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Packing tripods: narrowing the density gap *

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Abstract

In 1984, S. K. Stein and his co-authors posed a problem concerning simple three-dimensional shapes, known as semicrosses, or tripods. By definition, a tripod is formed by a corner and the three adjacent edges of an integer cube. How densely can one fill the space with non-overlapping tripods of a given size? In particular, is it possible to fill a constant fraction of the space as the tripod size tends to infinity? In this paper, we settle the second question in the negative: the fraction of the space that can be filled with tripods must be infinitely small as the size grows. We also make a step towards the solution of the first question, by improving the currently known asymptotic lower bound on tripod packing density, and by presenting some computational results on small-size packings.

1 Introduction

In [7, 15, 13], S. K. Stein and his co-authors posed a problem concerning simple three-dimensional polyominoes, called "semicrosses" in [7, 15], and "tripods" in [13].

A *tripod* of order n is formed by a corner and the three adjacent edges of an integer $n \times n \times n$ cube (see Figure 1). How densely can one fill the space with non-overlapping tripods of order n? In particular, is it possible to keep a constant fraction of the space filled as $n \to \infty$? Despite their simple formulation, these two questions appear to be yet unsolved. In this paper, we settle the second question in the negative: the density of tripod packing must approach zero as tripod order tends to infinity. It is easy to prove (see [15]) that this result implies similar results in dimensions higher than three. We also make a step towards the solution of the first question, by improving the currently known asymptotic lower bound on tripod packing density, and by presenting some computational results on low-order packings.

It is worth noting that *lattice* packing of tripods is currently understood much better than general packing. The main results on lattice packing can

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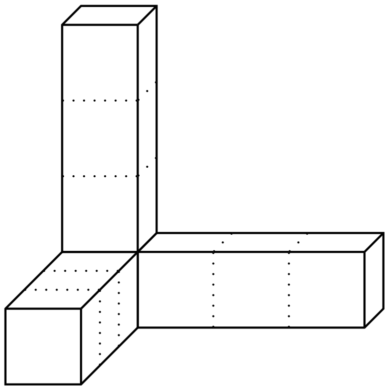

Figure 1: A tripod of order $n = 4$

be found in [15]. However, these results do not generalise to non-lattice packing, which seems to require a radically different approach.

Instead of dealing with the problem of packing tripods in space directly, we address an equivalent problem, also introduced in [7, 15, 13]. In this alternative setting, tripods of order n are packed with identical orientation and without overlap, so that their corner cubes are aligned with one of the unit cells of a fixed $n \times n \times n$ cube. The number of tripods in the packing is called the packing *size*. Following Stein, we denote by $f(n)$ the maximum size of a tripod packing for a given order n. The maximum fraction of space that can be filled with arbitrary non-overlapping tripods is proportional to $f(n)/n^2$. It is easy to see that for all n, $n \leq f(n) \leq n^2$. The questions we address can now be restated as follows: what is the asymptotic behaviour of $f(n)/n^2$ as $n \to \infty$? In particular, is $f(n) = \Theta(n^2)$ or $o(n^2)$? What is the highest $\alpha \leq 2$, such that $f(n) = \Omega(n^\alpha)$?

The only values of function f that have been known previously are $f(1)$ to $f(5)$: 1, 2, 5, 8, 11. The best previously known asymptotic lower bound for Stein's function is $f(n) = \Omega(n^{1.5235})$, and the best previously known upper bound is the trivial $f(n) \leq n^2$. Our goal is to narrow the gap between the lower and the upper bounds. In this paper, we approach this goal from two directions:

- improving the lower bound, computing large low-order packings, and combining them recursively to obtain infinite families of large packings;

- obtaining the first non-trivial upper bound on tripod packing density, using techniques of extremal graph theory.

The rest of this paper is organised as follows. Section 2 introduces computational techniques for obtaining optimal or near-optimal tripod packings of low order. Section 3 shows how such low-order packings can be combined into infinite families of large packings. It also introduces a new, surprisingly simple recursive packing technique, that leads to a new asymptotic lower

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02....3.
..0.2..3
14..5...
67......
..1..25.
..4.6.7.
...0.4.5
...1.6.7
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Figure 2: Best known tripod packings of order $2 \leq n \leq 8$

bound on tripod packing density. Section 4 uses extremal graph theory to prove the first non-trivial upper bound on tripod packing density.

