Alcom-FT Technical Report Series
ALCOMFT-TR-03-53

Solving order constraints in logarithmic space*

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Abstract

We combine methods of order theory, finite model theory, and universal algebra to study, within the constraint satisfaction framework, the complexity of some well-known combinatorial problems connected with a finite poset. We identify some conditions on a poset which guarantee solvability of the problems in (deterministic, symmetric, or non-deterministic) logarithmic space.

On the example of order constraints we study how a certain algebraic invariance property is related to solvability of a constraint satisfaction problem in non-deterministic logarithmic space.

Classification: computational complexity, logic in computer science

1 Introduction

A wide range of combinatorial search problems encountered in artificial intelligence and computer science can be naturally expressed as 'constraint satisfaction problems' [34], in which the aim is to find an assignment of values to a given set of variables subject to specified constraints. For example, the standard propositional satisfiability problem [36, 40] may be viewed as a constraint satisfaction problem where the variables must be assigned Boolean values, and the constraints are specified by clauses. Further examples include graph colorability, clique, and Hamiltonian circuit problems, conjunctive-query containment, and many others (see [20, 25]). One advantage of considering a common framework for all of these diverse problems is that it makes it possible to obtain generic structural results concerning the computational complexity of constraint satisfaction problems that can be applied in many different areas such as machine vision, belief maintenance, database theory, temporal reasoning, type reconstruction, graph theory, and scheduling (see, e.g., [12, 20, 25, 26, 34, 41]).

*Partially supported by the Future and Emerging Technologies programme of the EU under contract number IST-1999-14186 (ALCOM-FT).

The general constraint satisfaction problem (CSP) is **NP**-complete. Therefore, starting with the seminal paper by Schaefer [40], much work has been done on identifying restrictions on the problem that guarantee lower complexity (see, e.g., [3, 4, 5, 7, 8, 9, 12, 13, 14, 20, 21, 22, 23, 24, 25]).

The constraint satisfaction problem has several equivalent definitions (see, e.g., [5, 7, 14, 20, 25]). For the purposes of this paper, we define it, as in [8, 12, 20, 25], as the homomorphism problem: given two relational structures, $\mathcal{A}$ and $\mathcal{B}$, the question is whether there is a homomorphism from $\mathcal{A}$ to $\mathcal{B}$. One distinguishes uniform and non-uniform CSPs depending on whether both $\mathcal{A}$ and $\mathcal{B}$ are parts of the input or not (see, e.g., [25]). The case when the choice of $\mathcal{A}$ is restricted has been studied in connection with database theory (see, e.g., [13, 14, 16, 25]). In this paper, as in [8, 12, 20], we will consider non-uniform constraint satisfaction problems CSP($\mathcal{B}$) where the structure $\mathcal{B}$ is fixed.

We will concentrate on the much understudied case of constraint problems in logarithmic space complexity classes. The class **NL** of problems solvable in non-deterministic logarithmic space has received much attention in complexity theory. It is known that $\mathbf{NL} \subseteq \mathbf{NC}$, and so problems in **NL** are highly parallelizable.

Despite the large amount of tractable constraint satisfaction problems identified so far, to the best of our knowledge, only two families of concrete problems CSP($\mathcal{B}$) are known to be in **NL**. The first family consists of restricted versions of the Boolean satisfiability problem: the so-called bijunctive constraints [40], including the 2-SATISFIABILITY problem, and implicative Hitting-Set Bounded constraints [7, 8]. The second family consists of implicational constraints [24] which are a generalization of bijunctive constraints to non-Boolean problems, and which, in turn, have been slightly generalized in [8]. Dalmau introduced 'bounded path duality', a general sufficient condition for a CSP($\mathcal{B}$) to be in **NL**, which is shown to be satisfied by all problems from the two families [8]. However, this condition is not easy to apply; in general, it is not known to be decidable.

Our main motivation in this paper is to clarify the relation between solvability of a CSP in non-deterministic logarithmic space and the algebraic condition of invariance under a near-unanimity operation that is also satisfied by all concrete problems above. It is known that many important properties of CSPs can be captured by algebraic invariance conditions (see, e.g., [9, 12, 20]), and that the condition mentioned above exactly corresponds to the following property: for some l, every instance of CSP($\mathcal{B}$) has precisely the same solutions as the system of all its subinstances of size l [12, 21]. It is unknown whether this property is sufficient and/or necessary for a CSP($\mathcal{B}$) to be in **NL**. We answer this question for problems connected with posets by showing that, for such problems, the condition is sufficient but not necessary (and therefore it is not necessary in general).

In order theory, there is a rich tradition of linking combinatorial and algebraic properties of posets (see, e.g., [6, 27, 28, 29, 38, 44, 45]). We use such results in order to get combinatorial properties of posets that can be described in first-order logic with the transitive closure operator. As is known from finite model theory, sentences in this logic can be evaluated in non-deterministic logarithmic space. We use the poset retraction problem as a medium for applying finite model theory results to the considered CSPs. The notion of retraction plays an important role in order theory (see, e.g., [11, 27, 38]), and this problem has also been studied in computer science along with the problem of satisfiability of inequalities in a poset and the extendibility problem (see, e.g., [2, 12, 28, 37]). As intermediate results that are of independent interest, we describe two classes of posets for which the retraction problem is solvable in deterministic or (complete for) symmetric logarithmic space.

