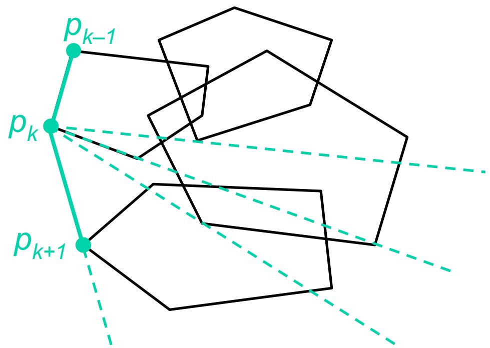

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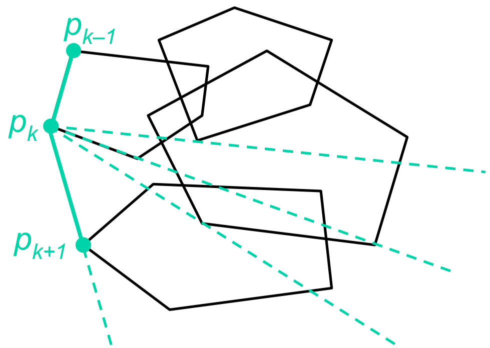

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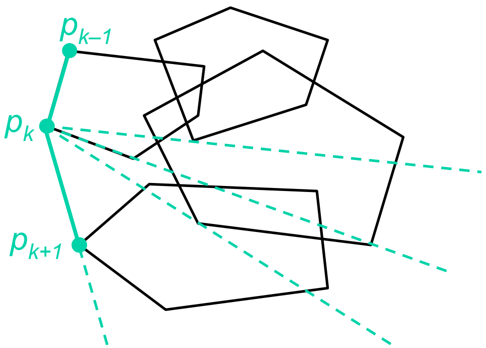

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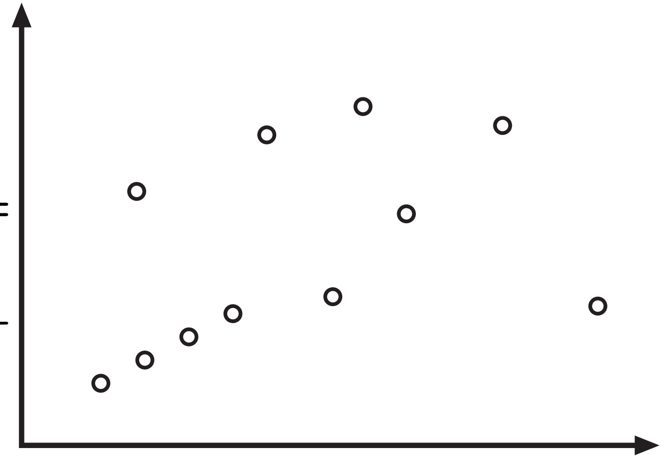
Computing Layers of Maxima



Def. of 'Layers of Maxima':

- Compute 'skyline' MAX($\mathcal{P}$) of $\mathcal{P}$.
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To achieve (in in-place setting):

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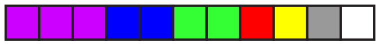

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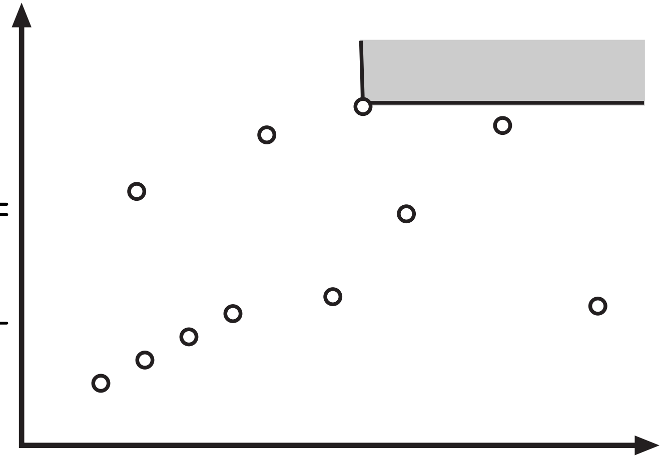

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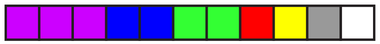

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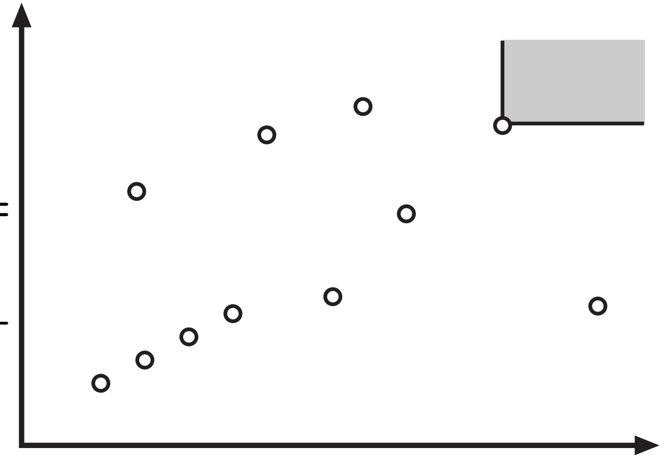

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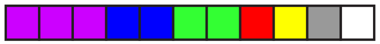

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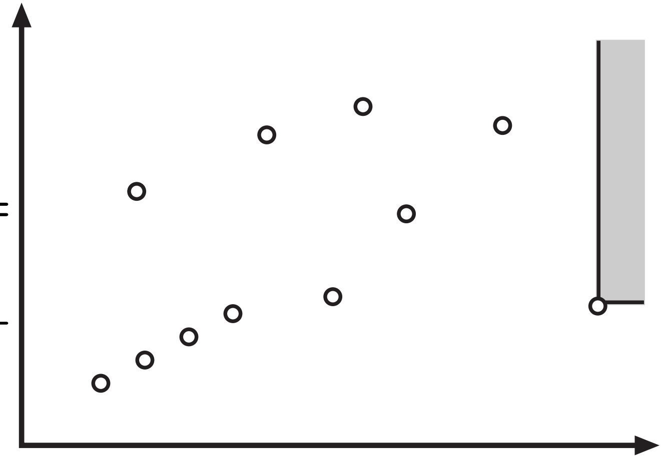

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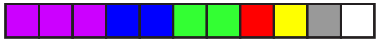

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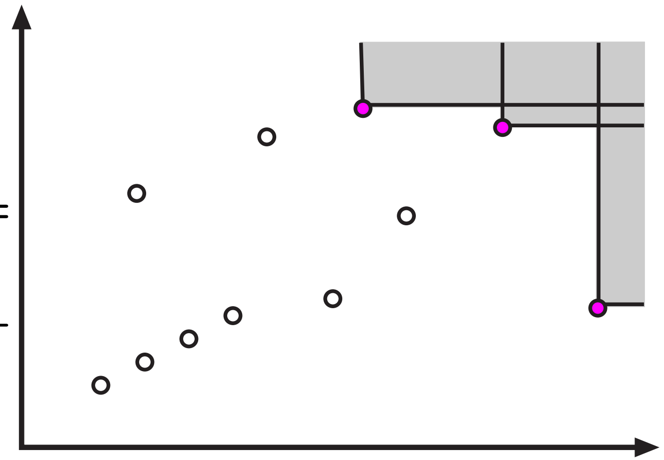

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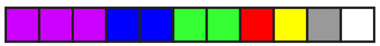

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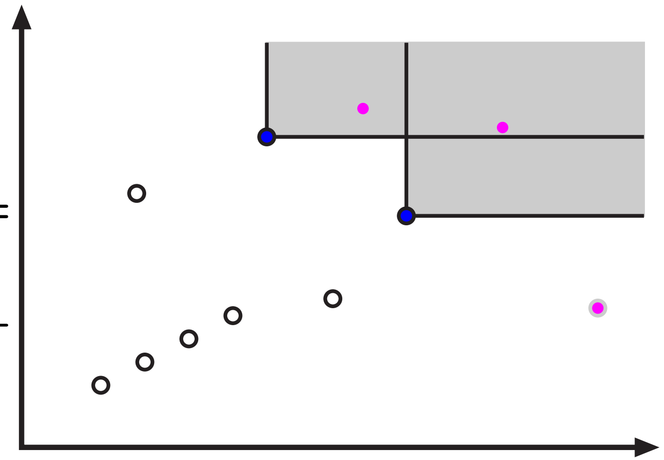

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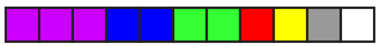

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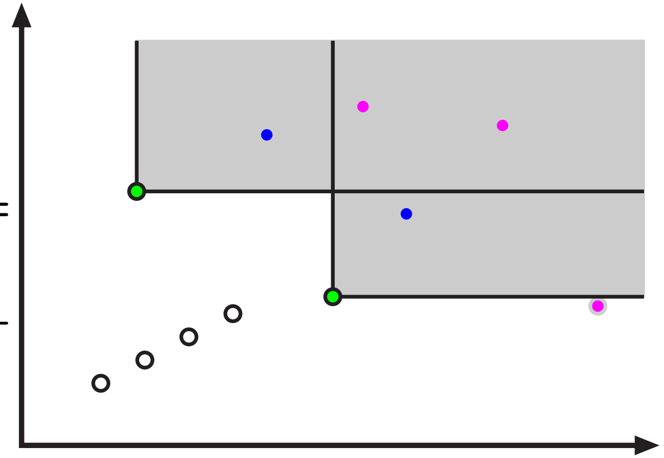

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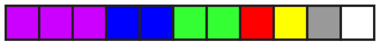

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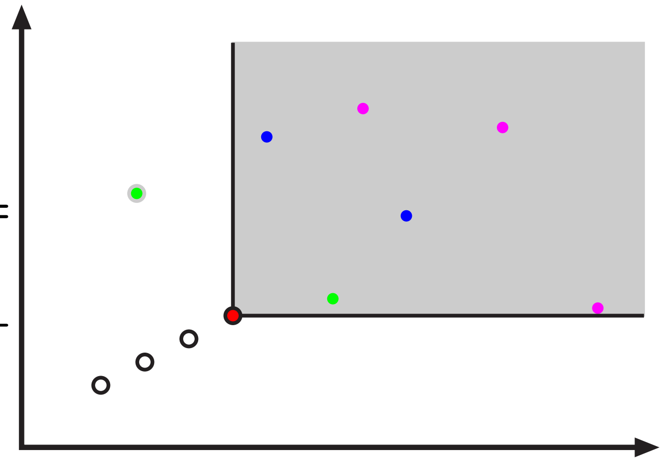

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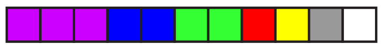