A preliminary version of this paper appeared as two separate conference papers [17, 18].

2 Low-order packings

For a given tripod packing of order n, the only known method of deciding its optimality is by brute-force search. Since the number of possible tripod packings grows exponentially, obtaining the precise value of the function $f(n)$ is difficult even for very moderate values of n.

One natural approach to computing tripod packing is by integer programming. The $n \times n \times n$ cube can be represented by an array of n^3 binary decision variables $p[i,j,k]$, $0 \leq i,j,k < n$. We can think of the array p as a three-dimensional chessboard, and of tripods as 3D "semi-rooks" (unidirectional rooks). The packing condition can now be expressed as follows: every chessboard cell can be attacked by a semi-rook at most once. This condition corresponds to a system of n^3 linear inequalities:

$$p[i,j,k] + \sum_{0 \leq i' < i} p[i',j,k] + \sum_{0 \leq j' < j} p[i,j',k] + \sum_{0 \leq k' < k} p[i,j,k'] \leq 1,$$

where $0 \leq i,j,k < n$. The above system can easily be programmed in a general-purpose integer programming solver, such as Xpress-MP [11]. Some optimisations are possible: for instance, we can always set the corner tripods $[0,0,0]$ and $[n-1,n-1,n-1]$. Figure 2 shows the resulting packings of order $2 \leq n \leq 8$, of which only those of order $n \leq 6$ are confirmed by Xpress-MP as optimal. Each packing is represented by a $n \times n$ matrix. A tripod in cell $[i,j,k]$ is represented by the number k in row i and column j of the matrix (with all indices running from 0 to $n-1$).

Note that the packings of order $n \leq 6$ have a very simple structure: apart from the corner tripods, they contain a triple of tripods $[0,1,2]$, $[1,2,0]$, $[2,0,1]$. The remaining tripods can be obtained by repeatedly translating this triple along the main diagonal of the cube, with simultaneous rotation by $60°$ in every step. The total number of tripods obtained by this construction is $3n-4$.

The packing of order 7 illustrates another, much more powerful method of constructing tripod packings. It can be obtained from the packing of

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$g(n)$	1	2	5	8	11	14	19	23	28	32	37	43	49	55	61

Table 1: Best known tripod packings of order $n \leq 15$

order 3 and size 5 by "blowing up" the two corner tripods to $3 \times 3 \times 3$ cubes, each containing a packing of size 5. The three non-corner tripods become "blown up" to $3 \times 3 \times 1$ cuboids, each containing a packing of size 3. The total number of tripods is $5 + 3 \cdot 3 + 5 = 19$.

In contrast with the previous packings, the packing of order 8 lacks any apparent structure (apart from the symmetry that we discuss below), and is larger than all known systematic packings of the same order. The existence of such "sporadic" packings motivates us to search exhaustively for large packings of a relatively low order.

In order to search the space of all packings efficiently, we need to exploit various symmetries of this space. For instance, a packing remains valid if the cube is rotated by 120° around its main diagonal. In fact, all packings in Figure 2 are invariant under such rotation: the existence of a tripod $[i, j, k]$ implies the existence of tripods $[j, k, i]$ and $[k, i, j]$. We call this property *3-symmetry.* All known optimal packings are either 3-symmetric, or can be transformed into 3-symmetric form without losing any tripods. Therefore, the following conjecture seems plausible.

Conjecture (3-symmetry). *For every n, there exists a 3-symmetric tripod packing of size $f(n)$.*

In other words, we conjecture that $f(n) = g(n)$, where $g(n)$ is the maximum possible size of a 3-symmetric packing. The 3-symmetry conjecture allows us to perform exhaustive search for much higher values of n. Table 1 shows the best known values of $g(n)$ (which are also the best known values of $f(n)$) for $n \leq 15$. These values are known to be optimal for $n \leq 14$.

Another transformation preserving the validity of a tripod packing is *mirror reflection*: a tripod $[i, j, k]$ is mapped to $[j, i, k]$. No non-trivial packing can be invariant under this transformation; still, breaking the mirror symmetry of the search space approximately halves the amount of search.

The last global transformation that preserves the validity of a packing is *duality*: a tripod $[i, j, k]$ is mapped to $[n - i - 1, n - j - 1, n - k - 1]$. All packings in Figure 2 except the last one are self-dual (i.e. invariant under duality). It might be tempting to strengthen our symmetry conjecture by considering packings that are both 3-symmetric and self-dual. However, since the value of $g(8) = 23$ is odd, it cannot correspond to any self-dual packing. Still, exploiting duality allows some partial symmetry breaking and further reduction in the amount of search.