The complexity class **SL** (symmetric log-space) appears in the literature less often than **L** and **NL**, so we say a few words about it here. It is the class of problems solvable by symmetric non-deterministic Turing machines in logarithmic space [30], and $\mathbf{L} \subseteq \mathbf{SL} \subseteq \mathbf{NL}$. This class can be characterized in a number of ways (see, e.g., [1, 15, 35]) and is known to be closed under complementation [35]. It contains such important problems as the UNDI-

RECTED st-CONNECTIVITY problem and the GRAPH 2-COLORABILITY problem [1, 35, 39]. It is known (see, e.g., [15]) that sentences of $\mathbf{FO} + \mathbf{STC}$, first-order logic with the symmetric transitive (reflexive) closure operator, can be evaluated in symmetric logarithmic space.

2 Constraint satisfaction problems

A *vocabulary* τ is a finite set of relation symbols or predicates. Every relation symbol $R \in \tau$ has an *arity* $ar(R) \geq 0$. A τ-structure $\mathcal{A}$ consists of a set A, called the *universe* of $\mathcal{A}$, and a relation $R^{\mathcal{A}}$ of arity $ar(R)$ for every $R \in \tau$. All structures considered in this paper are *finite*.

A *homomorphism* from a τ-structure $\mathcal{A}$ to a τ-structure $\mathcal{B}$ is a mapping $h : A \to B$ such that, for every $R \in \tau$, $(a_1, \ldots, a_{ar(R)}) \in R^{\mathcal{A}}$ implies $(h(a_1), \ldots, h(a_{ar(R)})) \in R^{\mathcal{B}}$. If there exists a homomorphism from $\mathcal{A}$ to $\mathcal{B}$, we write $\mathcal{A} \to \mathcal{B}$.

We consider the problem $\mathrm{CSP}(\mathcal{B})$ where only structure $\mathcal{A}$ is the input, and the question is whether $\mathcal{A} \to \mathcal{B}$. Of course, one can view $\mathrm{CSP}(\mathcal{B})$ as the class of all τ-structures $\mathcal{A}$ such that $\mathcal{A} \to \mathcal{B}$. So we have a class of problems parameterized by finite structures $\mathcal{B}$, and the ultimate goal of our research is to classify the complexity of such problems. The classic problems of this type are various versions of GRAPH COLORABILITY and SATISFIABILITY of logical formulas.

Example 1 (H-coloring) *If $\mathcal{B}$ is an undirected irreflexive graph H then* $\mathrm{CSP}(\mathcal{B})$ *is the* GRAPH H-COLORING *problem. This problem is* $\mathbf{SL}$*-complete if H is bipartite [35, 39], and* $\mathbf{NP}$*-complete otherwise [17]. If H is a complete graph K_k then* $\mathrm{CSP}(\mathcal{B})$ *is the* GRAPH k-COLORABILITY *problem.*

Example 2 (2-SAT) *Let $\mathcal{B}$ be the stucture with universe* $\{0, 1\}$ *and all at most binary relations over* $\{0, 1\}$*. Then* $\mathrm{CSP}(\mathcal{B})$ *is exactly the* 2-SATISFIABILITY *problem, it is* $\mathbf{NL}$*-complete [36].*

Example 3 (NAE-SAT) *Let $\mathcal{B}$ be the stucture with universe* $\{0, 1\}$ *and one ternary relation* $R = \{(a, b, c) \mid \{a, b, c\} = \{0, 1\}\}$*. Then* $\mathrm{CSP}(\mathcal{B})$ *is exactly the* NOT-ALL-EQUAL SATISFIABILITY *problem as defined in [40], it is* $\mathbf{NP}$*-complete.*

The most significant progress in classifying the complexity of $\mathrm{CSP}(\mathcal{B})$ has been made via methods of finite model theory [12, 25] and methods of universal algebra [3, 4, 5, 20, 21, 22, 23]. In [8, 9], both approaches are present.

The finite model theory approach aims at defining $\mathrm{CSP}(\mathcal{B})$ or its complement in various logics. Most of the known results in this direction make extensive use of the logic programming language Datalog.

A *Datalog program* over a vocabulary τ is a finite collection of rules of the form

$$t_0 :- t_1, \ldots, t_m$$

where each t_i is an atomic formula $R(v_1, \ldots, v_l)$. The predicates occuring in the heads of the rules are the *intensional database* predicates (IDBs), while all others are *extensional database* predicates (EDBs) and must belong to τ. One of the IDBs is designated as the goal predicate. A Datalog program is a recursive specification of the IDBs with semantics obtained via least fixed-points (see [42]).

For $0 \leq j \leq k$, (j, k)-Datalog is a collection of Datalog programs with at most k variables per rule and at most j variables per rule head. A Datalog program is *linear* if every rule has at most one IDB in its body.

