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Packing tripods: a computational approach *

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1 Introduction

In [4, 2], S. K. Stein and S. Szabó posed a problem concerning simple three-dimensional polyominoes, called “semicrosses” in [4], and “tripods” in [2]. A *tripod* of order n is formed by a corner and the three adjacent edges of an integer $n \times n \times n$ cube (see Figure 1). How densely can one fill the space with non-overlapping tripods of order n? In [6], the author showed that the fraction of the space occupied by tripods must be infinitely small as $n \to \infty$.

We now address a more general problem, also introduced in [4, 2]. Now, tripods of order n are to be packed without overlap, so that their corner cubes coincide with one of the unit cells of an $n \times n \times n$ cube, and all tripods have the same orientation. The number of tripods in the packing is called the packing *size*. Following Stein, we denote by $f(n)$ the maximum size of a tripod packing for a given order n.

The maximum fraction of space that can be filled with arbitrary non-overlapping tripods is proportional to $f(n)/n^2$. Our previous result can now be restated as follows: $f(n) = o(n^2)$ as $n \to \infty$. Our current goal is to study the behaviour of Stein's function $f(n)$ in more detail. In this paper, we approach this goal from two directions:

- computing good packings for small values of n;
- combining such packings recursively to obtain an asymptotic lower bound on $f(n)$.

The best previously known lower bound for Stein's function is $f(n) = \Omega(n^{1.5235})$. We show that this lower bound can be improved to $\Omega(n^{1.5287})$. We also make some conjectures on the structure of good packings, and suggest possible further improvements to our experiments.

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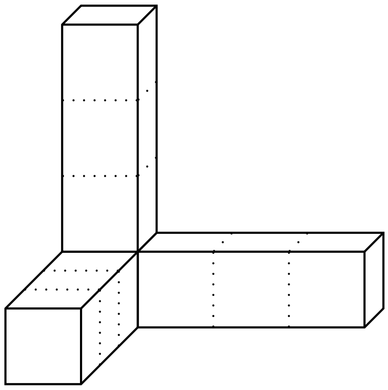

Figure 1: A tripod of order $n = 4$

0	0.	02.	02..	02...	02....	02.4...	02....3.
	.1	..0	..02	..02.	..02..	..05...	..0.2..3
		1.2	13..	13.4.	13.4..	1.26...	14..5...
			..13	..1.2	..1.24	012	67......
				..3.4	..35..	3...46.	..1..25.
					35	.3....4	..4.6.7.
						..3.5.6	...0.4.5
							...1.6.7

Figure 2: Best known tripod packings of order $n \leq 8$

2 Packings of small order

For a given tripod packing of order n, the only known method of deciding its optimality is by brute-force search. Since the number of possible tripod packings grows exponentially, obtaining the precise value of the function $f(n)$ is difficult even for very moderate n.

Figure 2 shows the best known tripod packings of order $n \leq 8$. Each packing is represented by a $n \times n$ matrix. A tripod in position (i, j, k) is represented by the number k in row i and column j of the matrix (here, all indices run from 0 to $n - 1$).

The packings with $n \leq 6$ are the only ones known to be optimal. Note that these optimal packings have a very simple structure: apart from the corner tripods $(0, 0, 0)$ and $(n - 1, n - 1, n - 1)$, they contain a triple of

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n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$g(n)$	1	2	5	8	11	14	19	23	28	32	37	43	49	55	61

Table 1: Best known tripod packings of order $n \leq 15$

tripods $(0,1,2)$, $(1,2,0)$, $(2,0,1)$. The remaining tripods can obtained by repeatedly translating this triple along the main diagonal of the cube, with simultaneous rotation by 60° in every step. The total number of tripods obtained by this construction is $3n-4$.

The packing of degree 7 illustrates another, much more powerful method of constructing tripod packings. This packing can be obtained from the packing of degree 3 by "blowing up" the two corner positions to $3 \times 3 \times 3$ subcubes, each with 5 tripods. The edge positions become "blown up" to $3 \times 3 \times 1$ cuboids; three of these cuboids contain 3 tripods each. The total number of tripods is 19. Every packing of order m and size d can be "blown up" to a packing of order m^2 and size d^2. By repeating this process, we can obtain a packing of order m^k and size d^k for any $k \geq 0$. Hence, $f(m^k) \geq d^k$, and therefore $f(n) = \Omega(n^{\log d / \log m})$. The highest value of the exponent given by the packings in Figure 2 is $\log 19 / \log 7 \approx 1.5131$.