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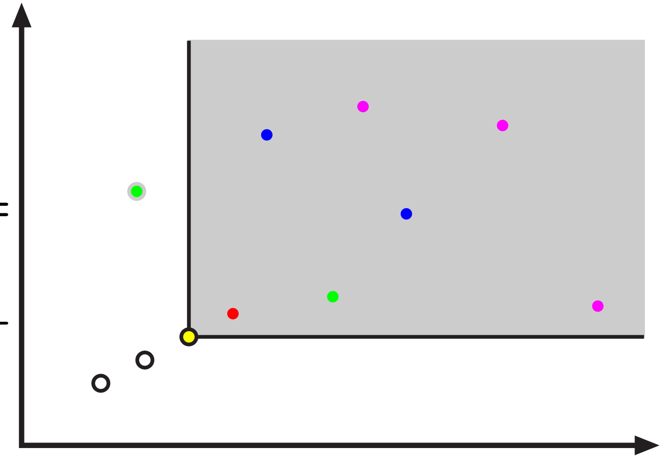

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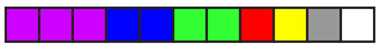

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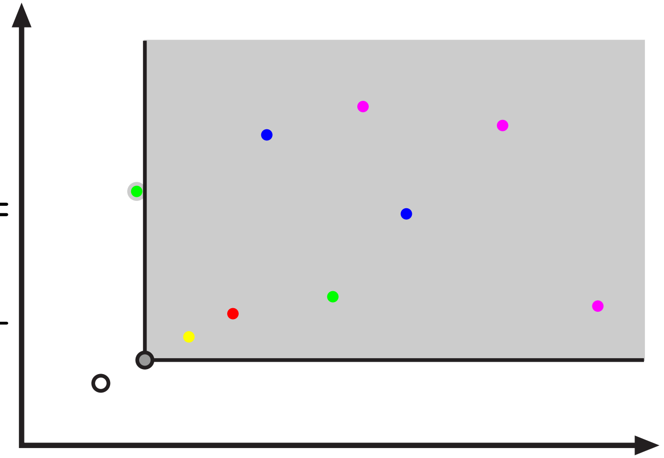

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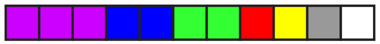

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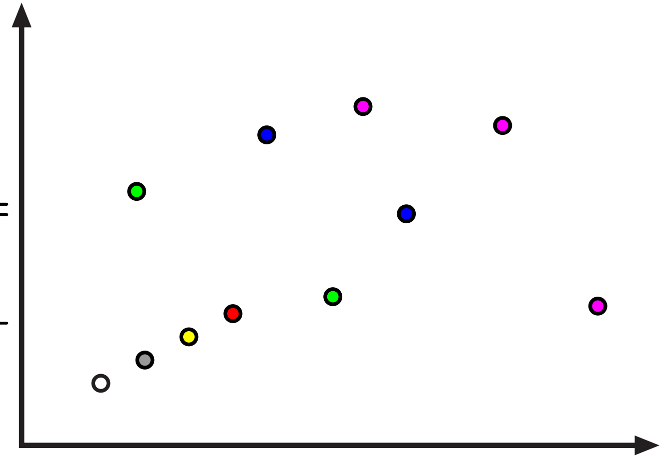

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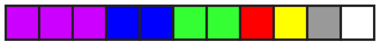

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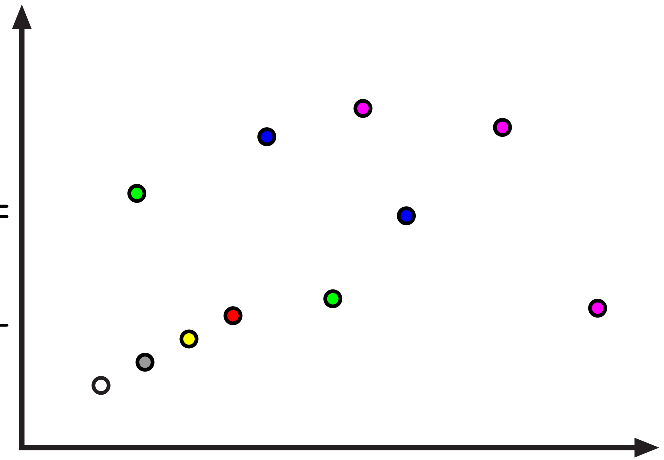

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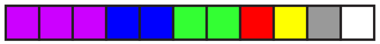

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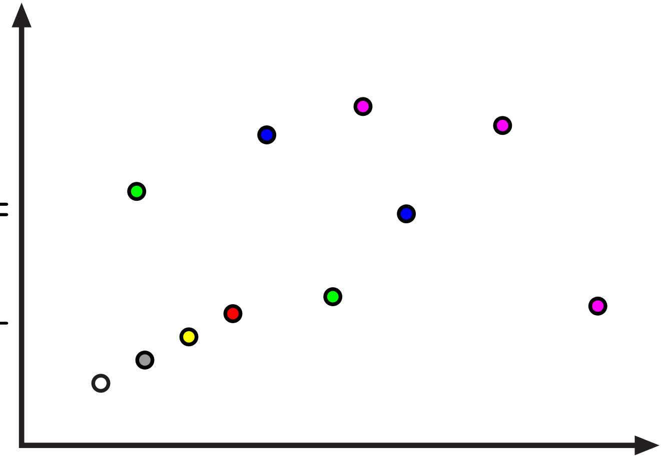

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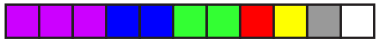

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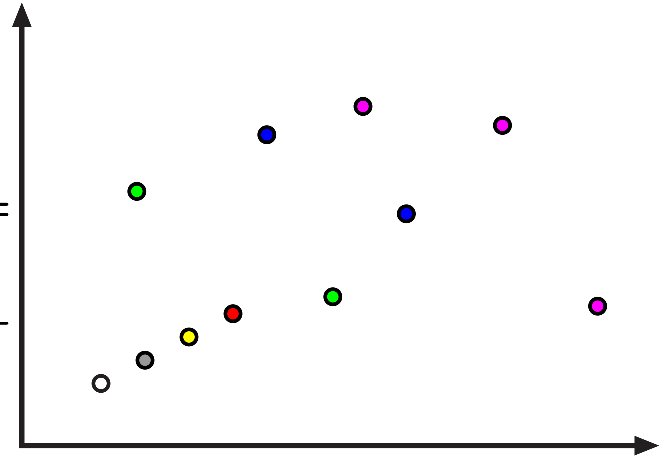

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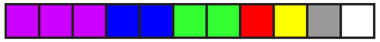

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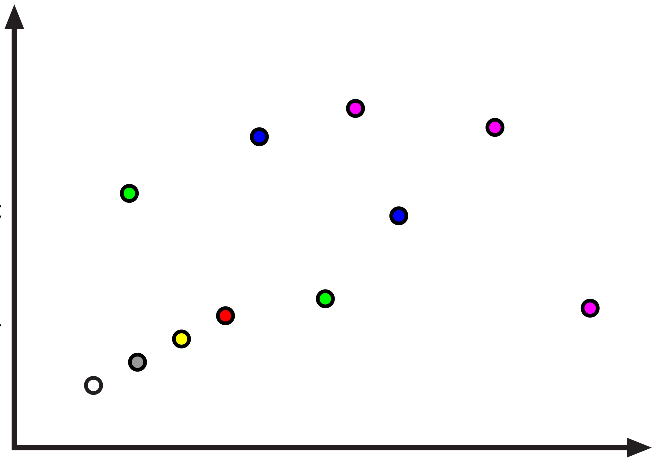

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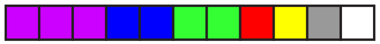

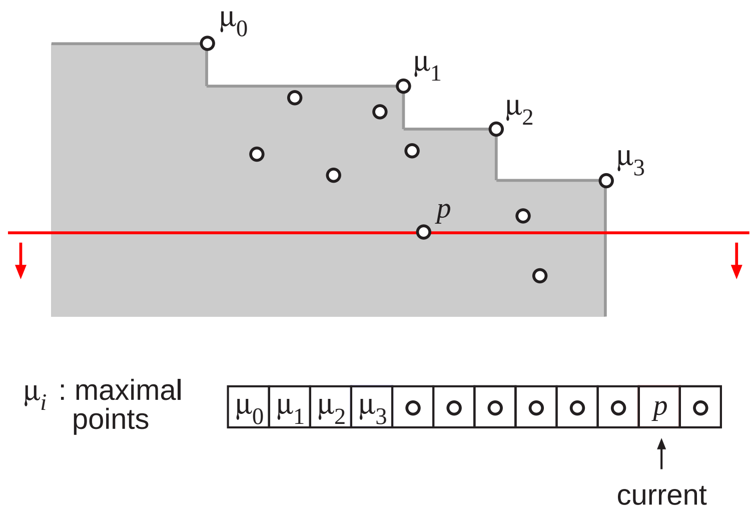

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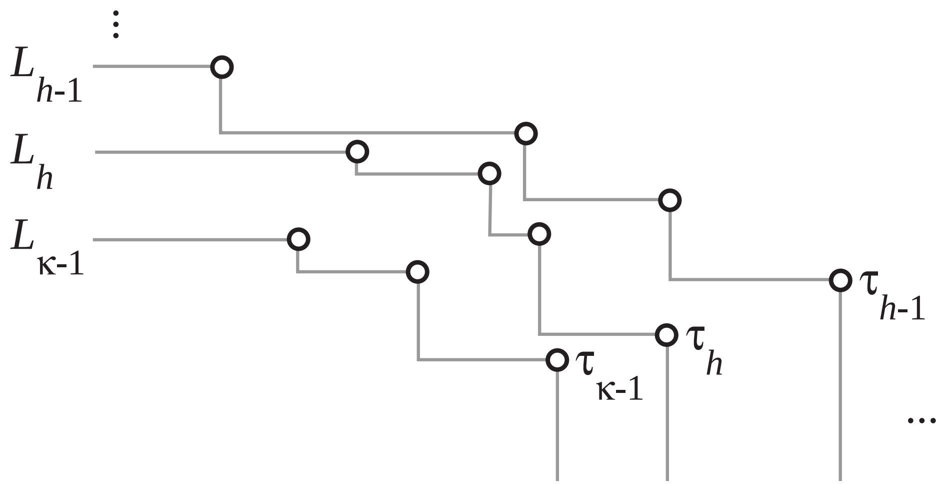

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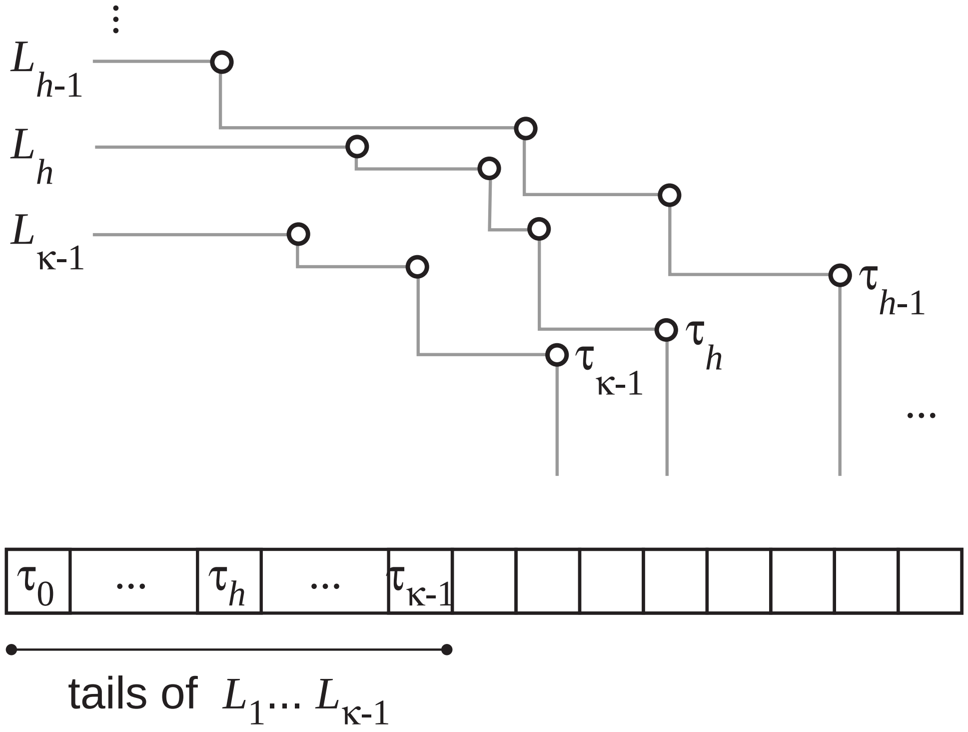