In [12], tractability of many constraint satisfaction problems was explained in the following way: if $\neg\mathrm{CSP}(\mathcal{B})$, the complement of $\mathrm{CSP}(\mathcal{B})$, is definable in (j, k)-Datalog for some j, k then $\mathrm{CSP}(\mathcal{B})$ is solvable in polynomial time. Dalmau [8] introduced "bounded path duality" for $\mathcal{B}$, a general sufficient condition for $\mathrm{CSP}(\mathcal{B})$ to be in **NL**. This condition is characterized in seven equivalent ways, one of them being definability of $\neg\mathrm{CSP}(\mathcal{B})$ in linear Datalog. However, only definability by linear $(1,k)$-Datalog programs is known to be decidable (for any fixed k) [8]. It is noted in that paper that all known concrete structures $\mathcal{B}$ with $\mathrm{CSP}(\mathcal{B})$ in **NL** have bounded path duality.

The algebraic approach to $\mathrm{CSP}(\mathcal{B})$ uses the notion of a polymorphism.

Definition 1 *Let R be an m-ary relation on B, and f an n-ary operation on B. Then f is said to be a* polymorphism *of R (or R is* invariant *under f) if, for every $(b_{11},\dots,b_{m1}),\dots,(b_{1n},\dots,b_{mn}) \in R$, we have*

$$f\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix} = \begin{pmatrix} f(b_{11},b_{12},\dots,b_{1n}) \\ f(b_{21},b_{22},\dots,b_{2n}) \\ \vdots \\ f(b_{m1},b_{m2},\dots,b_{mn}) \end{pmatrix} \in R.$$

An operation on B is said to be a polymorphism of a τ-structure $\mathcal{B}$ if it is a polymorphism of every $R^{\mathcal{B}}$, $R \in \tau$.

Example 4 *1) It is easy to check that both permutations on $\{0,1\}$ are polymorphisms of the relation R from Example 3, while the binary operation* min *is not.*

2) The polymorphisms of a poset are simply the monotone operations on it.

3) The dual discriminator *on B is defined by*

$$\mu_B(x,y,z) = \begin{cases} y & \text{if } y = z, \\ x & \text{otherwise.} \end{cases}$$

One can verify that $\mu_{\{0,1\}}$ is a polymorphism of the structure $\mathcal{B}$ from Example 2.

Let $\mathrm{Pol}(\mathcal{B})$ denote the set of all polymorphisms of $\mathcal{B}$. This set determines the complexity of $\mathrm{CSP}(\mathcal{B})$, as the following result shows.

Theorem 1 ([20]) *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be structures with the same universe B. If $\mathrm{Pol}(\mathcal{B}_1) \subseteq \mathrm{Pol}(\mathcal{B}_2)$ then $\mathrm{CSP}(\mathcal{B}_2)$ is polynomial-time reducible to $\mathrm{CSP}(\mathcal{B}_1)$.*

Moreover, if the equality relation $=_B$ on B can be expressed by using predicates of $\mathcal{B}_1$, conjunction, and existential quantification, then the above reduction is logarithmic-space.

Examples of classifying the complexity of $\mathrm{CSP}(\mathcal{B})$ by particular types of polymorphisms can be found in [3, 4, 5, 8, 9, 20, 21, 22, 23]. It follows from [22, 24] that if μ_B is a polymorphism of $\mathcal{B}$ then $\mathrm{CSP}(\mathcal{B})$ is in **NL**. This result is generalized in [8] to give a slightly more general form of ternary polymorphism guaranteeing that $\mathrm{CSP}(\mathcal{B})$ is in **NL**.

Definition 2 *Let f be an n-ary operation. Then f is said to be* idempotent *if $f(x,\dots,x) = x$ for all x, and it is said to be* Taylor *if, in addition, it satisfies n identities of the form*

$$f(x_{i1},\dots,x_{in}) = f(y_{i1},\dots,y_{in}), \quad i = 1,\dots,n$$

where $x_{ij}, y_{ij} \in \{x,y\}$ for all i,j and $x_{ii} \neq y_{ii}$ for $i = 1,\dots,n$.

For example, it is easy to check that any binary idempotent commutative operation and the dual discriminator are Taylor operations. The following result links hard problems with the absence of Taylor polymorphisms.

Theorem 2 ([28]) *Let $\mathcal{B}$ be a structure such that $\mathrm{Pol}(\mathcal{B})$ consists of idempotent operations. If $\mathcal{B}$ has no Taylor polymorphism then $\mathrm{CSP}(\mathcal{B})$ is **NP**-complete.*

3 Poset-related problems

In this section we introduce the particular type of problems we work with in this paper, the poset retraction problem and the order constraint satisfaction problem. Here and in the following the universes of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{P}$, $\mathcal{Q}$ etc. are denoted by A, B, P, Q etc., respectively.

Let $\mathcal{A}$, $\mathcal{B}$ be structures such that $B \subseteq A$. Recall that a *retraction* from $\mathcal{A}$ onto $\mathcal{B}$ is a homomorphism $h : \mathcal{A} \to \mathcal{B}$ that fixes every element of $\mathcal{B}$, that is $h(b) = b$ for every $b \in B$.

