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Sampling 10-colourings of the triangular lattice *

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Abstract

We use a new inductive argument to show that the spin-system consisting of the proper 10-colourings of the infinite triangular lattice has strong spatial mixing. This implies that it has a unique infinite-volume Gibbs measure and that Glauber dynamics is rapidly mixing on any finite piece of this lattice.

1 Introduction

This paper is concerned with (proper) colourings of an infinite lattice G. These colourings correspond to configurations in the zero-temperature antiferromagnetic Potts model. A colouring of the lattice is an assignment of colours from the set $\{1, \ldots, q\}$ to the vertices, or sites, of the lattice. It is *proper* if adjacent sites receive different colours. A measure μ on the set of proper colourings of G is an infinite-volume *Gibbs measure* (with respect to the uniform specification) if, for any finite set of vertices R, the conditional probability distribution $\mu(\cdot \mid \sigma_{\overline{R}})$ (conditioned on the colouring $\sigma_{\overline{R}}$ of all sites other than those in R) is the uniform distribution on those colourings of R that are consistent with the induced boundary colouring $\sigma_{\partial R}$. Infinite-volume Gibbs measures exist for any G. The problem of determining whether there is more than one infinite-volume Gibbs measure for a given "specification" is the DLR problem (Dobrushin, Lanford and Ruelle) from statistical physics (see [3]). A useful tool for determining whether an infinite-volume Gibbs measure is unique is the *Dobrushin uniqueness condition.* Informally, the Dobrushin constant α is the supremum, over lattice sites i, of the sum, over distinct lattice sites j, of the extent to which the colour of site i depends upon the colour of site j. See [21] for details. The Dobrushin uniqueness theorem says that if the Dobrushin constant is less than 1 then there is a unique infinite-volume Gibbs measure. Furthermore, in this measure, correlations decay exponentially.

An important notion in statistical physics is whether the system (as specified by the finite-volume Gibbs measures) satisfies *strong spatial mixing* [18]. We give a formal definition of this notion later. Informally, it says that for any finite set of vertices R, if you consider two different colourings $\mathcal{B}$ and $\mathcal{B}'$ of the boundary of R which differ at a single site y then the effect

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that this difference has on a subset Δ of R decays exponentially with the distance from Δ to y. Strong spatial mixing implies that there is a unique infinite-volume Gibbs measure with exponentially decaying correlations [18]. It is also very closely related to rapid mixing of *Glauber dynamics* on finite pieces of G. (See [11]).

We are interested in establishing strong spatial mixing directly in those situations where Dobrushin's condition seems too demanding. We consider the infinite triangular lattice. Salas and Sokal established the Dobrushin condition when q, the number of colours, is at least 11. This implies strong spatial mixing, and the proof also implies rapid mixing of Glauber dynamics on finite pieces of the lattice, via Bubley and Dyer's path coupling method [4]. Since the degree of the lattice is 6, rapid mixing for $q \geq 11$ also follows from a result of Vigoda [23].

We consider the case $q = 10$. In order to demonstrate that there is strong spatial mixing, we consider an arbitrary finite set of vertices R and two different colourings $\mathcal{B}$ and $\mathcal{B}'$ of the boundary of R which differ at a single site y. we show inductively that there is a coupling of the two conditional distributions in which, for every vertex $f \in R$, the probability of disagreement at f is exponentially small (as a function of its distance to the boundary discrepancy). Our inductive proof is based on discovering and solving a collection of mutual recurrences (Lemma 9). An important feature is that we prove inductively that a suitable coupling exists even though we do not construct an explicit coupling. This approach seems to be useful for proving strong spatial mixing and rapid mixing, in situations in which it is too complicated to explicitly construct couplings.

Our results are that the system has strong spatial mixing (Corollary 13)(and, therefore, a unique infinite-volume Gibbs measure), that a heat-bath block dynamics (on a very large but constant block size) has optimal $O(n \log n)$ (temporal) mixing (on any finite piece of the lattice)(Theorem 14), and that therefore *Glauber dynamics* is rapidly mixing (Theorem 18).

In the course of our proof we need to compute certain properties of colourings of a fixed five-vertex subset of the lattice. What we need to compute is essentially the number of colourings in which a specified vertex has a given colour. We do this by computation (Lemmas 6 and 7). Thus, it is interesting that our proof retains a computational element even though the direct computational approaches of [1, 5, 21] seem infeasible for us.

1.1 Glauber dynamics

The *Glauber dynamics* Markov chain is a natural algorithm for randomly sampling a proper colouring of a graph. This simple Markov chain is widely used in statistical physics and has been analysed extensively (see the references below). The definition of the Glauber dynamics is simple. The state space is the set of all proper colourings of the graph. To make a transition from a colouring σ, a vertex v is chosen uniformly at random. Then v is recoloured by picking a colour uniformly at random from the set of all colours which are not used at neighbours of v.

Jerrum [16] and, independently, Salas and Sokal [21], showed that Glauber dynamics is rapidly mixing on an n-vertex graph with degree at most Δ when q, the number of colours, is at least 2Δ. Vigoda [23] improved these results, showing that rapid mixing holds for $q \geq 11\Delta/6$. These mixing time bounds have been improved for various special cases such as cases in which the graph is required to have sufficiently large degree and girth [5, 8, 9, 13, 14, 15, 19]. Salas and Sokal's proof technique was Dobrushin uniqueness, which was discussed earlier. The rest of these proofs relied on the *coupling* method, though some of the couplings were constructed for related dynamics, which were compared to Glauber dynamics using the comparison method of Diaconis and Saloff-Coste [6]. Increasingly sophisticated techniques have been used to construct couplings, including path coupling [4], coupling to a stopping time [9, 14], and even construction of a non-Markovian multiple-step coupling [15]. The latter led to the best-known result (due to Hayes and Vigoda [15]), which gives rapid mixing for $q > (1+\varepsilon)\Delta$ for graphs with high degree and girth. In this paper we take a different approach. Our interest is in situations in which it may be too difficult to construct an explicit coupling. As described in Section 1, we use an inductive argument to show that the desired coupling exists.

There is a fair amount of existing work for the special case in which we want to sample colourings of a finite piece of a regular infinite lattice. Salas and Sokal [21] contains results for a number of lattices. Here we summarise the known results for the square lattice and for the triangular lattice, which we study. Salas and Sokal [21] establish strong spatial mixing (and rapid mixing) for the square lattice for $q \geq 7$. This result also follows from the more general result of Bubley et al [5]. Rapid mixing for $q = 3$ has been shown by Luby, Randall and Sinclair [17] for the fixed-boundary case and by the authors [12] for the free-boundary case. In a recent result, Achlioptas et al [1] have shown rapid mixing for $q = 6$. They show rapid mixing of Glauber dynamics, optimal mixing of heat-bath dynamics, and strong spatial mixing. It is not known whether Glauber dynamics (or heat bath) is rapidly mixing for $q = 4$ and $q = 5$, though strong spatial mixing is conjectured for these cases by Salas and Sokal [21].