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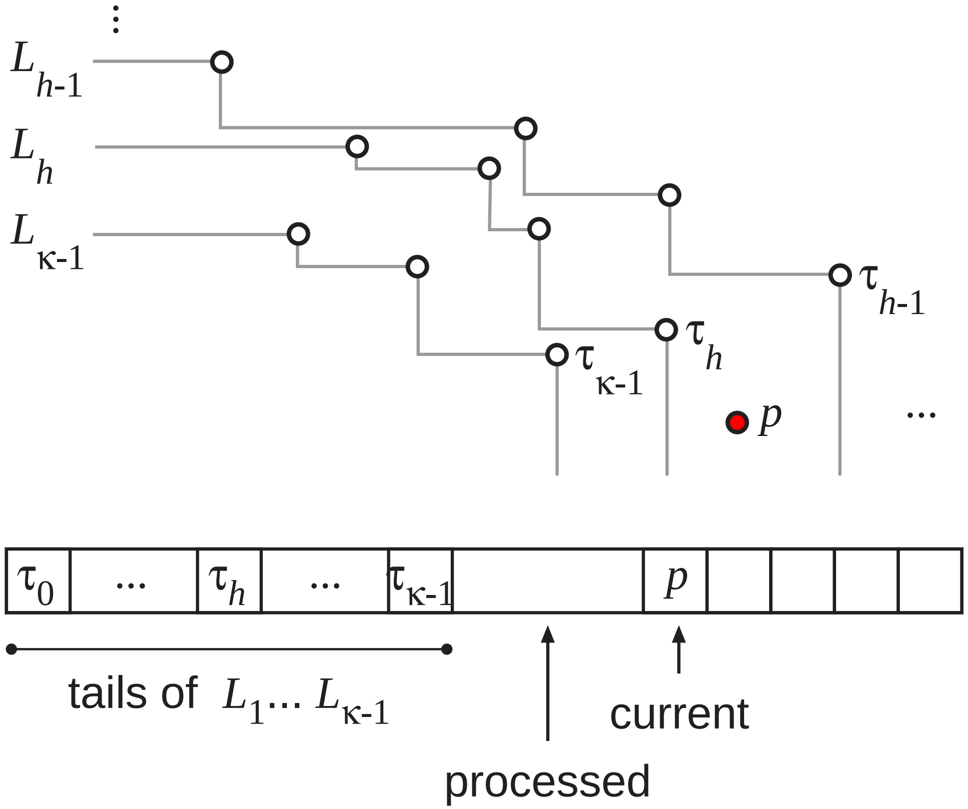

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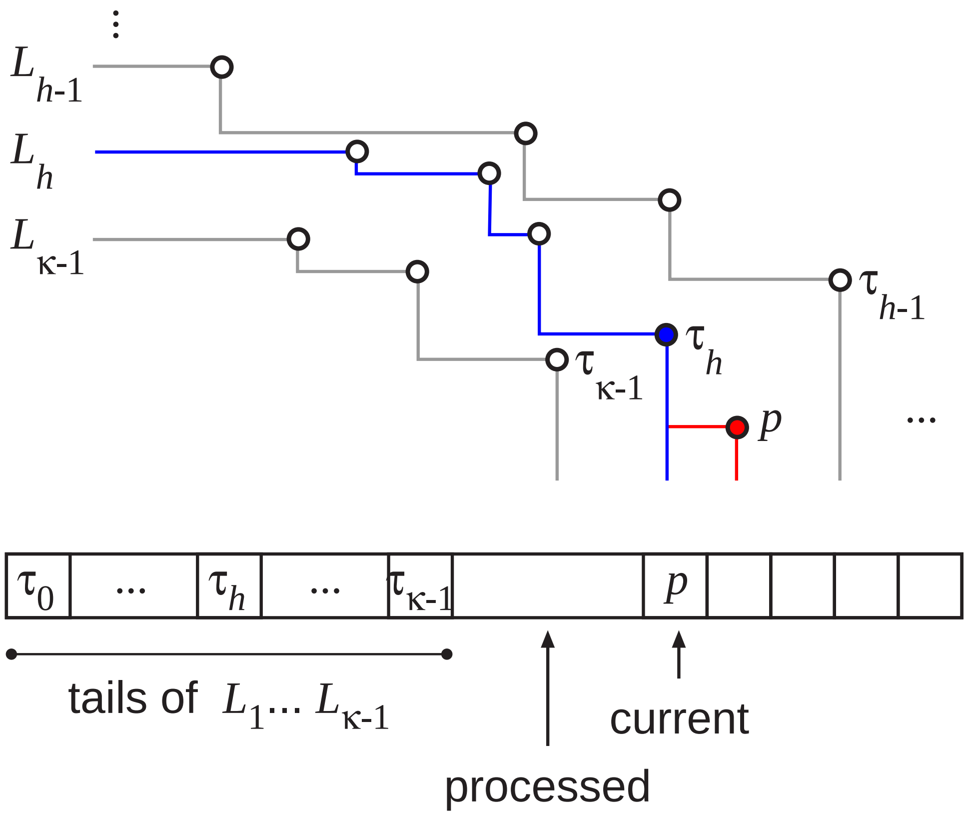

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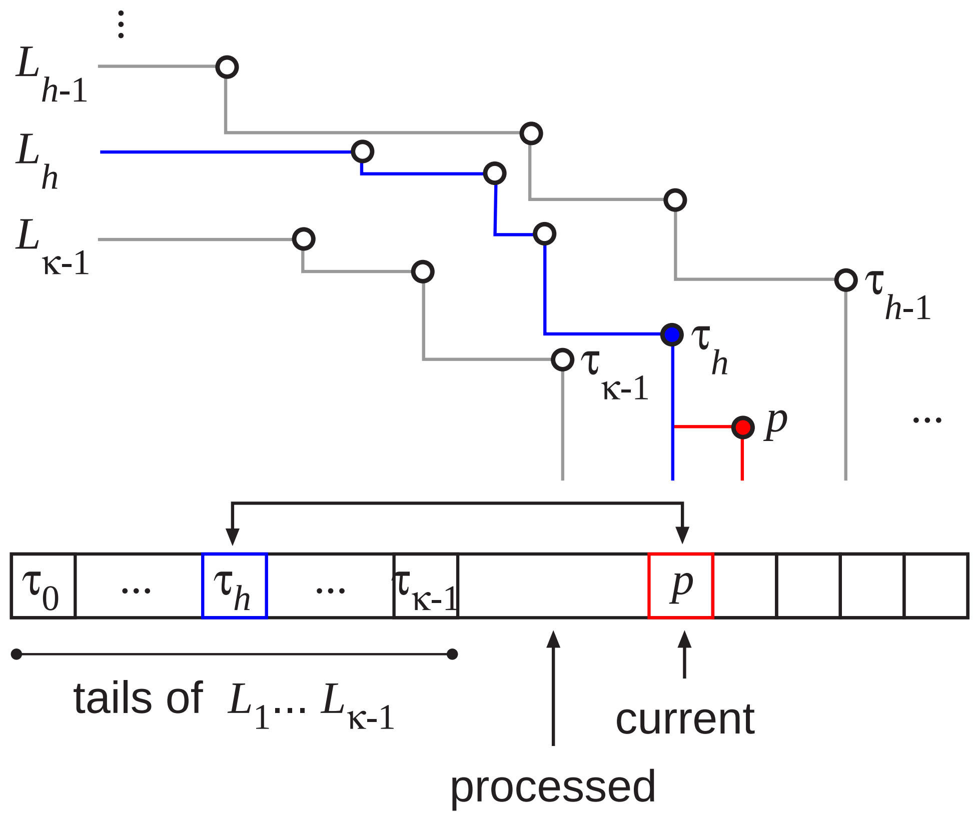

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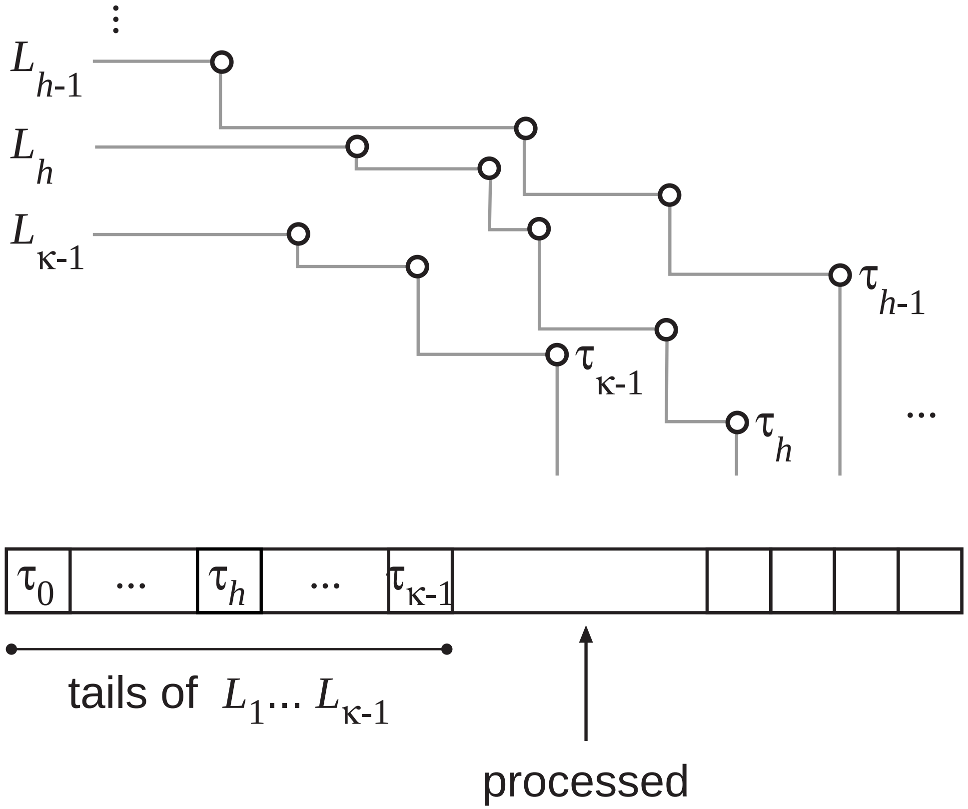