Fix a poset $\mathcal{P}$. An instance of the *poset retraction problem* $\mathrm{PoRet}(\mathcal{P})$ is a poset $\mathcal{Q}$ such that the partial order of $\mathcal{P}$ is contained in the partial order of $\mathcal{Q}$, and the question is whether there is a retraction from $\mathcal{Q}$ onto $\mathcal{P}$. Again, we can view $\mathrm{PoRet}(\mathcal{P})$ as the class of all posets $\mathcal{Q}$ with positive answer to the above question, and then $\neg\mathrm{PoRet}(\mathcal{P})$ is the class of all instances $\mathcal{Q}$ with the negative answer. This problem was studied in [2, 12, 37]. For example, it was proved in [12] that, for every structure $\mathcal{B}$, there exists a poset $\mathcal{P}$ such that $\mathrm{CSP}(\mathcal{B})$ is polynomial-time equivalent to $\mathrm{PoRet}(\mathcal{P})$.

In [37], a poset $\mathcal{P}$ is called *TC-feasible* if $\neg\mathrm{PoRet}(\mathcal{P})$ is definable (in the class of all instances of $\mathrm{PoRet}(\mathcal{P})$) by a sentence in a fragment of $\mathbf{FO} + \mathbf{TC}$, first-order logic with the transitive closure operator, over the vocabulary τ_1 that contains one binary predicate R interpreted as partial order and the constants c_p, $p \in P$, always interpreted as the elements of $\mathcal{P}$. The fragment is defined by the condition that negation and universal quantification are disallowed. Since any sentence in $\mathbf{FO} + \mathbf{TC}$ is verifiable in non-deterministic logarithmic space [15] and $\mathbf{NL} = \mathbf{CoNL}$ [19], TC-feasibility of a poset $\mathcal{P}$ implies that $\mathrm{PoRet}(\mathcal{P})$ is in $\mathbf{NL}$. Note that even if $\neg\mathrm{PoRet}(\mathcal{P})$ is definable by a sentence in full $\mathbf{FO} + \mathbf{TC}$ then, of course, $\mathrm{PoRet}(\mathcal{P})$ is still in $\mathbf{NL}$.

Let $P = \{p_1, \ldots, p_n\}$ and let τ_2 be a vocabulary containing one binary predicate R and n unary predicates $S_1, \ldots, S_n$. We denote by $\mathcal{P}_P$ the τ_2-structure with universe P, $R^{\mathcal{P}_P}$ being the partial order $\leq$ of $\mathcal{P}$, and $S_i^{\mathcal{P}_P} = \{p_i\}$. We call problems of the form $\mathrm{CSP}(\mathcal{P}_P)$ *order constraint satisfaction problems*. Note that if $\mathcal{B}$ is simply a poset then $\mathrm{CSP}(\mathcal{B})$ is trivial because every mapping sending all elements of an instance to a fixed element of $\mathcal{B}$ is a homomorphism. The main difference between $\mathrm{CSP}(\mathcal{P}_P)$ and $\mathrm{PoRet}(\mathcal{P})$ is that an instance of $\mathrm{CSP}(\mathcal{P}_P)$ is not necessarily a poset, but an arbitrary structure over τ_2.

The following theorem will provide us with a way of obtaining order constraint satisfaction problems in $\mathbf{NL}$.

Theorem 3 *Let $\mathcal{P}$ be a poset. If $\neg\mathrm{PoRet}(\mathcal{P})$ is definable (within the class of all posets containing $\mathcal{P}$) by a τ_1-sentence in $\mathbf{FO} + \mathbf{TC}$ then $\mathrm{CSP}(\mathcal{P}_P)$ is in $\mathbf{NL}$.*

Proof. Let $\mathcal{A}$ be an instance of $\mathrm{CSP}(\mathcal{P}_P)$. We can assume that $A \cap P = \emptyset$. Let τ_3 be $\tau_1 \cup \tau_2$, the vocabulary obtained by extending the vocabulary τ_2 with constant symbols c_p, $p \in P$, that again will always be interpreted as the elements of P. Let $\mathcal{A}'$ be the τ_3-structure with the universe $A \cup P$ and such that $R^{\mathcal{A}'} = R^{\mathcal{A}} \cup R^{\mathcal{P}_P}$, and $S_i^{\mathcal{A}'} = S_i^{\mathcal{A}} \cup \{p_i\}$ for $i = 1, \ldots, n$. It is obvious that $\mathcal{A} \to \mathcal{P}_P$ if and only if there is a retraction from $\mathcal{A}'$ to $\mathcal{P}'_P$ (where $\mathcal{P}'_P$ is $\mathcal{P}_P$ considered as τ_3-structure with naturally interpreted constant symbols), and that $\mathcal{A}'$ and $\mathcal{P}'_P$ can be obtained in constant time. We shall build a τ_3-sentence Φ such that $\mathcal{A}' \models \Phi$ if and only if there is no retraction from $\mathcal{A}'$ onto $\mathcal{P}'_P$.