In addition to these global symmetries, it is also possible to reduce the search by considering transformations that affect individual tripods. For example, it is easy to see that a packing only needs be considered if it satisfies the following property: no tripod $[i, j, k]$ can be moved to a cell $[i, j, k']$ with $k' < k$. This property, which we call *gravity*, is easy to enforce (even in conjunction with 3-symmetry), and provides significant further savings.

By running a branch-and-bound code, custom-written in C and including all the above optimisations, we have been able to compute Table 1 (and the corresponding packing matrices) in about 40 hours on a Sun Ultra 5 workstation. In the next section, we describe how these results feed into the search for the best asymptotic lower-bound packing exponent.

3 Packing asymptotics: The lower bound

3.1 Uniform blow-up

For a long time, the best asymptotic lower bound on the function $f(x)$ was the one by D. Hickerson [14]. Hickerson's construction is based on the recursive technique, an example of which we have already seen in Figure 2 for $n = 7$. We begin with a simple "external" packing, and replace every tripod with a suitably chosen "internal" packing. Both the external and the internal packings are in general cuboids, not necessarily cubes. When all the internal packings are identical, we call the resulting process *uniform blow-up*; otherwise, *non-uniform blow-up*.

Every packing of order m and size d can be blown up uniformly to a packing of order m^2 and size d^2. By repeating this process, we can obtain a packing of order m^k and size d^k for any $k \geq 0$. Hence, $f(m^k) \geq d^k$, and therefore $f(n) = \Omega(n^{\log d / \log m})$. The best lower bound given by the packings in Figure 2 is $f(n) = \Omega(n^{\log 19 / \log 7}) = \Omega(n^{1.5131})$.

Hickerson's lower bound is obtained in two stages. In the first stage, finite non-uniform blow-up is performed to obtain a large, relatively low-order packing. Hickerson ran a few steps of this process by hand, using just two external cuboid packings: one of order $2 \times 2 \times 1$ and size 2, another of order 3 and size 5. The resulting packing had order 255 and size 4639. In the second stage, this packing is used as a basis for infinite uniform blow-up, implying $f(n) = \Omega(n^{\log 4639 / \log 255}) = \Omega(n^{1.5235})$.

Hickerson's method has been programmed by G. Stoneman as part of his undergraduate project [16]. Additionally, a database of cuboid packings extending Table 1 was compiled and used for the selection of internal packings. One of the resulting recursive packings had order 255 and size 4776, implying $f(n) = \Omega(n^{\log 4776 / \log 255}) = \Omega(n^{1.5287})$. This is the best currently known asymptotic lower bound obtained by uniform blow-up.

3.2 Non-uniform blow-up

Since the finite version of non-uniform blow-up proves to be very useful in Hickerson's method, it is natural also to consider infinite non-uniform blow-up. This simple idea, surprisingly overlooked by previous researchers, turns out to be more successful at obtaining asymptotic lower bounds than all previous approaches.

Let us take the packing of order 3 and size 5 as the external packing. We blow up the two corner tripods to cube packings of order $3m$, and the three non-corner tripods to cuboid packings of order $3m \times 3m \times m$, where m is an arbitrary parameter. Each non-corner cuboid packing is filled with three cube packings of order m. We obtain a packing of order $(3+1+3)m = 7m$ and size $f(3m) + 3 \cdot 3f(m) + f(3m) = 2f(3m) + 9f(m)$. The asymptotic value of $f(n)$ can now be found from the recurrence equation

$$f(n) = 2f\big(\lfloor 3n/7 \rfloor\big) + 9f\big(\lfloor n/7 \rfloor\big)$$

This equation belongs to a standard class of divide-and-conquer recurrences, for which the Master Theorem [12] (see also [19, 8]) yields the asymptotic solution

$$f(n) = \Omega(n^{1.5340}),$$

where the value of the exponent is obtained as the unique solution of the equation $7^\alpha = 2 \cdot 3^\alpha + 9$.

The above lower bound is a significant improvement on the previously known lower bounds. Interestingly, it is obtained without any computation, apart from solving a single exponential equation. A more elaborate application of non-uniform blow-up will probably require new computational techniques, and will lead to further improvements.