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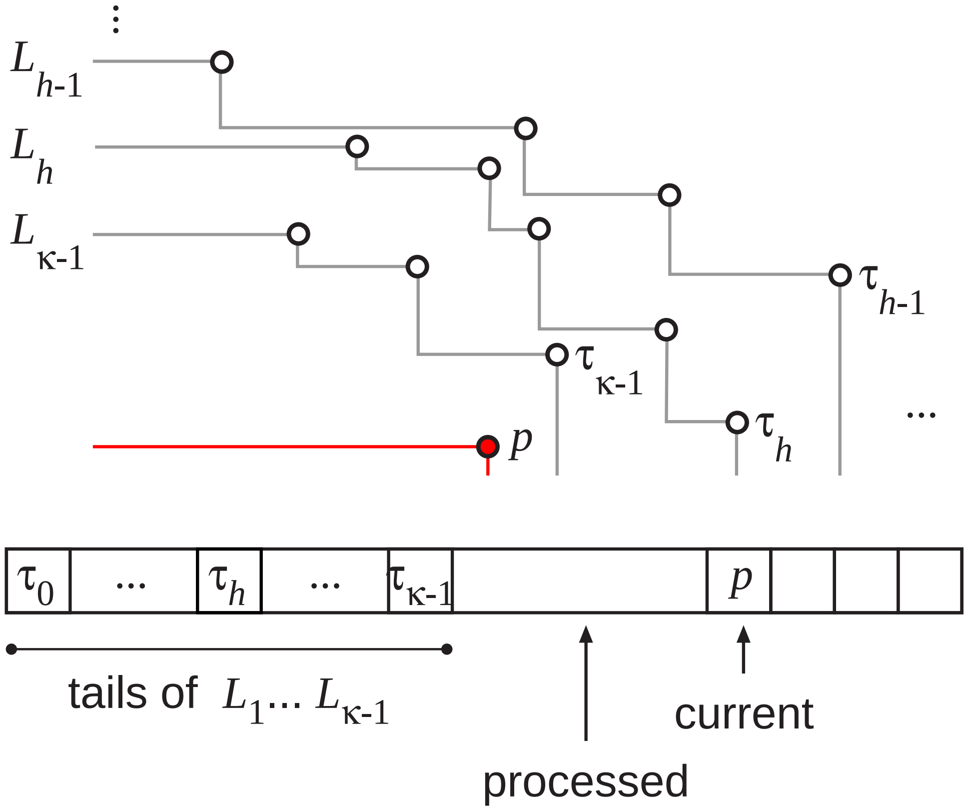

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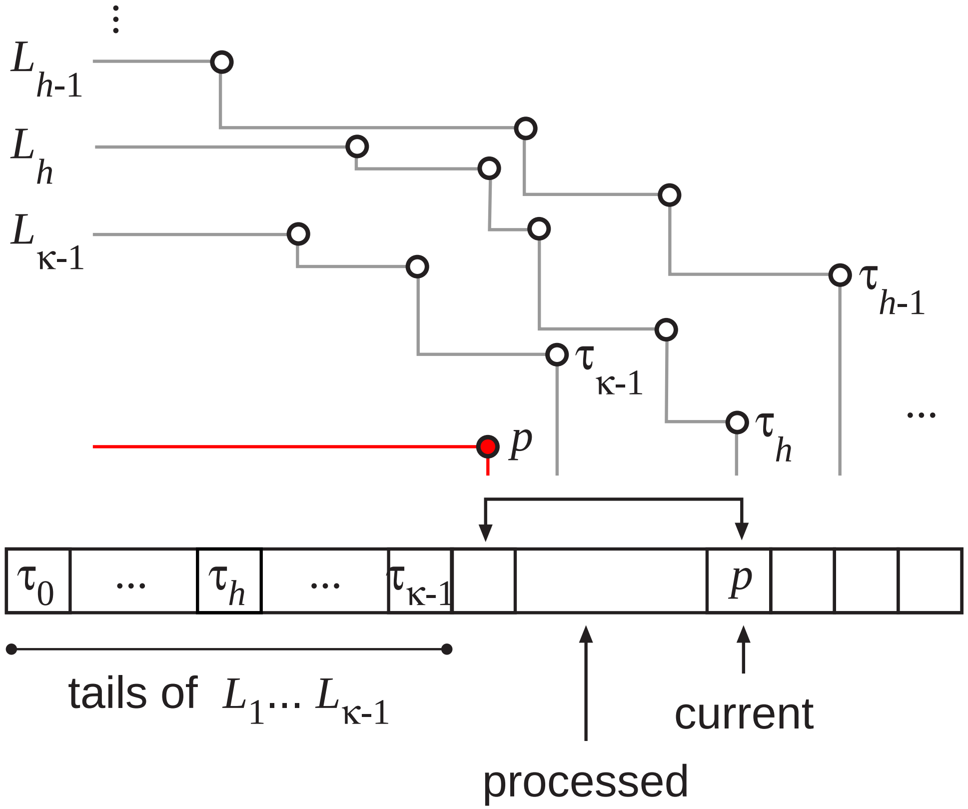

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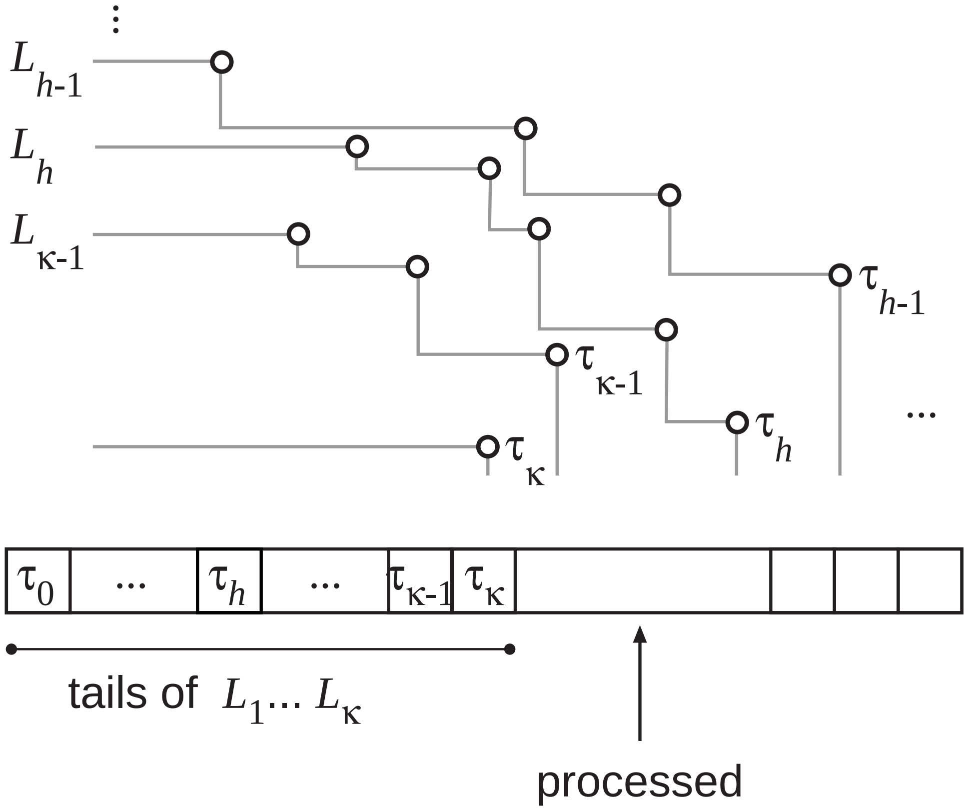

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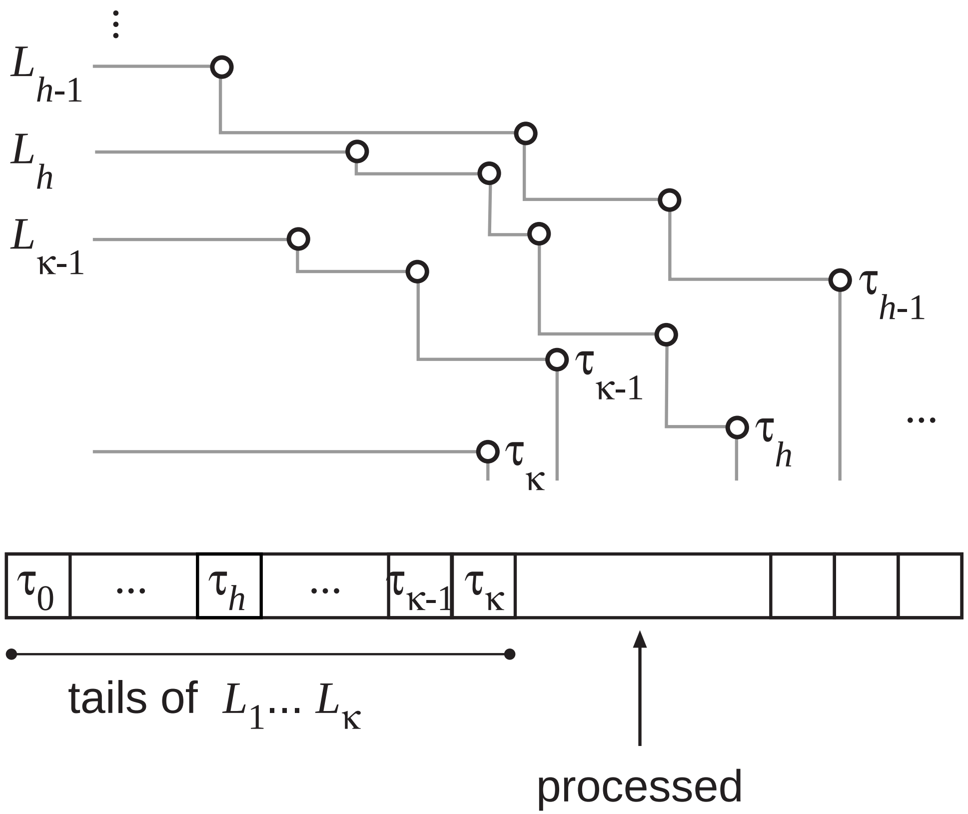