In contrast with the previous packings, the packing of degree 8 lacks any obvious structure (apart from the symmetry that we discuss below). This "sporadic" packing appears to be better than all known systematic packings of the same order. This motivates us in searching exhaustively for good packings of relatively low degrees.

In order to search the space of all packings efficiently, we need to exploit the various symmetries of this space. For instance, a packing remains valid if the cube is rotated by 120° around its main diagonal. In fact, all packings in Figure 2 are invariant under such rotation: the existence of a tripod i, j, k implies the existence of tripods j, k, i and k, i, j. We call this property *3-symmetry.* All known optimal packings are either 3-symmetric, or can be transformed into 3-symmetric form without losing any tripods. Therefore, the following conjecture seems plausible.

Conjecture (3-symmetry). *For every n, there exists a 3-symmetric tripod packing of size $f(n)$.*

In other words, we conjecture that $f(n) = g(n)$, where $g(n)$ is the maximum possible size of a 3-symmetric packing. The 3-symmetry conjecture allows us to perform exhaustive search for much higher values of n. Table 1 shows the best known values of $g(n)$ for $n \leq 15$. These values are known to be optimal up to and including $g(14)$.

Another transformation preserving the validity of a tripod packing is *mirror reflection*: a tripod (i, j, k) is mapped to (j, i, k). No non-trivial packing can be invariant under this transformation; still, breaking the mirror symmetry of the search space approximately halves the amount of search.

The last global transformation that preserves the validity of a packing is *duality*: a tripod (i, j, k) is mapped to $(n-i-1, n-j-1, n-k-1)$. All packings in Figure 2 except the last one are self-dual (i.e. invariant under duality). It might be tempting to strengthen our symmetry conjecture by considering packings that are both 3-symmetric and self-dual. However, since the value of $g(8) = 23$ is odd, it cannot correspond to any self-dual packing. Still, exploiting duality allows some more symmetry breaking and further reduction in the amount of search.

In addition to these global symmetries, it is also possible to reduce the search by considering transformations that affect individual tripods. For example, it is easy to see that a packing only needs be considered if it satisfies the following property: no tripod (i, j, k) can be moved to a position (i, j, k') with $k' < k$. This property, which we call *gravity*, is easy to enforce (even in conjunction with 3-symmetry), and provides significant further savings.

By running a branch-and-bound algorithm with all the above optimisations, we have been able to compute Table 1 (and the corresponding packing matrices) in about 40 hours on a Sun Ultra 5 workstation. In the next section, we describe how these results feed into the search for the best asymptotic lower-bound packing exponent.

3 Packing asymptotics

Previous to this work, the best asymptotic lower bound on the function $f(x)$ has been given by D. Hickerson [3]. Hickerson's construction is based on recursion similar to the one we have seen in the example of $n = 7$. We start with a simple "external" packing, and blow up each position by substituting suitably chosen "internal" packings. Both the external and the internal packings are in general cuboids with unequal dimensions. Hickerson ran a few levels of recursion by hand, using just two external cuboid packings: one of order $2\times2\times1$ and size 2, another of order $3\times3\times3$ and size 5. He obtained a packing of order 255 and size 4639, implying $f(n) = \Omega(n^{\log 4639/\log 255}) = \Omega(n^{1.5235})$.

Hickerson's method has been programmed by G. Stoneman as part of his final-year undergraduate project [5]. Additionally, a database of cuboid packings generalising Table 1 was compiled and used for selection of internal packings. One of the resulting recursive packings had order 255 and size 4776, implying $f(n) = \Omega(n^{\log 4776/\log 255}) = \Omega(n^{1.5287})$. This is the best currently known asymptotic lower bound, which corresponds to the densest known infinite family of tripod packings.

4 Conclusion

We have presented a systematic computational approach to Stein's tripod packing problem, which resulted in a new asymptotic lower bound on the maximum size of a packing: $f(n) = \Omega(n^{1.5287})$. There remains a substantial gap between this lower bound and the upper bound $f(n) = o(n^2)$ from [6]. It is likely that the lower bound can be further improved by a more sophisticated application of the methods described in this paper. For instance, report [5] describes an exhaustive search for recursive packings, using just two external cuboids. Using more external cuboids could yield even better results. Furthermore, it may be possible to collect statistics on cuboid dimensions in order to direct the search to the more promising regions of the cuboid dimension space. A proof or disproof of the 3-symmetry conjecture, which has eluded us so far, could provide new valuable insights into the structure of tripod packing. Finally, our computation techniques could be extended to other challenging combinatorial problems.

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