4 Packing asymptotics: The upper bound

4.1 Preliminaries

Throughout this section we use the standard language of graph theory, slightly adapted for our convenience. A graph G is defined by its set of nodes $V(G)$ and its set of edges $E(G)$. All graphs we consider are simple and undirected. *The order* of a graph G is the number of its nodes $|V(G)|$; the *size* is the number of its edges $|E(G)|$. A graph H is *a subgraph* of G, denoted $H \subseteq G$, iff $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$. Subgraph $H \subseteq G$ is *a spanning subgraph* of G, denoted $H \sqsubseteq G$, iff $V(H) = V(G)$.

A complete graph on r nodes is denoted K_r. The term *k-partite graph* is synonymous with "k-coloured". Complete bipartite and tripartite graphs are denoted K_{rs} and K_{rst}, where r, s, t are sizes of the colour classes. Graph $K_2 = K_{11}$(short for $K_{1,1}$) is a single edge; we call its complement $\bar{K}_2$ (an

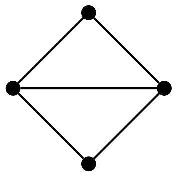

Figure 3: The diamond graph K_{121}

empty graph on two nodes) *a nonedge*. Graph $K_3 = K_{111}$ is called *a triangle*; graph K_{121} is called *a diamond* (see Figure 3). We call a k-partite graph *equi-k-partite*, if all its colour classes are of equal size.

The density of a graph is the ratio of its size to the size of a complete graph of the same order: if $G \sqsubseteq K_n$, then

$$\mathrm{dens}(G) = |E(G)|/|E(K_n)| = |E(G)|/\binom{n}{2}.$$

The *bipartite density* of a bipartite graph $G \sqsubseteq K_{mn}$ is

$$\mathrm{dens}_2(G) = |E(G)|/|E(K_{mn})| = |E(G)|/(mn).$$

Similarly, the *tripartite density* of a tripartite graph $G \sqsubseteq K_{mnp}$ is

$$\mathrm{dens}_3(G) = |E(G)|/|E(K_{mnp})| = |E(G)|/(mn + np + pm).$$

Let H be an arbitrary graph. Graph G is called *H-covered*, if every edge of G belongs to a subgraph isomorphic to H. Graph G is called *H-free*, if G does not contain any subgraph isomorphic to H. In particular, we will be interested in triangle-covered diamond-free graphs.

Let us now establish an upper bound on the density of an equitripartite diamond-free graph.

Lemma 1. *The tripartite density of an equitripartite diamond-free graph G with $|V(G)| = 3n \geq 6$ is at most $3/4$.*

Proof. By Dirac's generalisation of Turán's theorem (see e.g. [5, p. 300], [6]), we have $|E(G)| \leq (3n)^2/4$. Since $|E(K_{nnn})| = 3n^2$, the lemma follows immediately. ■

The upper bound of $3/4$ given by Lemma 1 is not the best possible. However, that bound will be sufficient to obtain the results of this paper. In fact, any constant upper bound strictly less than 1 would suffice.

4.2 The Regularity Lemma and the Blow-up Lemma

In most definitions and theorem statements below, we follow [10, 9, 2].

For a graph G, and node sets $X, Y \subseteq V(G)$, $X \cap Y = \emptyset$, we denote by $G(X, Y) \subseteq G$ the bipartite subgraph induced by the set pair X, Y. Let F

be a bipartite graph with colour classes A, B. Given some $\epsilon > 0$, graph F is called *ϵ-regular*, if for any $X \subseteq A$ of cardinality $|X| \geq \epsilon \cdot |A|$, and any $Y \subseteq B$ of cardinality $|Y| \geq \epsilon \cdot |B|$, we have

$$|\mathrm{dens}_2(F(X,Y)) - \mathrm{dens}_2(F)| \leq \epsilon.$$

Let G denote an arbitrary graph. We say that G admits *an ϵ-partitioning of order m*, if $V(G)$ can be partitioned into m disjoint subsets of equal cardinality, called *supernodes*, such that for all pairs of supernodes A, B, the bipartite subgraph $G(A,B)$ is ϵ-regular. The ϵ-regular subgraphs $G(A,B)$ will be called *superpairs*.