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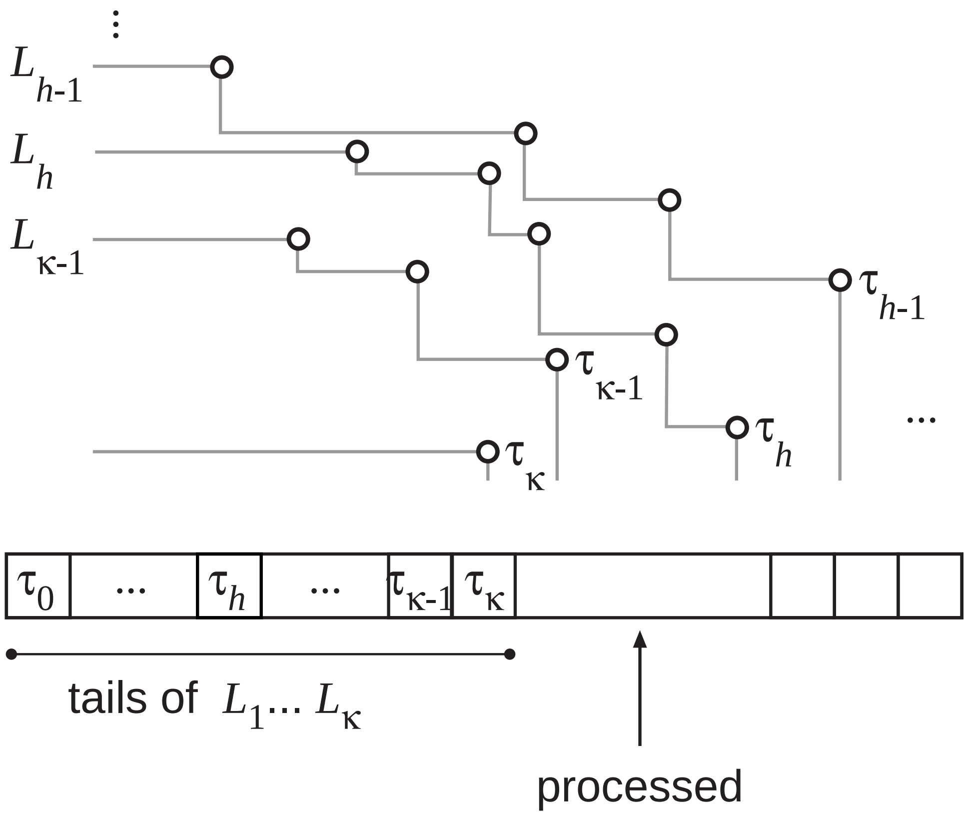

- In-place, $\mathcal{O}(\log n)$ time per point $\Rightarrow \mathcal{O}(n \log n)$ overall.
- Use this algorithm to count points on topmost layers. $\Longrightarrow$

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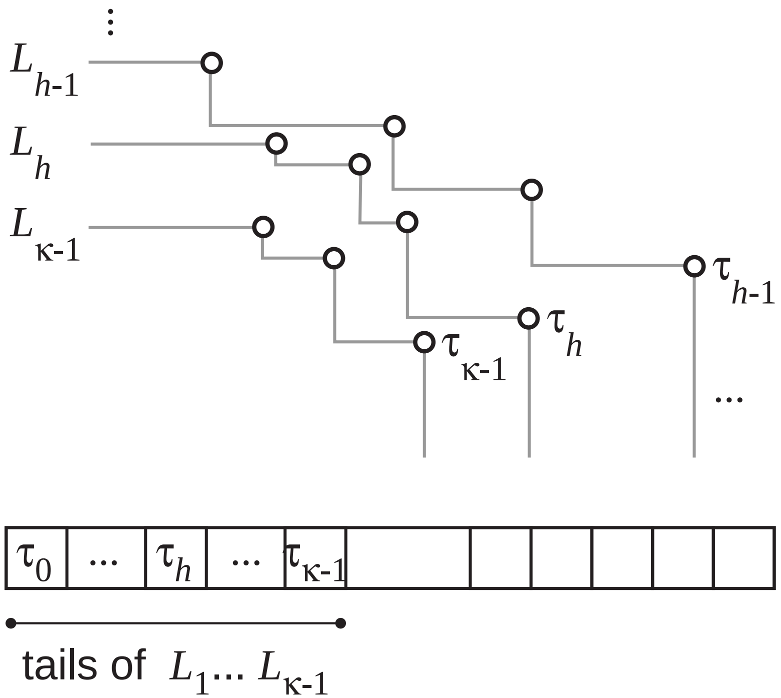
- Fix $\kappa \in \omega(1)$ and run essentially the same algorithm.
- Increment a global counter per "tail"-update.



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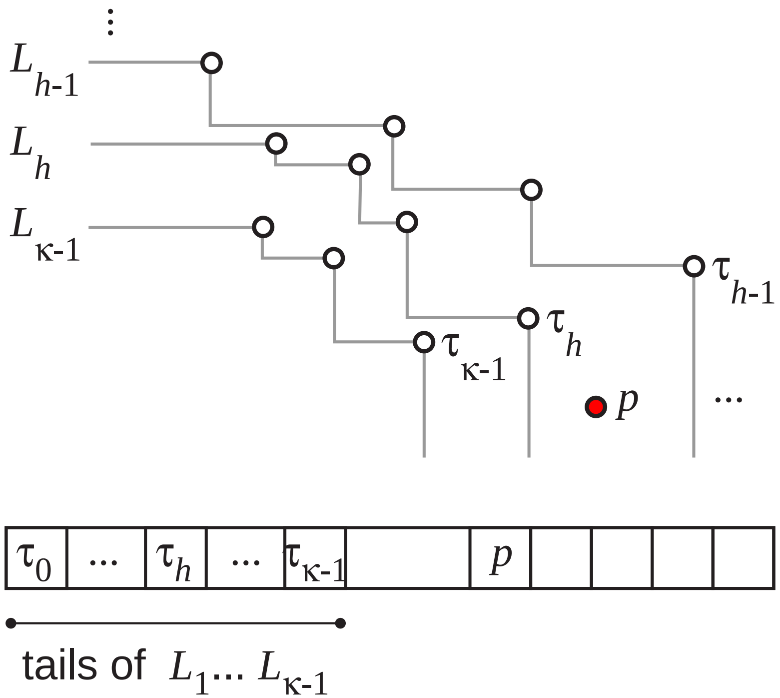

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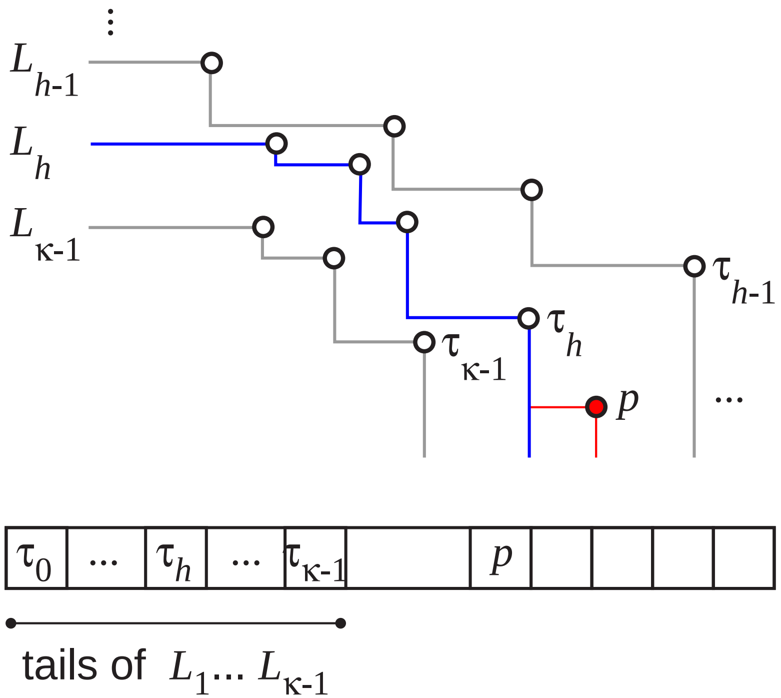

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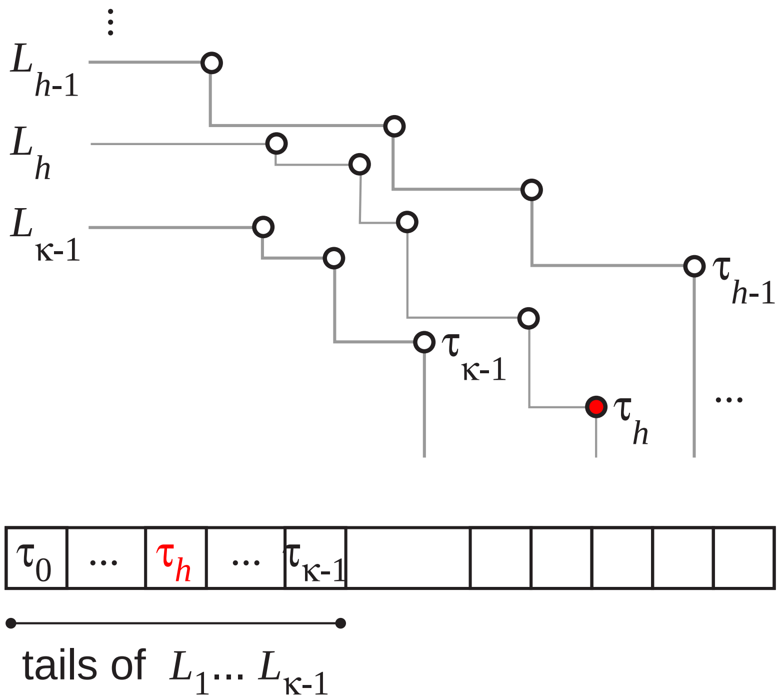

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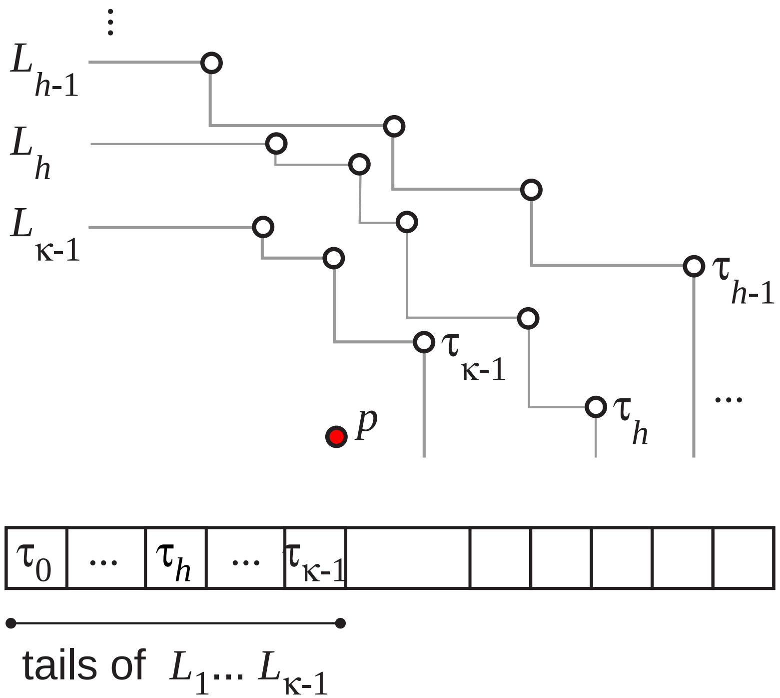
A closer look:

- Query: Is some point p not on topmost κ layers? $\mathcal{O}(1)$/point



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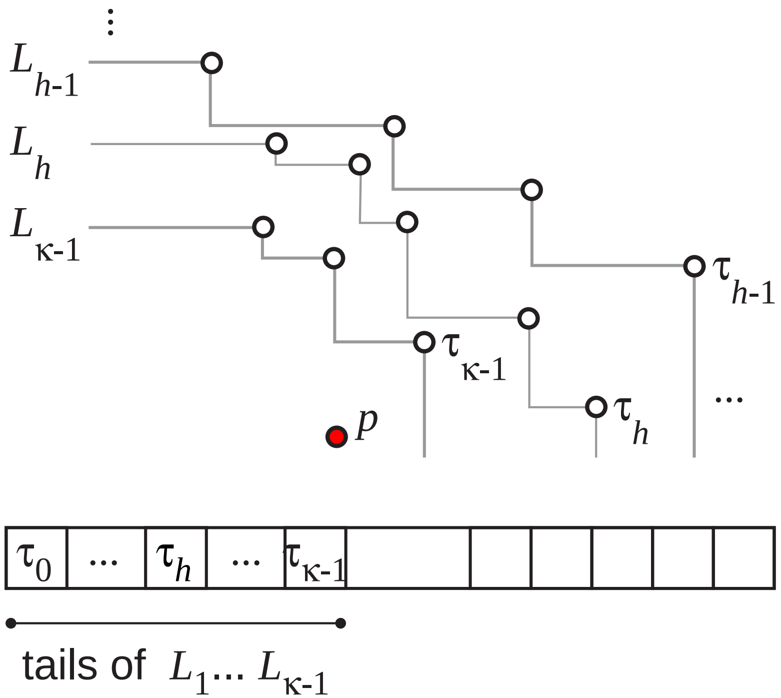

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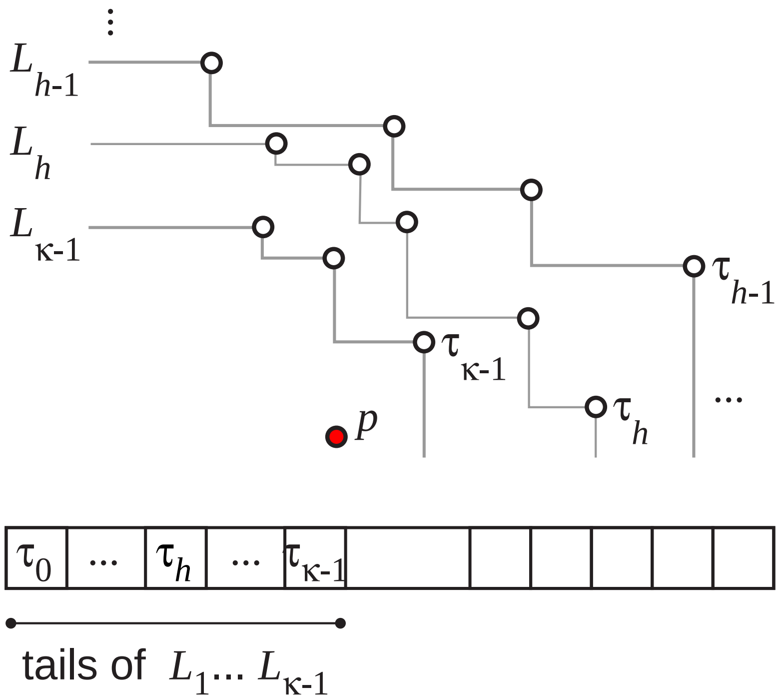

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To compute each $|L_i|$. . .

- . . . we need to increment a counter c_i for each update of τ_i.

Wait a minute!

- Did you say "$\kappa \in \omega(1)$"?
- Where/how to store κ counters?

Finding “extra” space

Space needed: $\kappa \in \omega(1)$ counters.

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Partitioning the input array:

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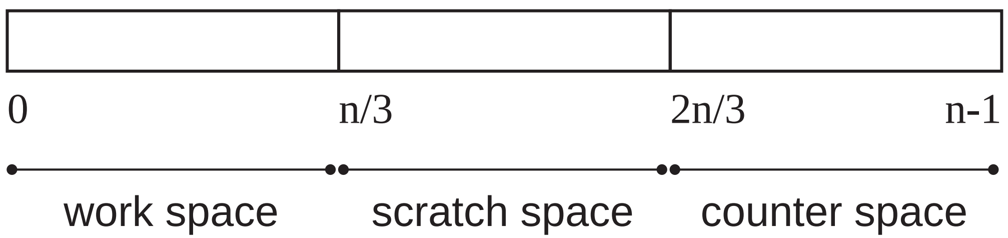

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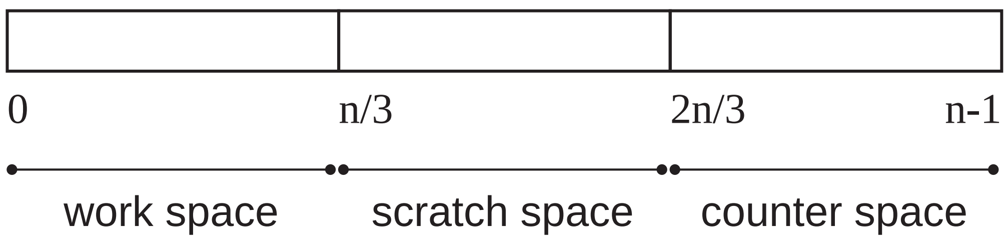
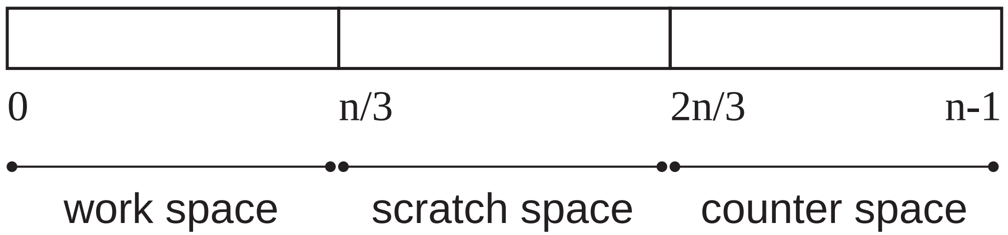


Extracting the Topmost κ Layers

Setup:

- Compute size c_i of i-th layer, $0 \leq i < \kappa = \frac{1}{6}n/\log_2 n$.



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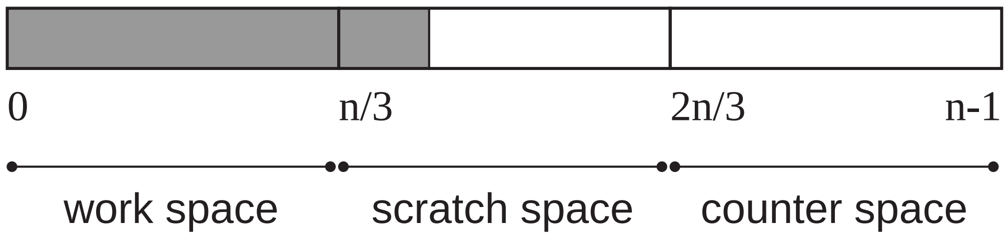
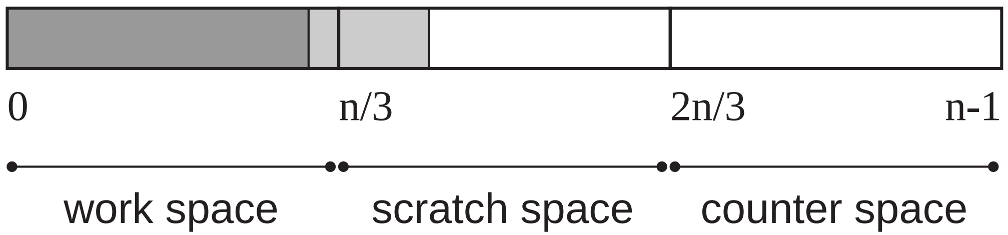


Extracting the Topmost κ Layers

Setup:

- Compute size c_i of i-th layer, $0 \leq i < \kappa = \frac{1}{6}n/\log_2 n$.
- Compute maximal j s.t. $\sum_{i=0}^{j} c_i \leq \frac{1}{3}n$.



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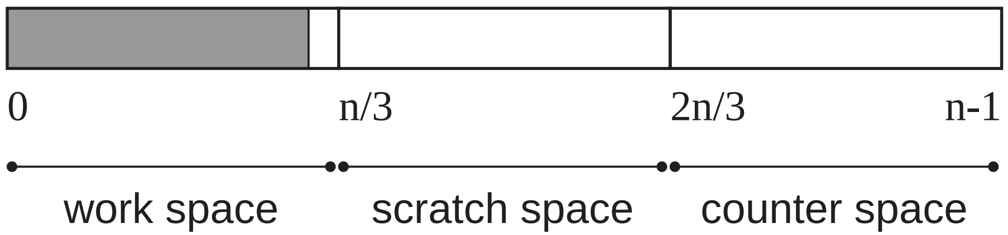
- Combine extraction with *counting sort*.



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Extracting the topmost j layers:

- Combine extraction with *counting sort*.
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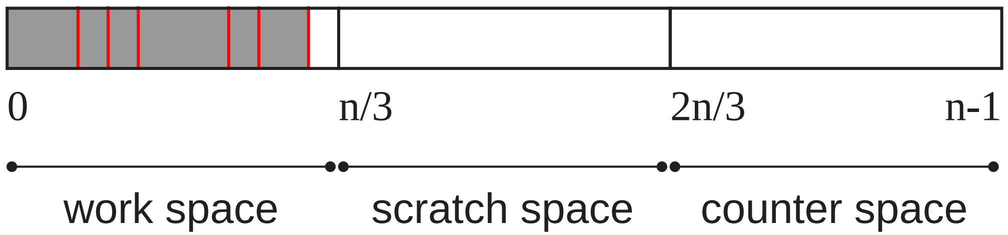
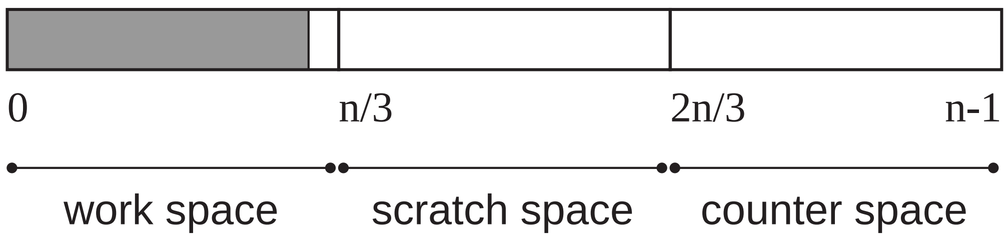
- Move constructed layers to front.



Extracting All Layers–I

First phase, i.e., for earlier iterations:

- Construct (groups of $\frac{1}{6} \cdot \frac{n}{\log n}$) layers in first part of the array.