Define binary predicate R_1 by the formula

$$R_1 = \mathrm{TC}_{x,y}[R(x,y) \vee \bigvee_{i=1}^{n} (S_i(x) \wedge S_i(y))]. \tag{1}$$

Denote by ϱ the quasiorder on $A \cup P$ defined by R_1 in $\mathcal{A}'$.

Let $\Psi = \bigvee R_1(c_{p_i}, c_{p_j})$ where the disjunction is taken over all pairs $(p_i, p_j) \in P^2$ such that $p_i \not\leq p_j$. Note that if Ψ holds in $\mathcal{A}'$ then there is no retraction from $\mathcal{A}$ onto $\mathcal{P}'_P$.

Assume that $\mathcal{A}' \not\models \Psi$. Then all elements of P are in different classes of the equivalence relation $\varepsilon = \varrho \cap \varrho^{-1}$. Let Q be a complete set of representatives of ε-classes such that

$P \subseteq Q$. Denote by $\sqsubseteq$ the partial order on Q induced by ϱ. Then Q with $\sqsubseteq$ can be considered as a τ_1-structure $\mathcal{Q}$, that is, as an instance of PoRet($\mathcal{P}$). Note that $\mathcal{A}'$ with unary relations removed and with $R^{\mathcal{A}'}$ replaced by ϱ can also be considered as a τ_1-structure (but not as an instance of PoRet($\mathcal{P}$)). Denote this structure by $\mathcal{A}_1$.

Claim 1. If $\mathcal{A}' \not\models \Psi$ then the following conditions are equivalent:

1. there is a retraction from $\mathcal{A}'$ onto $\mathcal{P}'_P$;
2. there is a retraction from $\mathcal{A}_1$ onto $\mathcal{P}$;
3. there is a retraction from $\mathcal{Q}$ onto $\mathcal{P}$.

To prove the claim, note that, since $\mathcal{P}$ is a poset, every retraction from $\mathcal{A}'$ onto $\mathcal{P}'_P$ or from $\mathcal{A}_1$ onto $\mathcal{P}$ must map all elements from one class of ε to the same element.

Claim 2. For every τ_1-sentence ψ of the transitive closure first-order logic without equality, $\mathcal{A}_1 \models \psi$ if and only if $\mathcal{Q} \models \psi$.
Let $f : A \cup P \to Q$ be the mapping such that $f(a) = q$ if and only if $(a, q) \in \varepsilon$. Note that f is well-defined and $f(p) = p$ for every $p \in P$.

Let $\varphi(x_1, \ldots, x_m)$ be a τ_1-formula of the above logic with free variables $x_1, \ldots, x_m$. One can show by induction on the construction of φ that, for every $a_1, \ldots, a_m \in A \cup P$, we have $\varphi^{\mathcal{A}_1}(a_1, \ldots, a_m) = true$ if and only if $\varphi^{\mathcal{Q}}(f(a_1), \ldots, f(a_m)) = true$. Indeed, this is obvious for all atomic formulas (that are formulas of the form $R(x_1, x_2)$, $R(x, c_p)$, and $R(c_p, x)$), and the induction step can be easily checked by cases. Claim 2 is proved.

Let Θ be a τ_1-sentence in $\mathbf{FO} + \mathbf{TC}$ defining $\neg$PoRet($\mathcal{P}$). We may assume that Θ does not contain equality, because every subformula of the form $\theta_1 = \theta_2$ can be replaced by $R(\theta_1, \theta_2) \wedge R(\theta_2, \theta_1)$, and, of course, the new sentence would also define $\neg$PoRet($\mathcal{P}$). Now let $\Theta_R^{R_1}$ be the τ_3-sentence obtained from Θ by substituting the expression for R_1 given in (1) for each occurence of R, and let $\Phi = \Psi \vee \Theta_R^{R_1}$. It follows from Claims 1 and 2 that $\mathcal{A}' \models \Phi$ if and only if there is no retraction from $\mathcal{A}'$ onto $\mathcal{P}'_P$. Since Φ is a sentence of $\mathbf{FO} + \mathbf{TC}$, and $\mathcal{A}'$ and $\mathcal{P}'_P$ can obtained from $\mathcal{A}$ in constant time, we conclude that CSP($\mathcal{P}_P$) is in **NL**. $\square$

By analysing the above proof one can show that the problems PoRet($\mathcal{P}$) and CSP($\mathcal{P}_P$) are polynomial-time equivalent. Invoking the result from [12] mentioned in this section, we get the following statement.

Corollary 1 *For every structure $\mathcal{B}$, there exists a poset $\mathcal{P}$ such that* CSP($\mathcal{B}$) *and* CSP($\mathcal{P}_P$) *are polynomial-time equivalent.*

It is not known whether the equivalence can be made logarithmic-space, and so the corollary cannot now help us in studying CSPs in **NL**. We believe that Corollary 1 is especially interesting in view of Theorem 1 because polymorphisms of a poset may be easier to analyse. In particular, note that all polymorphisms of $\mathcal{P}_P$ are idempotent, and so, according to Theorem 2, in order to classify the complexity of CSP($\mathcal{B}$) up to polynomial-time reductions, it suffices to consider posets with a Taylor polymorphism.