For different choices of supernodes A, B, the density of the superpair $G(A,B)$ may differ. We will distinguish between superpairs of "low" and "high" density, determined by a carefully chosen threshold. For a fixed d, $0 \leq d \leq 1$, we call a superpair $G(A,B)$ *a superedge*, if $\mathrm{dens}_2(G(A,B)) \geq d$, and *a super-nonedge*, if $\mathrm{dens}_2(G(A,B)) < d$. Now, given a graph G, and its ϵ-partitioning of order m, we can build a high-level representation of G by a graph of order m, which we will call *a d-map* of G. The d-map M contains a node for every supernode of G. Two nodes of M are connected by an edge, if and only if the corresponding supernodes of G are connected by a superedge. Thus, edges and nonedges in M represent, respectively, superedges and super-nonedges of G. For a node pair (edge or nonedge) e in G, we denote by $\mu(e)$ the corresponding pair in the d-map M. We call $\mu : E(G) \to E(M)$ *the mapping function*. The union of all superedges $\mu^{-1}(E(M)) \subseteq E(G)$ will be called *the superedge subgraph of G*. Likewise, the union of all super-nonedges in G will be called *the super-nonedge subgraph of G*.

We rely on the following fact, which is a restricted version of the Blow-up Lemma (see [9]).

Theorem 1 (Blow-up Lemma). *Let $d > \epsilon > 0$. Let G be a graph with an ϵ-partitioning, and let M be its d-map. Let H be a subgraph of M with maximum degree $\Delta > 0$. If $\epsilon \leq (d-\epsilon)^{\Delta}/(2+\Delta)$, then G contains a subgraph isomorphic to H.*

Proof. See [9]. ■

Since we are interested in diamond-free graphs, we take H to be a diamond. We simplify the condition on d and ϵ, and apply the Blow-up Lemma in the following form: if $\epsilon \leq (d-\epsilon)^3/5$, and G is diamond-free, then its d-map M is also diamond-free.

Our main tool is Szemerédi's Regularity Lemma. Informally, it states that any graph can be transformed into a graph with an ϵ-partitioning by removing a small number of nodes and edges. Its precise statement, slightly adapted from [10], is as follows.

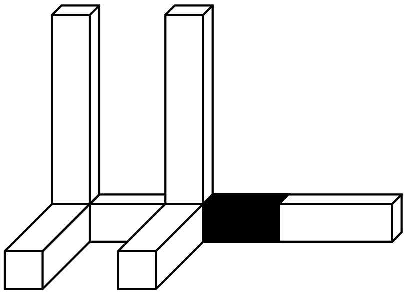

Figure 4: An axial collision

Theorem 2 (Regularity Lemma). *Let G be an arbitrary graph. For every $\epsilon > 0$ there is an $m = m(\epsilon)$ such that for some $G_0 \subseteq G$ with $|E(G) \setminus E(G_0)| \leq \epsilon \cdot |V(G)|^2$, graph G_0 admits an ϵ-partitioning of order at most m.*

Proof. See e.g. [10, 2]. ■

The given form of the Regularity Lemma is slightly weaker than the standard one. In particular, we allow to remove a "small" number of nodes and edges from the graph G, whereas the standard version only allows to remove a "small" number of nodes (with adjacent edges), and then a "small" number of superpairs. In our context, the difference between two versions is insignificant.

Note that if $|E(G)| = o(|V(G)|^2)$, the statement of the Regularity Lemma becomes trivial. In other words, the Regularity Lemma is only useful for dense graphs.

4.3 Tripods do not pack densely

Consider a packing of tripods of order n in an $n \times n \times n$ cube, of the type described in the Introduction (no overlaps, similar orientation, all tripods aligned with cube cells). A tripod in such a packing is uniquely defined by the coordinates of its corner cell $[i, j, k]$, $0 \leq i, j, k < n$. Moreover, if two of the three coordinates i, j, k are fixed, then the packing may contain at most one tripod with such coordinates — otherwise, the two tripods with an equal pair of coordinates would form an *axial collision*, depicted in Figure 4.

We represent a tripod packing by an equitripartite triangle-covered graph $G \sqsubseteq K_{nnn}$ as follows. Three color classes $U = \{u_i\}$, $V = \{v_j\}$, $W = \{w_k\}$, $0 \leq i, j, k < n$, correspond to the three dimensions of the cube. A tripod $[i, j, k]$ is represented by a triangle $\{(u_i, v_j), (v_j, w_k), (w_k, u_i)\}$. To prevent axial collisions, triangles representing different tripods must be edge-disjoint. Hence, if m is the number of tripods in the packing, then the representing graph G contains $3m$ edges.