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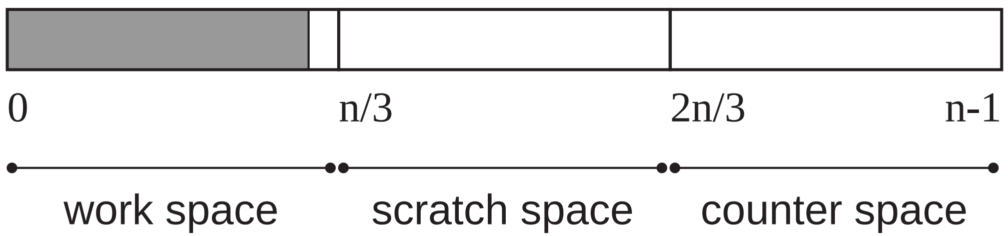

Analysis:

- Cost per iteration that processes all ξ points on $\frac{1}{6} \cdot \frac{n}{\log n}$ layers: $\mathcal{O}(n + \xi \log n)$.
- Invariant: Keep unprocessed(!) points in sorted $<_y$-order.

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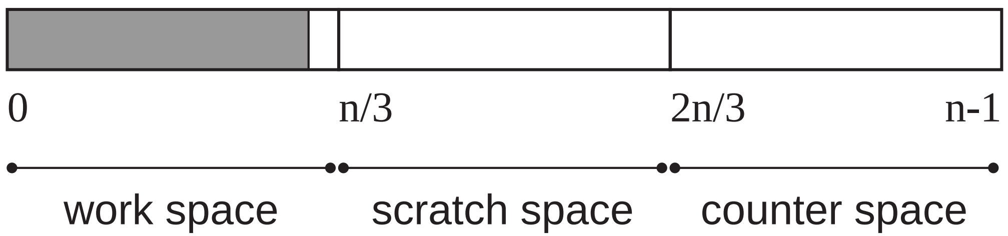

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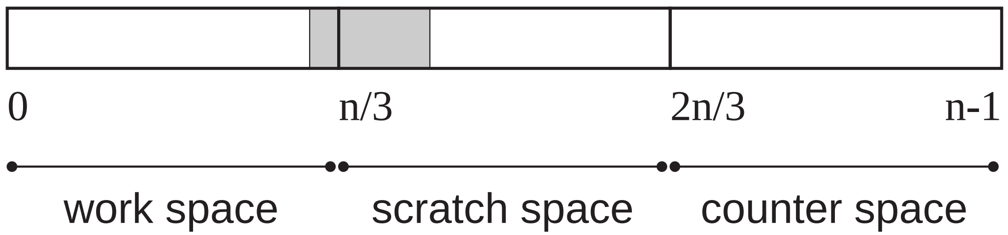
- Invariant: Keep unprocessed(!) points in sorted $<_y$-order.
- No more than $\mathcal{O}(\log n)$ such iterations, i.e., $\mathcal{O}(n \log n)$ global cost.

Second phase (with a grain of salt):

- Construct (in $\mathcal{O}(n \log n)$ time) one layer crossing the boundary between work and scratch space using the skyline algorithm.



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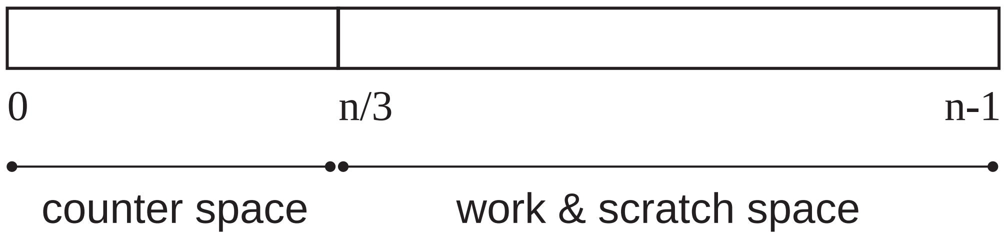

Third phase i.e., for later iterations:

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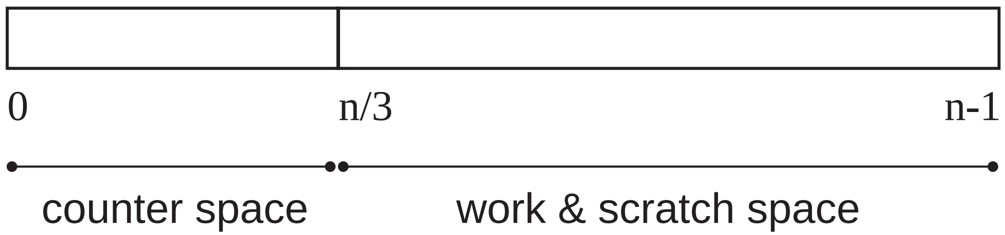

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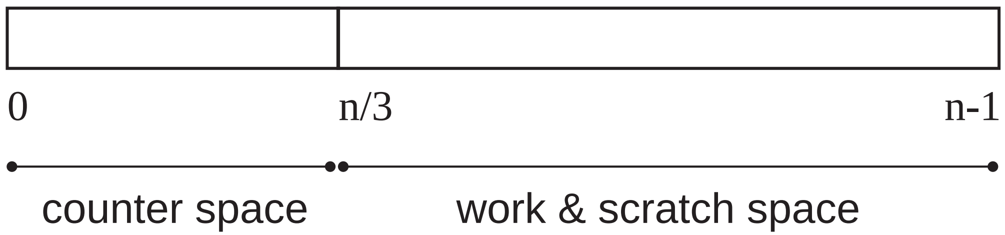

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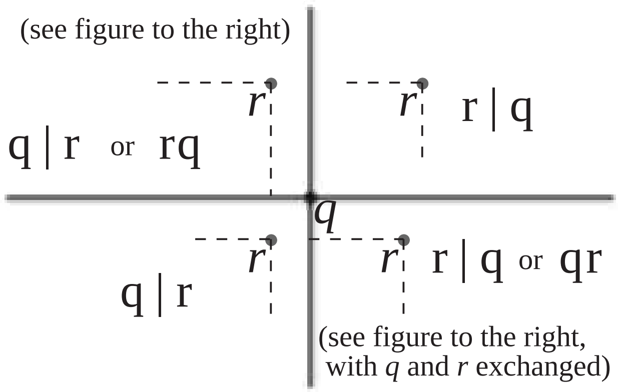
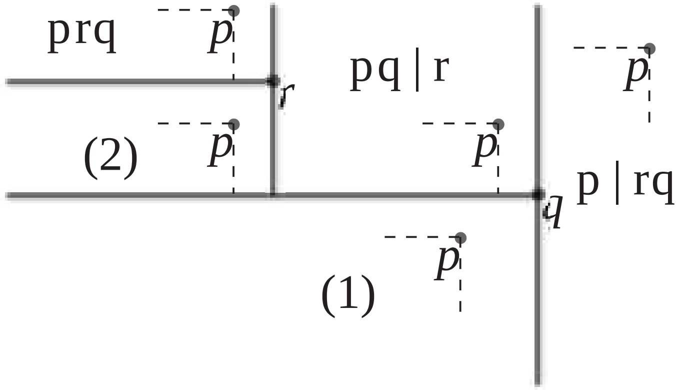
- $\mathcal{O}(n + \xi \log n)$ time to process $\frac{1}{6} \cdot \frac{n}{\log n}$ layers with ξ points.
- Whenever scratch space is too small, perform skyline computations on subarrays of geometrically decreasing size $\Rightarrow \mathcal{O}(n \log n)$.

Repairing the Layer Order

Finishing up:

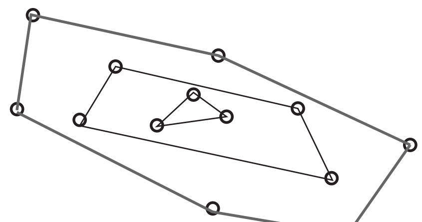
- Bit-encoding corrupts layer order (locally).
- Repairing after last iteration by linear time sweep:
- Each bit-neighbour pair (q, r) can be correctly ordered by only looking at q, r and predecessor p:





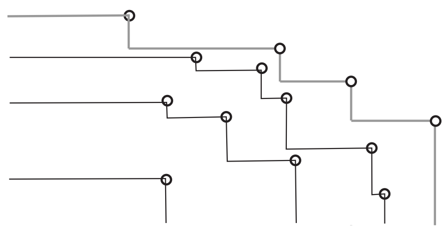
- $\Rightarrow$ Layer order repairable by linear scan.
- $\Rightarrow \mathcal{O}(n \cdot \log_2 n)$ in-place computation of layers of maxima.

Convex layers computation



Ideas applicable also to convex layers problem?

- Applicable:



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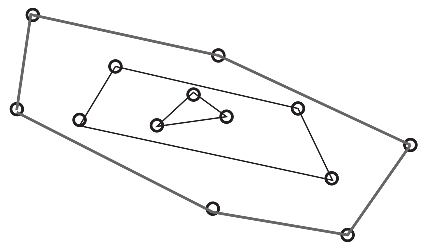
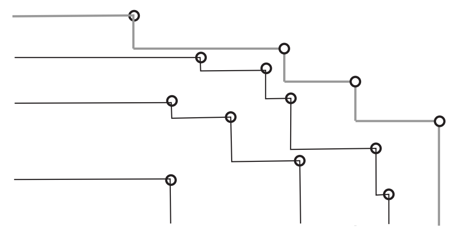


Ideas applicable also to convex layers problem?

- Applicable:
  - Sweep-framework.



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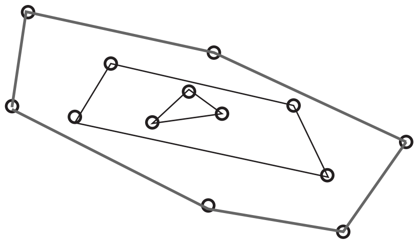

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- Applicable:
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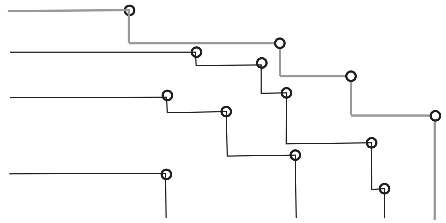
  - Maintainance of tails for each layer.



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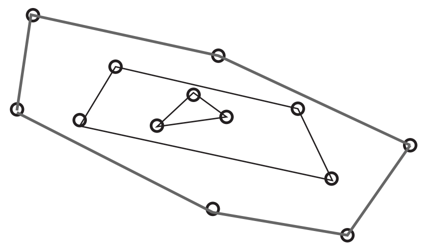

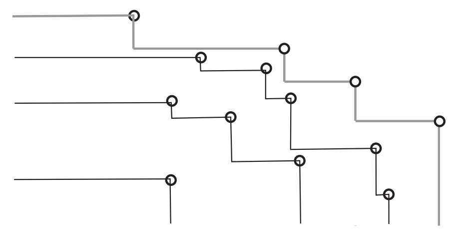

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- A point changes its convex layer $\mathcal{O}(n)$ times during sweep.

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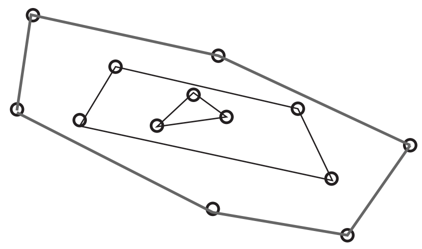

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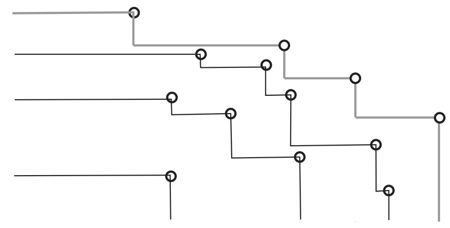

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Solutions (for 2D layers and 3D hull)?