Other poset-related problems studied in the literature are the satisfiability of inequalities, denoted $\mathcal{P}$-SAT, and the extendibility problem Ext($\mathcal{P}$). An instance of $\mathcal{P}$-SAT consists of a system of inequalities involving constants from $\mathcal{P}$ and variables, and the question is whether this system is satisfiable in $\mathcal{P}$. The problem $\mathcal{P}$-SAT plays an important role in type reconstruction (see, e.g., [2, 18, 37]), it was shown to be polynomial-time equivalent to PoRet($\mathcal{P}$) for the same $\mathcal{P}$ [37]. An instance of Ext($\mathcal{P}$) consists of a poset $\mathcal{Q}$ and a partial map f from $\mathcal{Q}$ to $\mathcal{P}$, and the question is whether f extends to a homomorphism from $\mathcal{Q}$ to $\mathcal{P}$. The problem Ext($\mathcal{P}$) was studied in [28], where it was shown to be polynomial-time equivalent to CSP($\mathcal{P}_P$), so all four poset-related problems are polynomial-time equivalent.

It is easy to see that $\mathrm{PoRet}(\mathcal{P})$ is the restriction of $\mathrm{Ext}(\mathcal{P})$ to instances where Q contains P and $f = \mathrm{id}_P$ is the identity function on P, while the problem $\mathrm{Ext}(\mathcal{P})$ can be viewed as the restriction of $\mathrm{CSP}(\mathcal{P}_P)$ to instances where the binary relation is a partial order. Of course, if $\mathrm{CSP}(\mathcal{P}_P)$ is in **NL** then so are the other three problems.

4 Near-unanimity polymorphisms

In this section we prove that posets with a polymorphism of a certain form, called *near-unanimity* operation, give rise to order constraint satisfaction problems in **NL**.

Definition 3 *A* near-unanimity (NU) *operation is an l-ary ($l \geq 3$) operation satisfying*

$$f(y, x, \ldots, x) = f(x, y, x, \ldots, x) = \cdots = f(x, \ldots, x, y) = x$$

for all x, y.

Near-unanimity operations have attracted much attention in order theory and universal algebra (see, e.g., [10, 27, 28, 29, 38, 44, 45]). For example, the posets having a ternary NU polymorphism (known as a *majority* operation) are precisely retracts of direct products of fences [38], where a fence is a poset on $\{a_0, \ldots, a_k\}$ such that $a_0 < a_1 > a_2 < \ldots a_k$ or $a_0 > a_1 < a_2 > \ldots a_k$, and there are no other comparabilities. Posets with an NU polymorphism (of some arity) are characterized in a number of ways in [27, 29].

It was proved in [12, 21] (using different terminology) that if a structure $\mathcal{B}$ has an NU polymorphism then $\mathrm{CSP}(\mathcal{B})$ is solvable in polynomial time. In [12], such problems are said to have *bounded strict width*, while in [21] they are shown to be related to the so-called *strong consistency*, a notion from artificial intelligence. Moreover, it was shown in these papers that the presence of an $(l+1)$-ary NU polymorphism f is equivalent to the l-Helly property for $\mathcal{B}$ and to definability of $\neg\mathrm{CSP}(\mathcal{B})$ by an (l, k)-Datalog program with a special property. Not going into formal definitions here, the intuition behind these properties is that every relation invariant under f is decomposable into its l-fold projections, and so a mapping from $\mathcal{A}$ to $\mathcal{B}$ that is not a homomorphism can be shown to be such using only at most l-element subsets of $\mathcal{A}$.

Interestingly, all known concrete structures $\mathcal{B}$ with $\mathrm{CSP}(\mathcal{B})$ in **NL** have an NU polymorphism. It is mentioned in [8] that all two-element structures $\mathcal{B}$ with an NU polymorphism (which precisely correspond to the first family mentioned in the introduction) have bounded path duality, and so the correponding problems $\mathrm{CSP}(\mathcal{B})$ belong to **NL**. All known concrete structures $\mathcal{B}$ that have at least three elements and such that $\mathrm{CSP}(\mathcal{B})$ in **NL** have a ternary NU polymorphism derived from dual discriminators (see Example 4) [8], these are the problems from the second family.

Our next result shows that every order constraint satisfaction problem with a near-unanimity polymorphism is in **NL**. Note that by [10], for every $l \geq 3$, there exists a poset that has an l-ary NU polymorphism, but no such polymorphism of smaller arity. Moreover, it was proved in [27] that it can be decided in polynomial time whether a poset has an NU polymorphism, while in [12] this was shown to be true for any fixed arity l for general structures.

In the following we deal mostly with posets, and so we use symbol $\leq$ (rather than R) to denote partial order. Recall that a poset is called *connected* if its comparability graph is connected.