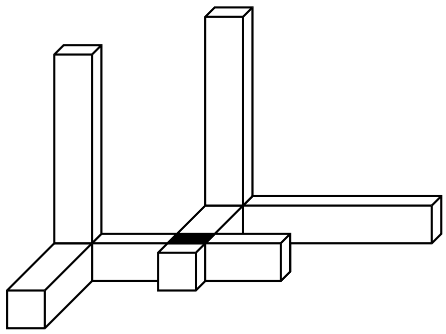

Figure 5: A simple collision

We now prove that the graph G is diamond-free. In general, G might contain a triangle with three edges coming from three different tripods; such a triangle would give rise to three diamonds. To prove that such a situation is impossible, we must consider, apart from axial collisions, also *simple collisions*, depicted in Figure 5.

Lemma 2. *A tripod packing graph is diamond-free.*

Proof. It is sufficient to show that the tripod packing graph does not contain any triangles apart from those representing tripods. Suppose the contrary: there is a triangle $\{(u_i, v_j), (v_j, w_k), (w_k, u_i)\}$, which does not represent any tripod. Then its three edges must come from triangles representing three different tripods; denote these tripods $[i, j, k']$, $[i, j', k]$, $[i', j, k]$, where $i \neq i'$, $j \neq j'$, $k \neq k'$. Consider the differences $i' - i$, $j' - j$, $k' - k$, all of which are non-zero. At least two of these three differences must have the same sign; without loss of generality assume that $i' - i$, $j' - j$ are of the same sign. Thus, we have either $i' < i$, $j' < j$, or $i' > i$, $j' > j$. In both cases, the tripods $[i, j', k]$, $[i', j, k]$ collide. Hence, our assumption must be false, and the triangle $\{(u_i, v_j), (v_j, w_k), (w_k, u_i)\}$ not representing any tripod cannot exist. Therefore, no triangles in G can share an edge — in other words, G is diamond-free. ∎

Thus, tripod packing graphs are equitripartite, triangle-covered and diamond-free. Note that these graph properties are invariant under any permutation of graph nodes within colour classes, whereas the property of a tripod packing being overlap-free is not invariant under permutation of indices within each dimension. Hence, the converse of Lemma 2 does not hold. However, even the loose characterisation of tripod packing graphs by Lemma 2 is sufficient to obtain our result.

The following theorem is a special case of an observation attributed to Szemerédi by Erdős (see [3], [1, p. 48]). Since Szemerédi's proof is apparently unpublished, we give an independent proof of our special case.

Theorem 3. *Consider an equitripartite, triangle-covered, diamond-free graph of order $3n$. The maximum density of such a graph tends to 0 as $n \to \infty$.*

Proof. Suppose the contrary: for some constant $d > 0$, and for an arbitrarily large n (i.e. for some $n \geq n_0$ for any n_0), there is a tripartite, triangle-covered, diamond-free graph G of order $3n$, such that $\mathrm{dens}_3(G) \geq d > 0$. The main idea of the proof is to apply the Regularity Lemma and the Blow-up Lemma to the graph G. This will allow us to "distil" from G a new graph, also triangle-covered and diamond-free, with tripartite density higher than $\mathrm{dens}_3(G)$ by a constant factor $\lambda > 1$. Repeating this process, we can raise the density to $\lambda^2 d$, $\lambda^3 d$, etc., until the density becomes higher than 1, which is an obvious contradiction.

Let us now fill in the details of the "distilling" process. We start with a constant γ, $0 < \gamma < 1$; its precise numerical value will be determined later. Select a constant $\epsilon > 0$, such that $\epsilon \leq (\gamma d - \epsilon)^3/5$, as required by the Blow-up Lemma. By the Regularity Lemma, graph G admits an ϵ-partitioning, the order of which is constant and independent of the order of G. Denote by M the γd-map of this partitioning, and let $\mu : G \to M$ be the mapping function. From now on, we restrict the term "superpair" to such subgraphs $G(A, B)$, where A, B are supernodes from different colour classes (when A, B are from the same colour class, the graph $G(A, B)$ is trivially empty).