- for 2D convex layers?
- for 3D convex hull?

Computing the skyline in 3D

Time-Optimal algorithm [Kung et al., 1975]:

- Do divide-and-conquer along z.
- For each conquer-step:
  - Problem broken down to 2D ...
  - Divide in upper and lower part.
  - Simultanious $y-$sweep over upper and lower maxima and exploit:
  - For each maximum of $\mathcal{P}_{|z>\zeta}$: on skyline (of whole input).
  - For each maximum of $\mathcal{P}_{|z\leq\zeta}$: on skyline $\Rightarrow$ x-larger than actual tail.

...	Maxima of $\mathcal{P}_{\vert z\leq\zeta}$		Maxima of $\mathcal{P}_{\vert z>\zeta}$		...

ℓ_b ℓ'_b $\lfloor(\ell_b+\ell_e)/2\rfloor$ ℓ'_e ℓ_e

$\Downarrow$

...	Maxima of $\mathcal{P}_{\vert z\leq\zeta} \cup \mathcal{P}_{\vert z>\zeta}$		...

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Making it in-place:

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- For each conquering: Explicit reconstruction of skyline bounds.

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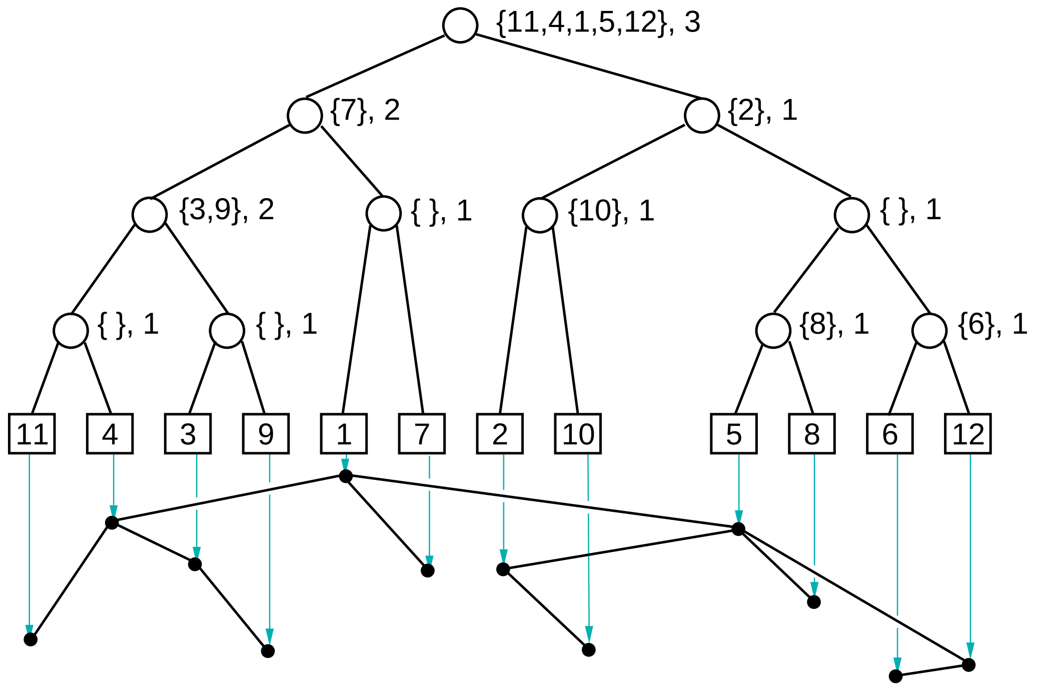
Convex hull and layers computation

Solutions (for 2D layers and 3D hull):

- Build recursively defined hull data structure . . .



Convex hull and layers computation

Solutions (for 2D layers and 3D hull):

- Build recursively defined hull data structure . . .
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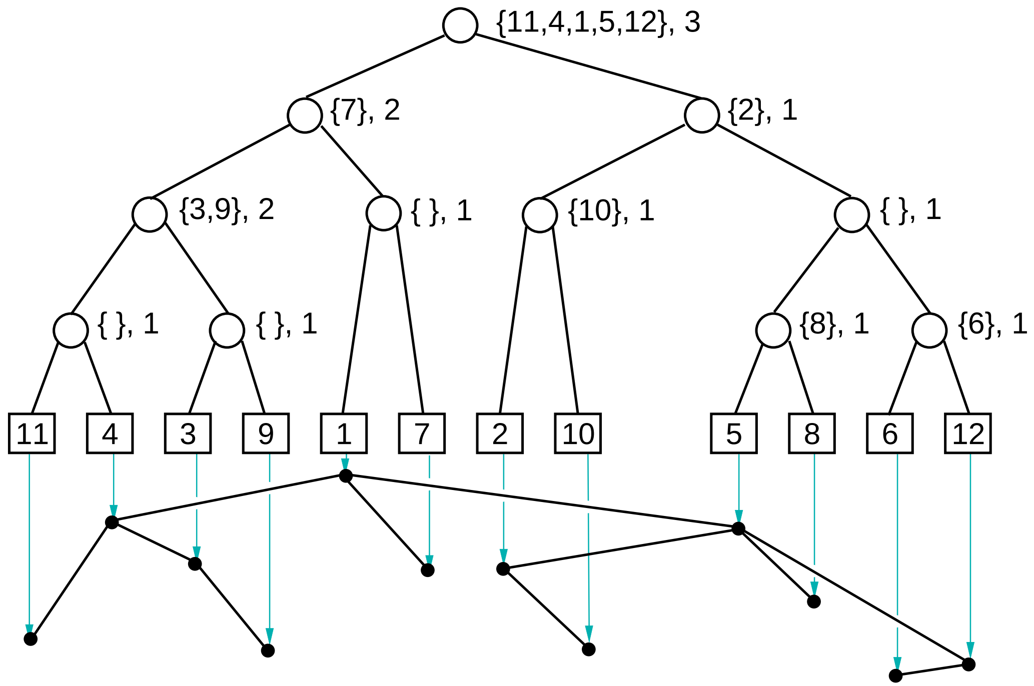

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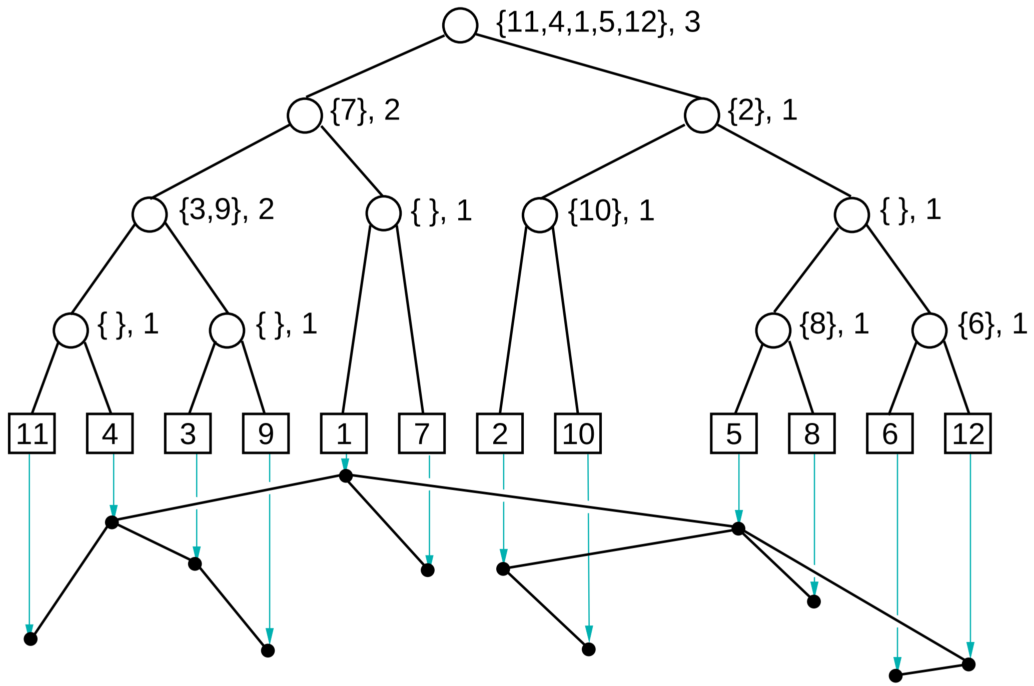
- $\Rightarrow$ 2D Convex layers in $\mathcal{O}(n \log_2 n)$ time [Chazelle, 1985].



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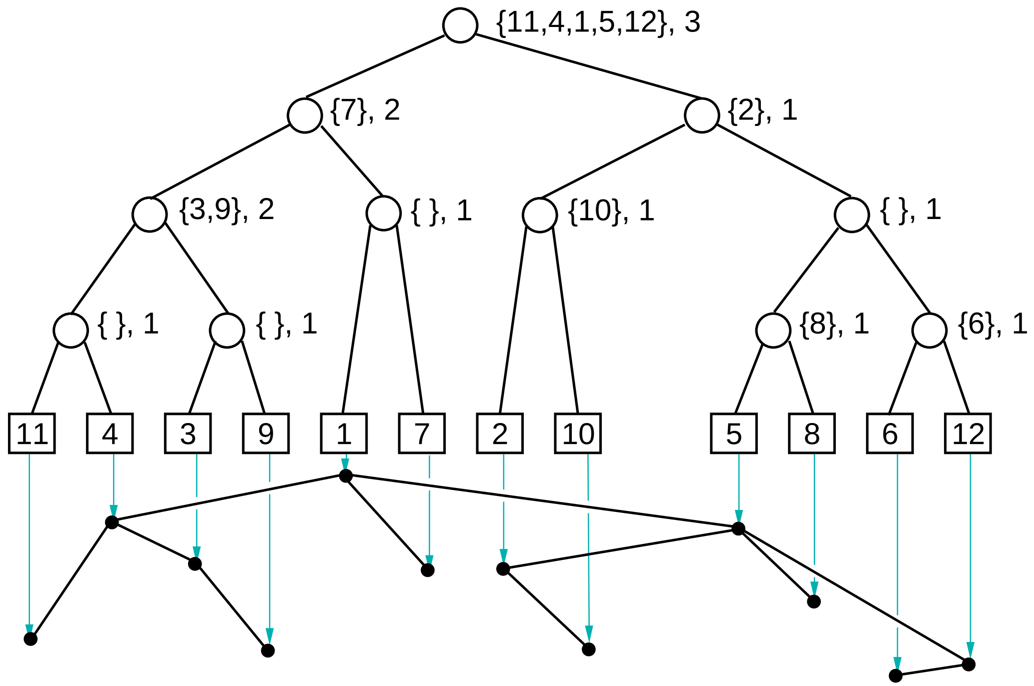
- $\Rightarrow$ 3D Convex hull in $\mathcal{O}\left(n \log_2^3 n\right)$ time and $\mathcal{O}(1)$ space [Brönnimann et al., 2004a].



Overview

1. Introduction: Motivation for implicit computation
2. Skylines and convex hulls

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