For the triangular lattice the best known result is that Glauber dynamics is rapidly mixing for $q \geq 11$. This follows from Vigoda's $11/6$ result [23] since the degree is 6. It was also proved independently by Salas and Sokal [21]. We consider $q = 10$ and show rapid mixing of Glauber dynamics, optimal mixing of heat-bath dynamics, and strong spatial mixing. The latter is conjectured for $q \geq 5$ by Salas and Sokal [21]. We note that our method also gives an alternative proof of the 6-colouring result of [1].

2 Definitions

We are interested in colouring finite sets of vertices of the infinite triangular lattice. It is convenient to work with the dual presentation of the lattice in which we colour hexagonal faces of the infinite hexagonal lattice. A finite piece of this lattice is depicted in Figure 1. Each internal face has degree 6. Each edge of the lattice is adjacent to two faces. A *region* R of the lattice is a, not necessarily connected, subset of the faces. A *colouring* of the region R is a function from R to the set of *colours* $Q = \{1, \ldots, 10\}$.

If R is a non-empty finite region of the lattice then let ∂R be the boundary of *faces* around R. In particular, a face f is in ∂R if it is not in R, but it is adjacent to some face in R.

We will define a colouring of ∂R to be a function from ∂R to the set $\{0\} \cup Q$. The colour "0" corresponds to an unconstrained boundary face — this notion will be explained shortly.

Given a colouring $\mathcal{B}$ of ∂R, a colouring C of R is said to be proper if

- adjacent faces in R receive different colours, and
- faces in R receive colours different from adjacent boundary faces.

Note that if a face in ∂R has a colour in Q, this coloured boundary face constrains the set of proper colourings of the region. On the other hand, a face in ∂R that is coloured with colour 0 provides no constraint – thus our set-up generalizes both the fixed-boundary scenario

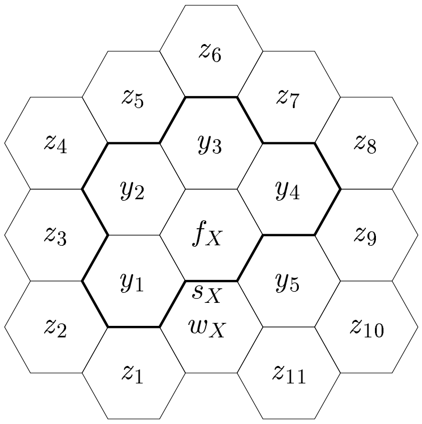

Figure 1: The triangular lattice

(in which all boundary faces have colours in Q) and the free-boundary scenario (in which all boundary faces have colour 0). Let $S(\mathcal{B})$ be the set of proper colourings of R and let $\pi_{\mathcal{B}}$ be the uniform distribution on $S(\mathcal{B})$.

We will be interested in Markov chains for sampling from $\pi_{\mathcal{B}}$. We will use the following notation. For any positive integer k and any face f let $W(f,k)$ denote the size-k^2 region of the lattice consisting of a length-k row going diagonally right and up from f and the $k-1$ rows parallel to this vertically below. For example in Figure 1, $W(y_2,1) = \{y_2\}$, $W(y_2,2) = \{y_2, y_3, y_1, f_X\}$, and $W(y_2,3) = \{y_2, y_3, z_7, y_1, f_X, y_4, z_1, w_X, y_5\}$.

Consider a problem instance $(R,\mathcal{B})$ and a positive integer k. Let

$$R^* = \{f \mid W(f,k) \cap R \text{ is non-empty}\}.$$

Let $\mathcal{M}_k$ be the *heat-bath* Markov chain with state space $S(\mathcal{B})$ and the following transition probabilities. $\mathcal{M}_k$ makes a transition from a state $\sigma \in S(\mathcal{B})$ by choosing a face $f \in R^*$ uniformly at random. Let R_f be the region consisting of $W(f,k) \cap R$ and let $\mathcal{B}_f$ be the colouring of ∂R_f induced by σ and $\mathcal{B}$. To make the transition from σ, recolour the faces in R_f by sampling from $\pi_{\mathcal{B}_f}$.

Note that $\mathcal{M}_1$ is the same as the *Glauber dynamics* Markov chain defined in Section 1.1. It is easy to see that for any k the Markov chain $\mathcal{M}_k$ is ergodic and that it is reversible with respect to the uniform distribution $\pi_{\mathcal{B}}$, which is its unique stationary distribution.

The *variation distance* between distributions θ_1 and θ_2 on Ω is

$$d_{\mathrm{tv}}(\theta_1,\theta_2) = \frac{1}{2}\sum_i |\theta_1(i) - \theta_2(i)| = \max_{A\subseteq\Omega} |\theta_1(A) - \theta_2(A)|.$$

Consider a problem instance consisting of a region R and a colouring $\mathcal{B}$ of ∂R. Let n be the size of the instance, namely $|R|$. Let P_k denote the transition matrix of $\mathcal{M}_k$. Given a distinguished state $\sigma \in S(\mathcal{B})$, the mixing time of $\mathcal{M}_k$ (as a function of the deviation ε from stationarity) is

$$\tau_\sigma(\mathcal{M}_k,\varepsilon) = \min\{t > 0 : d_{\mathrm{tv}}(P_k^t(\sigma,\cdot), \pi_{\mathcal{B}}(\cdot)) \leq \varepsilon\}.$$

The mixing time of $\mathcal{M}_k$ is $\tau(\mathcal{M}_k,\varepsilon) = \max_\sigma \tau_\sigma(\mathcal{M}_k,\varepsilon)$.

In this paper, we show that there is a constant k such the heat-bath dynamics $\mathcal{M}_k$ has optimal temporal mixing. That is, for any fixed ε, $\tau(\mathcal{M}_k, \varepsilon) \in O(n \log n)$. This implies that Glauber dynamics is rapidly mixing. In particular, $\tau(\mathcal{M}_1, \varepsilon) \in O(n^2 \log n)$. Finally, we show that the stationary distribution of $\mathcal{M}_k$ has strong spatial mixing.

3 Exponential decay and edge discrepancies

Let R be a non-empty finite region of the lattice. For most of the technical part of the paper it will be convenient to consider the edge-boundary of R rather than the boundary ∂R of faces surrounding R. Here is the notation that we will use. The *boundary* of the region R is the collection of all lattice edges that have exactly one face in R. A *colouring* of the boundary is a function from the set of edges in the boundary to the set $\{0\} \cup Q$.

Let R be a finite region and let B be a colouring of its boundary. A colouring C of R is said to be *proper* if

- adjacent faces in R receive different colours, and
- faces in R receive colours different from adjacent boundary edges.

Let $S(B)$ denote the set of proper colourings of R and let π_B be the uniform distribution on $S(B)$. We will be interested in studying how much $S(B)$ varies when we change the boundary colouring B by recolouring a single edge. This small change to the boundary is formalised in the following notation.

A *boundary pair* X consists of

- a non-empty finite region R_X of the lattice, and
- a distinguished edge s_X on the boundary of R_X, and
- a pair (B_X, B'_X) of colourings of the boundary of R_X which differs only on the edge s_X. We require that the two colours $B_X(s_X)$ and $B'_X(s_X)$ are both in Q. That is, the two boundary colourings differ on the colouring of edge s_X, but this edge is not an unconstrained edge (with colour 0) in either boundary colouring.