Consider the superedge subgraph $\mu^{-1}(E(M))$. Let $G_\triangle \sqsubseteq \mu^{-1}(E(M)) \sqsubseteq G$ be a spanning subgraph of G, consisting of all triangles *completely contained* in $\mu^{-1}(M)$; in other words, each triangle in $G_\triangle$ is completely contained in some supertriangle of G. We claim that $G_\triangle$ contains a significant fraction of all triangles (and, hence, of all edges) in G. Indeed, the bipartite density of a super-nonedge is by definition at most γd, hence the super-nonedge subgraph has at most $3\gamma d \cdot n^2$ edges. Every triangle not completely contained in $\mu^{-1}(E(M))$ must have at least one edge in the super-nonedge subgraph; since triangles in G are edge-disjoint, the total number of such triangles cannot exceed $3\gamma d \cdot n^2$. By the initial assumption, the total number of triangles in G is at least $d \cdot n^2$, therefore the number of triangles in $G_\triangle$ must be at least $(1 - 3\gamma) d \cdot n^2$. By selecting a sufficiently small γ, we can make the number of triangles in $G_\triangle$ arbitrarily close to $d \cdot n^2$. For the rest the proof, let us fix the constant γ within the range $0 < \gamma < 1/12$, e.g. $\gamma = 1/24$. As a corresponding ϵ we can take e.g. $\epsilon = (\gamma d/2)^3/5 = d^3/(5 \cdot 48^3)$.

It only remains to observe that, since graph G is diamond-free, its γd-map M is diamond-free by the Blow-up Lemma. We can always assume that $|V(M)| \geq 6$; otherwise, $|V(M)| = 3$, and we can modify the ϵ-partitioning by splitting every supernode in two. By Lemma 1, $\mathrm{dens}_3(M) \leq 3/4$. This means that among all superpairs of G, the fraction of superedges is at most $3/4$. All edges of $G_\triangle$ are contained in superedges of G, therefore the average bipartite density of a superedge in $G_\triangle$ is at least $4/3 \cdot \mathrm{dens}_3(G_\triangle)$. In particular, there must be some superedge in $G_\triangle$ with at least such density. Since

$G_\triangle$ consists of edge-disjoint supertriangles, this superedge is contained in a unique supertriple $F \subseteq G_\triangle$, with

$$\mathrm{dens}_3(F) \geq 4/3 \cdot \mathrm{dens}_3(G_\triangle) \geq 4/3 \cdot (1-3\gamma)d = \\ 4/3 \cdot (1 - 3 \cdot 1/24)d = 7/6 \cdot d.$$

In our previous notation, we have $\lambda = 7/6 > 1$.

We define the supertriple F to be our new "distilled" equitripartite triangle-covered diamond-free graph. Graph order has only been reduced by a constant factor, equal to the order of the ϵ-partitioning. By taking the original graph G large enough, the "distilled" graph F can be made arbitrarily large. Its density $\mathrm{dens}_3(F) \geq \lambda d = 7/6 \cdot d > d$. By repeating the whole process, we can increase the graph density to $(7/6)^2 \cdot d$, $(7/6)^3 \cdot d$, ..., and eventually to values higher than 1, which contradicts the definition of density (in fact, values higher than 3/4 will already contradict Lemma 1). Hence, the initial assumption of existence of arbitrarily large equitripartite triangle-covered diamond-free graphs with constant positive density must be false. Negating this assumption, we obtain our theorem. ∎

The upper bound $o(n)$ on tripod packing density is now an easy corollary of Lemma 2 and Theorem 3.

Corollary 1. *Consider a tripod packing of order n. The maximum size of such a packing $f(n) = o(n^2)$.*

A tighter analysis of the parameters involved in the application of the Regularity Lemma yields $f(n) = O\big(n^2/(\log n)^{1/15}\big)$.

5 Conclusion

In this paper, we have improved our understanding of tripod packing asymptotics. We have increased the lower bound on maximum packing size $f(n)$ to $\Omega(n^{1.5340})$. We have also obtained the first non-trivial upper bound $f(n) = o(n^2)$; more precisely, $f(n) = O\big(n^2/(\log n)^{1/15}\big)$.

Despite our efforts, there remains a substantial gap between the lower and the upper bounds. It is likely that the lower bound can be further improved by a more sophisticated application of the methods described here. Any improvement in the upper bound looks much more difficult. A proof or disproof of the 3-symmetry conjecture, which has eluded us so far, could provide new valuable insights into the structure of tripod packing. Finally, our computational techniques could be refined to produce better small-order packings, or extended to other interesting combinatorial problems.

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