An important tool used in the proof of the next theorem is the notion of a poset *zigzag* introduced in [44]. Intuitively, a $\mathcal{P}$-zigzag is a minimal obstruction for the extendibility problem $\mathrm{Ext}(\mathcal{P})$. More formally, a poset $\mathcal{X} = (X, \leq^{\mathcal{X}})$ is said to be contained in a poset $\mathcal{Q} = (Q, \leq^{\mathcal{Q}})$ if $\leq^{\mathcal{X}} \subseteq \leq^{\mathcal{Q}}$, and it is said to be properly contained if $\mathcal{X} \neq \mathcal{Q}$. A *$\mathcal{P}$-zigzag* is a pair $(\mathcal{Q}, f)$ such that f is a partial mapping from $\mathcal{Q}$ to $\mathcal{P}$ that cannot be extended to a full homomorphism, but, for every poset $\mathcal{X}$ properly contained in $\mathcal{Q}$, the mapping $f|_X$ is extendible to a full homomorphism from $\mathcal{X}$ to $\mathcal{P}$.

Theorem 4 *Let $\mathcal{P}$ be a poset with a near-unanimity polymorphism. Then* PoRet($\mathcal{P}$) *is in* **SL** *and* CSP($\mathcal{P}_P$) *is in* **NL**. *If, in addition, $\mathcal{P}$ is connected then* PoRet($\mathcal{P}$) *is in* **L**.

Proof. Suppose that $\mathcal{P}$ is connected. By [29], a connected poset has a near-unanimity polymorphism if and only if the number of its zigzags is finite. Let $(\mathcal{Q}, f)$ be a $\mathcal{P}$-zigzag. Denote by Q' the set of all $q \in Q$ such that $f(q)$ is undefined. For every $q \in Q$, set $q_f = x_q$ if $q \in Q'$ and $q_f = c_p$, where $p = f(q)$, otherwise. Suppose $Q' = \{q_1, \dots, q_k\}$. Denote by $\varphi_{(\mathcal{Q},f)}$ the sentence

$$\exists x_{q_1} \dots \exists x_{q_k} (\bigwedge q_f \leq q'_f)$$

where the conjunction is taken over all pairs (q, q') such that $q \leq^{\mathcal{Q}} q'$. Since $\varphi_{(\mathcal{Q},f)}$ mimics a zigzag, no poset satisfying this sentence has a retraction onto $\mathcal{P}$. Consider the sentence

$$\Phi_{\mathcal{P}} = \bigvee \varphi_{(\mathcal{Q},f)}$$

where the disjunction is taken over all $\mathcal{P}$-zigzags. Recall that PoRet($\mathcal{P}$) is a restriction of Ext($\mathcal{P}$), and so every pair $(\mathcal{X}, \mathrm{id}_P)$ such that $\mathcal{X}$ is in $\neg$PoRet($\mathcal{P}$) contains a $\mathcal{P}$-zigzag. It follows that $\Phi_{\mathcal{P}}$ defines $\neg$PoRet($\mathcal{P}$). Since $\Phi_{\mathcal{P}}$ is a first-order sentence, it is verifiable in deterministic logarithmic space (see, e.g., [15]). We conclude that PoRet($\mathcal{P}$) is in **L** (in fact, it is even in $\mathbf{AC^0}$).

Assume now that $\mathcal{P}$ is disconnected, and $\mathcal{P}_1, \dots, \mathcal{P}_n$ are the connected components of $\mathcal{P}$. Consider

$$\Phi = (\bigvee_{i=1}^{n} \Phi_{\mathcal{P}_i}) \vee (\bigvee (\mathrm{STC}_{x,y}[x \leq y](c_{p_1}, c_{p_2})))$$

where $\Phi_{\mathcal{P}_i}$'s are as defined above and the disjunction in the second parentheses is taken over all pairs (p_1, p_2) such that p_1 and p_2 belong to different connected components of $\mathcal{P}$. It is easy to show that Φ defines $\neg$PoRet($\mathcal{P}$). Since Φ belongs to **FO + STC**, we conclude that PoRet($\mathcal{P}$) is in **SL**.

Since $\mathrm{STC}_{x,y}[\psi(x,y)]$ is equivalent to $\mathrm{TC}_{x,y}[\psi(x,y) \vee \psi(y,x)]$, it remains to apply Theorem 3 to obtain that CSP($\mathcal{P}_P$) is in **NL**. $\square$

Theorem 4 generalizes a result from [37] where it was shown that if poset $\mathcal{P}$ has the 2-Helly property then PoRet($\mathcal{P}$) is in **NL**. Moreover, together with Theorem 2 it completely covers the classification of complexity of PoRet($\mathcal{P}$) for bipartite posets given in [37].

5 Series-parallel posets

In this section we exhibit the first concrete examples of structures without NU polymorphism but with constraint satisfaction problem solvable in non-deterministic logarithmic space. We deal with the poset-related problems for series-parallel posets. These posets have been studied in computer science because they play an important role in concurrency (see, e.g., [31, 32, 33]).

Proofs of some lemmas and (partially) of the theorem stated in this section are omitted due to space constraints.

Recall that a linear sum of two posets $\mathcal{P}_1$ and $\mathcal{P}_2$ is a poset $\mathcal{P}_1 + \mathcal{P}_2$ with the universe $P_1 \cup P_2$ and partial order given by $\leq^{\mathcal{P}_1} \cup \leq^{\mathcal{P}_2} \cup \{(p_1, p_2) \mid p_1 \in P_1, p_2 \in P_2\}$.