For any boundary pair X, we define f_X to be the face in R_X that is adjacent to s_X, and w_X to be the other face adjacent to s_X. See Figure 1, where the region R_X is surrounded by boldface lines. Let E_X be the set of lattice edges that are adjacent both to f_X and to another face in R_X. A *coupling* Ψ of π_{B_X} and $\pi_{B'_X}$ is a distribution on $S(B_X) \times S(B'_X)$ which has marginal distributions π_{B_X} and $\pi_{B'_X}$. For such a coupling Ψ, we define $1_{\Psi,f}$ to be the indicator random variable for the event that, when a pair of colourings is drawn from Ψ, the colour of f differs in these two colourings.

For any boundary pair X we define Ψ_X to be some coupling of π_{B_X} and $\pi_{B'_X}$ minimising $E[1_{\Psi,f_X}]$. For every pair of colours c and c' let $p_X(c, c')$ be the probability that, when a pair of colourings (C, C') is drawn from Ψ_X, f_X is coloured with colour c in C and with colour c' in C'.

We will restrict attention to a certain subset of boundary pairs. Intuitively, the idea is that these boundary pairs are the ones that are induced by a pair of colourings of ∂R_X. We say that a boundary pair is *good* if it is the case that any two adjacent edges on the boundary that share a face $f \notin R_X$ have the same colour in at least one of the two colourings B_X

and B'_X (and so in both of B_X and B'_X except when edge s_X is involved). For example in Figure 1, if R_X consists of the five faces enclosed by the thicker line and X is a good boundary pair then B_X and B'_X assign the same colour to the edges (y_1, z_3) and (y_2, z_3).

Suppose that X is a boundary pair and that f is a face in R_X. Let $d(f, s_X)$ denote the distance within R_X from f to s_X. Thus, $d(f_X, s_X) = 1$ and if face f in R_X adjoins f_X then $d(f, s_X) = 2$ and so on. The main technical result of this section is the following.

Lemma 1 *For every good boundary pair X there exists a coupling Ψ of π_{B_X} and $\pi_{B'_X}$ such that, for all $f \in R_X$,*

$$E[1_{\Psi,f}] \leq \frac{1}{\varepsilon}(1-\varepsilon)^{d(f,s_X)}.$$

Lemma 1 says that the effect of the discrepancy at the boundary edge s_X decays exponentially with the distance from s_X. It seems to be difficult to give an explicit construction of the coupling Ψ in Lemma 1. Instead, we define (see Lemma 9 below) some mutual recurrences. We use these recurrences to show by induction that the coupling exists.

In order to relate the coupling to the system of recurrences we need some tools. To aid our analysis, we will define a labelled tree T_X associated with each boundary pair X. The tree T_X is constructed as follows. Start with a vertex r which will be the root of T_X. For every pair of colours $c \in Q$ and $c' \in Q$, add an edge labelled $(p_X(c,c'), f_X)$ from r to a new node $r_{c,c'}$. If E_X is empty, $r_{c,c'}$ is a leaf. Otherwise, let $e_1, \ldots, e_k$ be the edges in E_X, ordered clockwise around the face f_X starting from s_X. For each $i \in \{1, \ldots, k\}$, let $X_i(c,c')$ be the boundary pair consisting of

- the region $R_X - f_X$, and
- distinguished edge e_i, and
- the colouring B of the boundary of $R_X - f_X$ that
  - agrees with B_X on common edges, and
  - colours $e_1, \ldots, e_{i-1}$ with colour c', and
  - colours $e_i, \ldots, e_k$ with colour c, and
- the colouring B' that agrees with B except that it colours e_i with colour c'.

Note that if X is good then $X_i(c,c')$ is good. Recursively construct $T_{X_i(c,c')}$, the tree corresponding to boundary pair $X_i(c,c')$. Add an edge with label $(1,\cdot)$ from $r_{c,c'}$ to the root of $T_{X_i(c,c')}$. That completes the construction of T_X.

We will say that an edge e of T_X is *degenerate* if the second component of its label is $\cdot$. For edges e and e' of T_X, we write $e \to e'$ to denote the fact that that e is an ancestor of e'. That is, either $e = e'$, or e is a proper ancestor of e'. Define the *level* of edge e to be the number of non-degenerate edges on the path from the root down to, and including, e. Suppose that e is an edge of T_X with label (p, f). We say that the *weight* $w(e)$ of edge e is p. Also the *name* $n(e)$ of edge e is f. The *likelihood* $\ell(e)$ of e is $\prod_{e':e'\to e} w(e)$. The *cost* $\gamma(f, T_X)$ of a face f in T_X, is $\sum_{e:n(e)=f} \ell(e)$.

Lemma 2 *For every boundary pair X there exists a coupling Ψ of π_{B_X} and $\pi_{B'_X}$ such that, for all $f \in R_X$, $E[1_{\Psi,f}] \leq \gamma(f, T_X)$.*

Proof. We will give a recursive construction of the coupling Ψ. Let (C, C') denote the random variable corresponding to a pair of colourings from Ψ. If $|R_X| = 1$ then $\Psi = \Psi_X$. Otherwise, let $e_1, \ldots, e_k$ be the edges in E_X, i.e., those that are adjacent both to f_X and to another face in R_X. We will use Ψ_X to couple the colouring of face f_X and we will recursively construct a different coupled colouring of the other faces in R_X. We will assign $C(f_X) = c$ and $C'(f_X) = c'$ with probability $p_X(c, c')$.

Let $X(c, c')$ be an "extended boundary pair" consisting of

- the region $R_X - f_X$, and
- the colouring B_c of the boundary of $R_X - f_X$ that agrees with B_X on common edges and colours edges in E_X with colour c, and
- the colour $B_{c'}$ of the boundary of $R_X - f_X$ that agrees with B_X on common edges and colours edges in E_X with colour c'.

We can complete the coupling Ψ by constructing (for each c and c') a coupling $\Psi_{c,c'}$ of π_{B_c} and $\pi_{B_{c'}}$. (The reader may verify that this ensures that the marginal distributions of Ψ are correct.)

The particular choice that we make for $\Psi_{c,c'}$ is either the perfect coupling if $c = c'$ or, if $c \neq c'$, the composition of $\Psi_1(c, c'), \ldots, \Psi_k(c, c')$ where $\Psi_i(c, c')$ is a recursively-constructed coupling for boundary pair $X_i(c, c')$.[1]

We will now show that, for all $f \in R_X$, $E[1_{\Psi,f}] \leq \gamma(f, T_X)$. The proof is by induction on $|R_X|$. If $f = f_X$ then $E[1_{\Psi,f}] = \gamma(f, T_X)$ by the construction of Ψ and T_X. This handles the base case, $|R_X| = 1$. Suppose $f \neq f_X$ and $|R_X| > 1$. Then

$$\begin{aligned} E[1_{\Psi,f}] = \sum_{c,c'} p_X(c,c') E[1_{\Psi_{c,c'},f}] &\leq \sum_{c,c'} p_X(c,c') \sum_{i=1}^{k} E[1_{\Psi_i(c,c'),f}] \\ &\leq \sum_{c,c'} p_X(c,c') \sum_{i=1}^{k} \gamma(f, T_{X_i(c,c')}) = \gamma(f, T_X), \end{aligned}$$

where the second inequality uses the inductive hypothesis. $\square$

Suppose that X is a boundary pair. Lemma 2 ensures that there is a coupling of π_{B_X} and $\pi_{B'_X}$ with substantial agreement as long as, for most faces $f \in R_X$, $\gamma(f, T_X)$ is small. A key ingredient from the construction of T_X which affects $\gamma(f, T_X)$ is the quantity $E[1_{\Psi_X, f_X}]$, which we denote $\nu(X)$. (Thus, $\nu(X) = \min_\Psi E[1_{\Psi, f_X}]$, where the minimum is over all couplings Ψ of π_{B_X} and $\pi_{B'_X}$.) An important part of our method is to determine good upper bounds on $\nu(X)$.

Let $d = B_X(s_X)$ and $d' = B'_X(s_X)$. Let $Q' = Q - \{d, d'\}$. For $c \neq d$, let $n_c(X)$ denote the number of colourings in $S(B_X)$ which colour f_X with colour c. Let $n_d(X)$ denote the number of colourings in $S(B'_X)$ which colour f_X with colour d. Let $N(X) = \sum_{c \in Q'} n_c(X)$. Define

$$\mu(X) = \frac{\max(n_d(X), n_{d'}(X))}{N(X) + \max(n_d(X), n_{d'}(X))}.$$

[1] The composition that we have in mind is the natural one — to choose a pair (σ_0, σ_k) from $\Psi_{c,c'}$ first choose (σ_0, σ_1) from $\Psi_1(c, c')$. Say $\sigma_0 = x_0$ and $\sigma_1 = x_1$. Then choose (σ_1, σ_2) from the conditional distribution of $\Psi_2(c, c')$, conditioned on $\sigma_1 = x_1$. Say that $\sigma_2 = x_2$. Now choose (σ_2, σ_3) from the conditional distribution of $\Psi_3(c, c')$ conditioned on $\sigma_2 = x_2$ and so on. This is the same as the composition that occurs in path coupling [4].

We use the following lemma to derive upper bounds on $\nu(X)$.

Lemma 3 *Suppose that X is a good boundary pair. Let R' be any subset of R_X which includes f_X. Let χ be the set of good boundary pairs $X' = (R_{X'}, s_{X'}, B_{X'}, B'_{X'})$ such that $R_{X'} = R'$, $s_{X'} = s_X$, $B_{X'}$ agrees with B_X on common edges, and $B'_{X'}$ agrees with B'_X on common edges. Then $\nu(X) \leq \max_{X' \in \chi} \mu(X')$.*

Proof. We will show
(i) $\nu(X) \leq \mu(X)$, and
(ii) $\mu(X) \leq \max_{X' \in \chi} \mu(X')$.

Let X be a good boundary pair. To shorten the notation we will use n_c to denote $n_c(X)$ and N to denote $N(X)$. Let $d = B_X(s_X)$ and $d' = B'_X(s_X)$. Let $Q' = Q - \{d, d'\}$.

For (i), we can construct a coupling Ψ of π_{B_x} and $\pi_{B'_X}$ which matches N colourings, each of which occurs with probability at least

$$\frac{1}{\max(|S(B_X)|, |S(B'_X)|)} = \frac{1}{N + \max(n_d, n_{d'})}.$$

Thus,

$$\nu(X) \leq 1 - \frac{N}{N + \max(n_d, n_{d'})} = \mu(X).$$

For (ii), let $W = R_X - R_{X'}$. Let H be the set of colourings of W. For $c \neq d$ and $\rho \in H$, let $n_{c,\rho}$ denote the number of colourings in $S(B_X)$ which colour f_X with colour c and W with colouring ρ. For $\rho \in H$, let $n_{d,\rho}$ denote the number of colourings in $S(B'_X)$ which colour f_X with colour d and W with colouring ρ. Let $N_\rho = \sum_{c \in Q'} n_{c,\rho}$. Then

$$\begin{aligned}
\mu(X) &= 1 - \frac{N}{N + \max(n_d, n_{d'})} \\
&= 1 - \frac{\sum_{\rho \in H} N_\rho}{\sum_{\rho \in H} N_\rho + \max(\sum_{\rho \in H} n_{d,\rho}, \sum_{\rho \in H} n_{d',\rho})} \\
&\leq 1 - \frac{\sum_{\rho \in H} N_\rho}{\sum_{\rho \in H} N_\rho + \sum_{\rho \in H} \max(n_{d,\rho}, n_{d',\rho})} \\
&= \frac{\sum_{\rho \in H} \max(n_{d,\rho}, n_{d',\rho})}{\sum_{\rho \in H}(N_\rho + \max(n_{d,\rho}, n_{d',\rho}))} \\
&\leq \max_{\rho \in H} \frac{\max(n_{d,\rho}, n_{d',\rho})}{N_\rho + \max(n_{d,\rho}, n_{d',\rho})} \\
&= \max_{\rho \in H} \mu(X'),
\end{aligned}$$

where $B_{X'}$ and $B'_{X'}$ are the colourings of the boundary of $R_{X'}$ that are induced by ρ. Note that X' is a good boundary pair, since X is, and the new edges in the boundary are induced by colourings of faces. $\square$

Let $\mu_1 = 31/136$ and $\mu_2 = 1111/4966$. We now give four lemmas bounding $\mu(X)$ for particular boundary pairs X.

Lemma 4 *Suppose X is a boundary pair with $|R_X| = 1$. Then $\mu(X) \leq 1/4$.*

Proof. The numerator in the definition of $\mu(X)$ is at most 1. The denominator is at least 4. $\square$

Lemma 5 *Suppose X is a boundary pair in which R_X consists of f_X and one neighbour y of f_X. Then $\mu(X) \leq 5/21$.*

Proof. Let E_1 be the set of edges of f_X except for s_X and the edge between f_X and y. Let E_2 be the set of edges of y except for the edge between f_X and y. Let U be the set of colours that B_X (and so also B'_X) assigns to edges in E_1, and similarly let V be the set of colours assigned to E_2. Let $d = B_X(s_X)$ and $d' = B'_X(s_X)$. Let $Q' = Q - \{d, d'\}$. To shorten the notation we will use n_c to denote $n_c(X)$ and N to denote $N(X)$. Let $q_1 = |Q' \setminus U|$ and let $q_2 = |Q \setminus V|$. We see that $q_1 \geq 8 - |U| \geq 4$ and $q_2 \geq 10 - |V| \geq 5$.

For $c \in Q$ we see that

$$\begin{aligned} n_c &= 0 && \text{if } c \in U \\ &= q_2 && \text{if } c \notin U \text{ and } c \in V \\ &= q_2 - 1 && \text{if } c \notin U \text{ and } c \notin V. \end{aligned}$$

Thence, $\max(n_d, n_{d'}) \leq q_2$ and

$$N = \sum_{c \in Q'} n_c \geq (q_2 - 1)|Q' \setminus U| = (q_2 - 1)q_1.$$

Since $\mu(X)$ is monotone increasing in $\max(n_d, n_{d'})$ and decreasing in N, we have

$$\mu(X) \leq \frac{q_2}{q_2 + (q_2 - 1)q_1} = \frac{1}{1 + (1 - 1/q_2)q_1} \leq \frac{1}{1 + (1 - 1/5)4} = \frac{5}{21}.$$

$\square$

Lemma 6 *Suppose X is a good boundary pair in which $R_X = \{f_X, y_1, y_2, y_3\}$ and f_X is adjacent to each of the y_is and $w_X \sim y_1 \sim y_2 \sim y_3$. Then $\mu(X) \leq \mu_1$.*

Proof. By computation. We considered every such good boundary pair X and calculated $\mu(X)$. $\square$

Lemma 7 *Suppose X is a good boundary pair in which $R_X = \{f_X, y_1, y_2, y_3, y_4\}$ and f_X is adjacent to each of the y_is and $w_X \sim y_1 \sim y_2 \sim y_3 \sim y_4$. Then $\mu(X) \leq \mu_2$.*

Proof. By computation. We considered every such good boundary pair X and calculated $\mu(X)$. $\square$

Let X be a boundary pair. For any $d \geq 1$, let E_d denote the set of level-d edges in T_X. Let $\Gamma_d(X) = \sum_{e \in E_d} \ell(e)$. In Lemma 10 below we will show that $\Gamma_d(X)$ is exponentially small in d. We start by deriving some simple recurrences. Let G be the set of good boundary pairs. Say that a good boundary pair X is in $\mathcal{N}_i$ (for $i \in \{0, \ldots, 5\}$) if at most i of the neighbours of f_X are in R_X. Let $\Gamma_d = \max_{X \in G} \Gamma_d(X)$.

Lemma 8 *Suppose $d > 1$. If $X \in \mathcal{N}_2$ then $\Gamma_d(X) \leq \Gamma_{d-1}/2$. If $X \in \mathcal{N}_3$ then $\Gamma_d(X) \leq 3\Gamma_{d-1}/4$. If $X \in \mathcal{N}_4$ then $\Gamma_d(X) \leq 20\Gamma_{d-1}/21$.*

Proof. If $X \in \mathcal{N}_2$ then by the definition of T_X, $\Gamma_d(X) \le \nu(X) \cdot 2 \cdot \Gamma_{d-1}$. Then by Lemma 3 with $R' = \{f_X\}$ and Lemma 4, $\nu(X) \le 1/4$. Similarly, if $X \in \mathcal{N}_3$ then $\Gamma_d(X) \le \nu(X) \cdot 3 \cdot \Gamma_{d-1} \le 3\Gamma_{d-1}/4$. Finally, suppose that exactly 4 neighbours of f_X are in R_X. Then $\Gamma_d(X) \le \nu(X) \cdot 4 \cdot \Gamma_{d-1}$. Apply Lemma 3 where R' contains f_X and one of its neighbours in R_X. By Lemma 5, $\nu(X) \le 5/21$. □

Next we define some subsets of G, the set of good boundary pairs. These subsets, like the $\mathcal{N}_i$s, depend only on R_X and s_X. Refer to Figure 1 to clarify these definitions. Let X be a good boundary pair.

- X is in U if there is a neighbour y_5 of f_X and of w_X that is not in R_X, and there is a neighbour $y_4 \neq w_X$ of f_X and of y_5 that is not in R_X.
- X is in V if there is a neighbour y_5 of f_X and of w_X that is not in R_X.
- X is in W if there is a neighbour y_5 of f_X and of w_X in R_X, and a neighbour $z_{11} \neq f_X$ of w_X and of y_5 that is not in R_X.

Let $U_d = \max_{X \in U} \Gamma_d(X)$, $V_d = \max_{X \in V} \Gamma_d(X)$, and $W_d = \max_{X \in W} \Gamma_d(X)$. The following lemma follows from the definition of T_X and the geometry of the lattice.

Lemma 9 *Suppose $d > 1$. Then*

$$\begin{aligned}
\Gamma_d &\le \max\left(\mu_2\left(\Gamma_{d-1} + 2V_{d-1} + 2W_{d-1}\right), \frac{20\Gamma_{d-1}}{21}\right), \\
U_d &\le \max\left(\mu_1\left(2V_{d-1} + W_{d-1}\right), \frac{\Gamma_{d-1}}{2}\right), \\
V_d &\le \max\left(\mu_2\left(2V_{d-1} + 2W_{d-1}\right), \frac{3\Gamma_{d-1}}{4}\right), \text{ and} \\
W_d &\le \max\left(\mu_2\left(\Gamma_{d-1} + V_{d-1} + 2W_{d-1} + U_{d-1}\right), \frac{20\Gamma_{d-1}}{21}\right).
\end{aligned}$$

Proof. Suppose $X \in G$. If $X \in \mathcal{N}_4$ then $\Gamma_d(X) \le 20\Gamma_{d-1}/21$ by Lemma 8. Otherwise an examination of Figure 1 reveals that

$$\Gamma_d(X) \le \nu(X)\left(\Gamma_{d-1} + 2V_{d-1} + 2W_{d-1}\right).$$

The two instances of V_{d-1} correspond to y_1 and y_5 in the picture (since w_X is not in R_X), the two instances of W_{d-1} correspond to y_2 and y_4, and the instance of Γ_{d-1} corresponds to y_3. Apply Lemma 3 where R' is the set containing f_X and the faces y_1, y_2, y_3 and y_4 from Figure 1. By Lemma 7, $\nu(X) \le \mu_2$. This proves the upper bound on Γ_d.

Suppose $X \in U$. If $X \in \mathcal{N}_2$ then $\Gamma_d(X) \le \Gamma_{d-1}/2$ by Lemma 8. Otherwise

$$\Gamma_d(X) \le \nu(X)\left(2V_{d-1} + W_{d-1}\right).$$

As in the upper bound on Γ_d, one instance of V_{d-1} corresponds to y_1 and one instance of W_{d-1} corresponds to y_2. An examination of Figure 1 reveals that since $X \in U$, y_3 corresponds to V_{d-1}. Apply Lemma 3 where R' is the set containing f_X and the faces y_1, y_2, and y_3 from Figure 1. By Lemma 6, $\nu(X) \le \mu_1$. This proves the upper bound on U_d.

Suppose $X \in V$. If $X \in \mathcal{N}_3$ then $\Gamma_d(X) \leq 3\Gamma_{d-1}/4$ by Lemma 8. Otherwise

$$\Gamma_d(X) \leq \nu(X)\left(2V_{d-1} + 2W_{d-1}\right).$$

This is the same as the upper bound on Γ_d except that, since $X \in V$, y_3 corresponds to W_{d-1} and y_4 to V_{d-1}. Apply Lemma 3 where R' is the set containing f_X and the faces y_1, y_2, y_3 and y_4 from Figure 1. By Lemma 7, $\nu(X) \leq \mu_2$. This proves the upper bound on V_d.

Suppose $X \in W$. If $X \in \mathcal{N}_4$ then $\Gamma_d(X) \leq 20\Gamma_{d-1}/21$ by Lemma 8. Otherwise

$$\Gamma_d(X) \leq \nu(X)\left(\Gamma_{d-1} + V_{d-1} + 2W_{d-1} + U_{d-1}\right).$$

This is the same as the upper bound on Γ_d except that, since $X \in W$, y_5 corresponds to U_{d-1}. Apply Lemma 3 where R' is the set containing f_X and the faces y_1, y_2, y_3 and y_4 from Figure 1. By Lemma 7, $\nu(X) \leq \mu_2$. This proves the upper bound on W_d. □

Let $\varepsilon = 0.001$.

Lemma 10 *For every good boundary pair X and every $d \geq 1$, $\Gamma_d(X) \leq (1-\varepsilon)^d$.*

Proof. Let $u = 15/26$, $v = 31/40$, and $w = 21/22$. We will prove by induction on d that $\Gamma_d \leq (1-\varepsilon)^d$ and $U_d \leq u(1-\varepsilon)^d$ and $V_d \leq v(1-\varepsilon)^d$ and $W_d \leq w(1-\varepsilon)^d$. For the base case, $d = 1$, note that for any good boundary pair X, $\Gamma_1(X) \leq \nu(X)$. Now apply Lemma 3 with $R' = \{f_X\}$ and by Lemma 4, we find that $\nu(X) \leq 1/4$. Thus, $\Gamma_1(x) \leq \min\{u, v, w, 1\}(1-\varepsilon)$.

The inductive step follows directly from Lemma 9 since the following inequalities hold. $\mu_2(1 + 2v + 2w) \leq 1 - \varepsilon$, $20/21 \leq 1 - \varepsilon$, $\mu_1(2v + w) \leq u(1-\varepsilon)$, $1/2 \leq u(1-\varepsilon)$, $\mu_2(2v + 2w) \leq v(1-\varepsilon)$, $3/4 \leq v(1-\varepsilon)$, $\mu_2(1 + v + 2w + u) \leq w(1-\varepsilon)$, and $20/21 \leq w(1-\varepsilon)$. □

We can now prove Lemma 1.

Proof. (Proof of Lemma 1) By Lemma 2, $E[1_{\Psi,f}] \leq \gamma(f, T_X)$. Furthermore, $\gamma(f, T_X)$ is at most $\sum_{d \geq d(f,s_X)} \Gamma_d(X)$. By Lemma 10, this is at most $\sum_{d \geq d(f,s_X)} (1-\varepsilon)^d$. □

4 Exponential decay, face discrepancies and strong spatial mixing

Lemma 1 shows that the effect of a discrepancy at a boundary edge decays exponentially with the distance from that edge. In this section we show that the same holds for a discrepancy at a boundary face. This enables us to show that the collection of finite-volume Gibbs measures corresponding to $\mathcal{M}_k$ has strong spatial mixing.

A *face-boundary pair X* consists of

- a non-empty finite region R_X of the lattice, and
- a distinguished face s_X in ∂R_X, and
- a pair $(\mathcal{B}_X, \mathcal{B}'_X)$ of colourings of ∂R which differs only on face s_X. We require that the two colours $\mathcal{B}_X(s_X)$ and $\mathcal{B}'_X(s_X)$ are both in Q. That is, the two boundary colourings differ on the colour of face s_X, but this face is not an unconstrained face (with colour 0) in either boundary colouring.

Let $d(f, s_X)$ be the distance within R_X from a face f to face s_X.

Lemma 11 *For every face-boundary pair X there is a coupling Ψ of $\pi_{\mathcal{B}_X}$ and $\pi_{\mathcal{B}'_X}$ such that, for all $f \in R$,*

$$E[1_{\Psi,f}] \leq \frac{10}{\varepsilon(1-\varepsilon)}(1-\varepsilon)^{d(f,s_X)}.$$

Proof. First suppose that s_X has a neighbour $y \notin R_X$. Let $e_1, \ldots, e_k$ be the boundary edges of R_X that are adjacent to s_X. Label these clockwise so that there is at least one non-boundary edge between e_k and e_1. Let X_i be the good boundary pair consisting of the region R_X, the distinguished edge e_i, a colouring B of the boundary of R_X that agrees with $\mathcal{B}_X$ except that edges $e_1, \ldots, e_{k-1}$ are coloured with colour $\mathcal{B}'_X(s_X)$ and $e_i \ldots, e_k$ are coloured with colour $\mathcal{B}_X(s_X)$ and a colouring B' that is the same as B except that it colours e_i with colour $\mathcal{B}'_X(s_X)$. We construct a coupling of $\pi_{\mathcal{B}_X}$ and $\pi_{\mathcal{B}'_X}$ by composing couplings $\Psi_1, \ldots, \Psi_k$ of $X_1, \ldots, X_k$. Since $k \leq 5$,

$$E[1_{\Psi,f}] \leq 5\frac{1}{\varepsilon}(1-\varepsilon)^{d(f,s_X)}.$$

Now we must deal with the case in which all neighbours of s_X are in R_X. A technical detail arises here because the natural induced boundary pairs are not all good. Let y be any neighbour of s_X. Let Ψ' be any coupling of $\pi_{\mathcal{B}_X}$ and $\pi_{\mathcal{B}'_X}$. Let (C, C') be the random variable corresponding to the pair of colourings in $S(\mathcal{B}_X) \times S(\mathcal{B}'_X)$ drawn from Ψ. We will choose the colour of y in C and C' according to Ψ'. To complete the construction of Ψ, for every pair (c, c'), we will let $\mathcal{B}_X(c)$ denote the face-boundary of $R_X - \{y\}$ which agrees with $\mathcal{B}_X$ except that y is coloured c and we will let $\mathcal{B}'_X(c')$ denote the face-boundary of $R_X - \{y\}$ which agrees with $\mathcal{B}'_X$ except that y is coloured c'. We will construct a coupling of $\mathcal{B}_X(c)$ and $\mathcal{B}'_X(c')$ by composing the couplings of up to 10 good boundary pairs (these boundary pairs correspond to discrepancies on the 5 boundary edges of face s_X and the up-to-5 boundary edges on face y). The $(1-\varepsilon)$ in the denominator comes from the fact that the distance from a face f to the discrepancy edge may be one less than $d(f, s_X)$. ☐

Let R be a non-empty finite region of the lattice and let $\Delta \subseteq R$. Let $\mathcal{B}$ be a colouring of ∂R. Let $\pi_{\mathcal{B},\Delta}$ denote the distribution on colourings of Δ induced by $\pi_{\mathcal{B}}$. Let d_{tv} denote *total variation distance.* The system of colourings that we study is said to have *strong spatial mixing*[11] if the following holds.

Definition 12 *The system has strong spatial mixing if there exist constants β and $\alpha > 0$ such that for any face-boundary pair X and any $\Delta \subseteq R_X$,*

$$d_{\text{tv}}(\pi_{\mathcal{B}_X,\Delta}, \pi_{\mathcal{B}'_X,\Delta}) \leq \beta|\Delta| \exp(-\alpha d(s_X, \Delta)).$$

Corollary 13 *The system has strong spatial mixing.*

Proof. The total variation distance of $\pi_{\mathcal{B}_X,\Delta}$ and $\pi_{\mathcal{B}'_X,\Delta}$ is at most the probability that the induced colourings differ in the coupling Ψ from Lemma 11. This is at most

$$\sum_{f\in\Delta} E[1_{\Psi,f}] \leq |\Delta| 10 \frac{1}{\varepsilon(1-\varepsilon)}(1-\varepsilon)^{d(s_X,\Delta)}. \quad (1)$$

Thus, we take $\beta = 10/(\varepsilon(1-\varepsilon))$ and $\alpha = -\log(1-\varepsilon)$. ☐

Remark. Note that the $|\Delta|$ factor is not crucial in the definition of strong spatial mixing. In particular, Observation 15 below can be used to show $\sum_{f\in\Delta} E[1_{\Psi,f}] \le O((1-\varepsilon)^{d(s_X,\Delta)})$. See the proof of Lemma 17.

5 Rapid mixing

The main result of this section is the following.

Theorem 14 *There is a positive integer k such that the heat-bath dynamics $\mathcal{M}_k$ has optimal mixing. That is, $\tau(\mathcal{M}_k,\varepsilon) \in O(n\log(n\varepsilon^{-1}))$.*

The following observations are useful in the proof of Theorem 14.

Observation 15 *Let f be a face of the infinite lattice. For any positive integer i, the number of faces at distance i from f is $6i$.*

Observation 16 *For any regions A and B, $\partial(A\cap B)\cap A \subseteq \partial B$.*

Proof. Suppose that face $v\in A$. We will suppose that $v\notin\partial B$ and show that $v\notin\partial(A\cap B)$. Suppose first that face v is neither in B nor in ∂B. In this case, it is clear that v is not in $\partial(A\cap B)$. So suppose that $v\in B$. Since $v\in A$, we cannot have $v\in\partial(A\cap B)$. □

Lemma 17 *There is a positive constant K such that, for every face-boundary pair X, there is a coupling Ψ of $\pi_{\mathcal{B}_X}$ and $\pi_{\mathcal{B}'_X}$ such that the expected Hamming distance of the coupled colourings of R_X is at most K.*

Proof. Let Ψ be the coupling from Lemma 11. Then the expected Hamming distance is

$$\sum_{f\in R_X} E[1_{\Psi,f}] \le \frac{10}{\varepsilon(1-\varepsilon)}\sum_{f\in R_X}(1-\varepsilon)^{d(f,s_X)}.$$

By Observation 15, this is at most

$$\frac{10}{\varepsilon(1-\varepsilon)}\sum_{i\ge 1} 6i(1-\varepsilon)^i,$$

and the sum converges. □

Proof. (Proof of Theorem 14.) We will use path coupling [4]. Consider a problem instance consisting of a non-empty region R with $|R|=n$ and a colouring $\mathcal{B}$ of ∂R. Let σ and τ be colourings in $S(B)$ with Hamming distance 1. Let v be the face at which colourings σ and τ disagree. We will describe a coupled transition from the pair (σ,τ) to a new pair of colourings (σ',τ'). In the coupling we will choose the same face f for the transition $\sigma\to\sigma'$ that we choose for the transition $\tau\to\tau'$.

There are k^2 faces $f\in R^*$ such that $v\in W(f,k)\cap R$. For each of these we can ensure that $\sigma'=\tau'$.

If the chosen face f is far from v in the sense that $v\notin W(f,k)\cap R$ and $v\notin\partial(W(f,k)\cap R)$ then we will couple the transitions from (σ,τ) so that σ' and τ' disagree only at v, and they have Hamming distance 1.

We next calculate an upper bound on the number of faces $f \in R^*$ such that $v \in \partial(W(f,k) \cap R)$. Applying Observation 16 with $A = R$ and $B = W(f,k)$ and noting that $v \in R$, we merely need an upper bound on the number of faces $f \in R^*$ such that $v \in \partial W(f,k)$. There are at most $4k+2$ such faces. If we choose such a face f then by Lemma 17 we can couple the transitions so that the expected Hamming distance of σ' and τ' is at most $K+1$.

Adding it all up, the expected Hamming distance of σ' and τ' is at most

$$1 - \frac{k^2}{|R^*|} + \frac{(4k+2)K}{|R^*|}.$$

If we choose k sufficiently large with respect to K and we let $\beta = 1/|R^*|$, we have that the expected Hamming distance of σ' and τ' is at most $1-\beta$.

By path coupling, the mixing time $\tau(\mathcal{M}_k, \varepsilon)$ is at most $\log(D\varepsilon^{-1})/\beta$, where D is the maximum Hamming distance between any two colourings in $S(\mathcal{B})$. The theorem follows from the observation that D and $|R^*|$ are in $O(n)$. $\square$

6 Glauber dynamics

The main result of this section is the following.

Theorem 18 *The Glauber dynamics $\mathcal{M}_1$ is rapidly mixing. In particular, $\tau(\mathcal{M}_1, \varepsilon) \leq O(n^2 \log n)\log(\varepsilon^{-1})$.*

We will prove Theorem 18 using the comparison method of Diaconis and Saloff-Coste [6]. The rest of the section contains the proof. Let k be a positive integer such that $\tau(\mathcal{M}_k, \varepsilon) \in O(n \log(n\varepsilon^{-1}))$ (see Theorem 14). We will compare the chain $\mathcal{M}_1$ with the chain $\mathcal{M}_k$.

6.1 The comparison method of Diaconis and Saloff-Coste

In this section we set up the machinery for the comparison method of Diaconis and Saloff-Coste [6]. In order to avoid modifying our dynamics to ensure that the transition matrices have non-negative eigenvalues, we use both comparison theorems of Diaconis and Saloff-Coste. This is a little non-standard, so we provide detailed definitions. For $i \in \{1,k\}$, let E_i be the set of pairs of distinct colourings (σ,τ) with $P_i(\sigma,\tau) > 0$. Let E_i^* be the set of all pairs (σ,τ) (distinct or not) with $P_i(\sigma,\tau) > 0$. We will sometimes refer to the members of E_i^* as "edges" because they are edges of the transition graph of $\mathcal{M}_i$.

For every edge $(\sigma,\tau) \in E_k^*$, let $\mathcal{P}_{\sigma,\tau}$ be the set of paths from σ to τ using transitions of $\mathcal{M}_1$. More formally, let $\mathcal{P}_{\sigma,\tau}$ be the set of paths $\gamma = (\sigma = \sigma_0, \sigma_1, \ldots, \sigma_k = \tau)$ such that

1. each (σ_i, σ_{i+1}) is in E_1^*, and
2. each $(\sigma', \tau') \in E_1^*$ appears at most twice on γ.[2]

We write $|\gamma|$ to denote the length of path γ. So, for example, if $\gamma = (\sigma_0, \ldots, \sigma_k)$ we have $|\gamma| = k$. Let $\mathcal{P} = \cup_{(\sigma,\tau)\in E_k^*} \mathcal{P}_{\sigma,\tau}$.

A *flow* is a function ϕ from $\mathcal{P}$ to the interval $[0,1]$ such that for every $(\sigma,\tau) \in E_k^*$,

$$\sum_{\gamma \in \mathcal{P}_{\sigma,\tau}} \phi(\gamma) = P_k(\sigma,\tau)\pi_{\mathcal{B}}(\sigma). \tag{2}$$

[2]This requirement is just there for technical reasons — we want to ensure that $\mathcal{P}_{\sigma,\tau}$ is finite.

The flow is said to be an *odd* flow if it is supported by odd-length paths. That is, for every $\gamma \in \mathcal{P}$, either $\phi(\gamma) = 0$ or $|\gamma|$ is odd.

Let $r((\sigma', \tau'), \gamma)$ be the number of times that the edge (σ', τ') appears on path γ. For every $(\sigma', \tau') \in E_1^*$, the *congestion* of edge (σ', τ') in the flow ϕ is the quantity

$$A_{\sigma', \tau'}(\phi) = \frac{1}{\pi_{\mathcal{B}}(\sigma') P_1(\sigma', \tau')} \sum_{\gamma \in \mathcal{P} : (\sigma', \tau') \in \gamma} r((\sigma', \tau'), \gamma) \, |\gamma| \, \phi(\gamma).$$

The *congestion* of the flow is the quantity

$$A(\phi) = \max_{(\sigma', \tau') \in E_1^*} A_{\sigma', \tau'}(\phi).$$

The comparison method of Diaconis and Saloff-Coste [6] enables one to compare the eigenvalues of the transition matrices P_1 and P_k using the quantity $A(\phi)$ for a flow ϕ. It is typical in the literature to assume that these eigenvalues are non-negative, and we could achieve this by modifying the dynamics with additional self-loops (see [22]). We prefer to obtain results about the original dynamics. Diaconis and Saloff-Coste's method makes this possible, but we have to be careful to ensure that the flow that we construct is an odd flow. This complicates our construction, but does not change the result. A proof of the following proposition can be found in [10] (Theorem 10). The proposition is similar to Proposition 4 of Randall and Tetali [20] except that the latter requires the self-loops. Both propositions are based closely on the ideas of Aldous [2], Diaconis and Stroock [7], and Sinclair [22].

Proposition 19 *Suppose that ϕ is an odd flow.*

$$\tau_\sigma(\mathcal{M}_1, \varepsilon) \leq A(\phi) \tau(\mathcal{M}_k, 1/(2e))(-\ln(\varepsilon \pi_\beta(\sigma))).$$

It is clear from Proposition 19 that to prove Theorem 18, we need only construct an odd flow ϕ with $A(\phi) \in O(1)$. We construct ϕ in Section 6.2 and bound $A(\phi)$ in Section 6.3.

6.2 Constructing an odd flow

Consider a problem instance consisting of a non-empty region R with $|R| = n$ and a colouring $\mathcal{B}$ of ∂R. We will now construct an odd flow ϕ.

For every pair $(\sigma, \tau) \in E_k$, let f be the lexicographically least face such that $W(f, k)$ contains the discrepancy between colourings σ and τ. Let $\gamma_{\sigma,\tau} \in \mathcal{P}_{\sigma,\tau}$ be the canonical path from σ to τ constructed as follows:

- Update the rows of $W(f, k)$ in order from top to bottom.
- Update each row from left to right.
- In order to update a given face $v \in W(f, k)$
  - If the colour of v is $\tau(v)$ then do nothing.
  - Otherwise, if any neighbours of v have colour $\tau(v)$, recolour these with the lexicographically first available colour.

– Recolour v with colour $\tau(v)$.

- Finally, if the length of the constructed path is not odd, then add three more transitions — colour f with the first available colour then with the next available colour that is not $\tau(f)$ then recolour it with $\tau(f)$.

Assign all of the flow from σ to τ to path $\gamma_{\sigma,\tau}$. That is, set $\phi(\gamma_{\sigma,\tau}) = P_k(\sigma,\tau)\pi_{\mathcal{B}}(\sigma)$.

For every $\sigma \in S(\mathcal{B})$ and every $v \in R$, Let $\gamma_v(\sigma)$ be the following length-3 path from σ to itself: Recolour v with the first available colour, then with the next available colour that is not $\sigma(v)$, then with $\sigma(v)$. Set $\phi(\gamma_v(\sigma)) = \frac{1}{n}P_k(\sigma,\sigma)\pi_{\mathcal{B}}(\sigma)$. Note that Equation (2) is satisfied, since there are n vertices $v \in R$.

6.3 Bounding $A(\phi)$

In this section we will show that $A(\phi) = O(1)$, which completes the proof of Theorem 18.

First note that $A_{\sigma',\sigma'}(\phi) = 0$, for all $(\sigma',\sigma') \in E_1^*$. Now let σ' and τ', where $(\sigma',\tau') \in E_1$, be colourings that disagree on face x. Now

$$\begin{aligned} A_{\sigma',\tau'}(\phi) &= \frac{1}{\pi_{\mathcal{B}}(\sigma')P_1(\sigma',\tau')} \left(\sum_{\substack{(\sigma,\tau)\in E_k \\ (\sigma',\tau')\in\gamma_{\sigma,\tau}}} r((\sigma',\tau'),\gamma_{\sigma,\tau})\,|\gamma_{\sigma,\tau}|\,P_k(\sigma,\tau)\,\pi_{\mathcal{B}}(\sigma) \right. \\ &\qquad \left. + \sum_{\substack{\sigma \\ (\sigma',\tau')\in\gamma_x(\sigma)}} r((\sigma',\tau'),\gamma_x(\sigma))\,|\gamma_x(\sigma)|\,\frac{1}{n}P_k(\sigma,\sigma)\pi_{\mathcal{B}}(\sigma) \right). \end{aligned}$$

Since $\pi_{\mathcal{B}}$ is uniform and all of the path lengths are $O(1)$ and all of the quantities $r((\sigma',\tau'),\gamma)$ are $O(1)$, this simplifies to

$$A_{\sigma',\tau'}(\phi) \le O(1) \times \left(\sum_{\substack{(\sigma,\tau)\in E_k \\ (\sigma',\tau')\in\gamma_{\sigma,\tau}}} \frac{P_k(\sigma,\tau)}{P_1(\sigma',\tau')} + \sum_{\substack{\sigma \\ (\sigma',\tau')\in\gamma_x(\sigma)}} \frac{P_k(\sigma,\sigma)}{nP_1(\sigma',\tau')} \right).$$

To see that the first sum is $O(1)$ note that there are only $O(1)$ pairs (σ,τ) in the summation (this holds since σ and τ agree with σ' except in a constant-sized window around x). Then, since $\sigma \neq \tau$, $P_k(\sigma,\tau) \in O(1/n)$. Finally, $P_1(\sigma',\tau') \in \Omega(1/n)$.

To see that the second sum is $O(1)$ note that there are only $O(1)$ colourings σ in the summation. Also, $P_k(\sigma,\sigma) \le 1$ and $P_1(\sigma',\tau') \in \Omega(1/n)$.

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