Definition 4 *A poset is called* series-parallel *if it can be constructed from singletons by using disjoint union and linear sum.*

Let $\mathbf{k}$ denote a k-antichain (that is, disjoint union of k singletons). A 4-*crown* is a poset isomorphic to $\mathbf{2}+\mathbf{2}$. The N-*poset* can be described as $\mathbf{2}+\mathbf{2}$ with one comparability missing (its Hasse diagram looks like the letter "N"). Series-parallel posets can be characterized as N-free posets, that is, posets not containing the N-poset as a subposet [43].

Denote by $\mathbb{A}$ the class of all series-parallel posets $\mathcal{P}$ with the following property: if $\{a, a', b, b'\}$ is a 4-crown in $\mathcal{P}$, a and a' being the bottom elements, then at least one of the following conditions holds:

1. there is $e \in P$ such that a, a', e, b, b' form a subposet in $\mathcal{P}$ isomorphic to $\mathbf{2}+\mathbf{1}+\mathbf{2}$;
2. $\inf_{\mathcal{P}}(a, a')$ exists;
3. $\sup_{\mathcal{P}}(b, b')$ exists.

Recall that $\inf_{\mathcal{P}}$ and $\sup_{\mathcal{P}}$ denote the greatest common lower bound and the least common upper bound in $\mathcal{P}$, respectively.

Theorem 5 *Let $\mathcal{P}$ be a series-parallel poset. If $\mathcal{P} \in \mathbb{A}$ then* PoRet($\mathcal{P}$) *is in* **SL** *and* CSP($\mathcal{P}_P$) *is in* **NL**. *Otherwise both problems are* **NP**-*complete (via polynomial-time reductions).*

The class $\mathbb{A}$ can be characterized by means of "forbidden retracts".

Lemma 1 *A series-parallel poset $\mathcal{P}$ belongs to $\mathbb{A}$ if and only if it has no retraction onto one of* $\mathbf{2}+\mathbf{2}$, $\mathbf{2}+\mathbf{2}+\mathbf{2}$, $\mathbf{1}+\mathbf{2}+\mathbf{2}+\mathbf{2}$, $\mathbf{2}+\mathbf{2}+\mathbf{2}+\mathbf{1}$, *and* $\mathbf{1}+\mathbf{2}+\mathbf{2}+\mathbf{2}+\mathbf{2}+\mathbf{1}$.

It was proved in [45] that a series-parallel poset $\mathcal{P}$ has an NU polymorphism if and only if it has no retraction onto one of $\mathbf{2}+\mathbf{2}$, $\mathbf{1}+\mathbf{2}+\mathbf{2}$, $\mathbf{2}+\mathbf{2}+\mathbf{1}$, and $\mathbf{1}+\mathbf{2}+\mathbf{2}+\mathbf{2}+\mathbf{1}$. Combining this result with Theorem 5 and Lemma 1 we can get, in particular, the following.

Corollary 2 *If $\mathcal{P}$ is one of* $\mathbf{1}+\mathbf{2}+\mathbf{2}$, $\mathbf{2}+\mathbf{2}+\mathbf{1}$, *and* $\mathbf{1}+\mathbf{2}+\mathbf{2}+\mathbf{2}+\mathbf{1}$ *then* PoRet($\mathcal{P}$) *is in* **SL** *and* CSP($\mathcal{P}_P$) *is in* **NL** *but $\mathcal{P}_P$ has no NU polymorphism.*

It should be noted here that poset $\mathcal{P} = \mathbf{2}+\mathbf{2}+\mathbf{1}$ was shown to be TC-feasible in [37].

For a poset $\mathcal{P}$, denote by $\overline{w}(\mathcal{P})$ and $\underline{w}(\mathcal{P})$ the biggest and the smallest, respectively, elements in $\mathcal{P}$ that are comparable with each element in $\mathcal{P}$.

Lemma 2 *Let $\mathcal{P} \in \mathbb{A}$ be connected. Then $\overline{w}(\mathcal{P})$ and $\underline{w}(\mathcal{P})$ exist.*

Let $A, B \subseteq \mathcal{P}$ be two antichains, and consider formula

$$\Theta_{A,B}(x) = (\bigwedge_{a \in A} x \leq c_a) \wedge (\bigwedge_{b \in B} x \geq c_b).$$

One of the A, B may be empty, and in this case the corresponding part of the formula disappears. For a poset $\mathcal{Q}$ such that $P \subseteq Q$, let $\mathcal{Q}_{\Theta_{A,B}}$ denote the set of all $x \in Q$ such that $\Theta_{A,B}(x)$ is true.

Lemma 3 *Let $\mathcal{P} \in \mathbb{A}$ and $A, B \subseteq P$ be two antichains (one of which may be empty). Then every connected component of the subposet $\mathcal{P}_{\Theta_{A,B}}$ belongs to $\mathbb{A}$.*

Proof of Theorem 5 (sketch). We begin with the **NP**-completeness part. Suppose that $\mathcal{P} \notin \mathbb{A}$. As noted before Corollary 1, the two problems are polynomial-time equivalent, so it suffices to show that CSP($\mathcal{P}_P$) is NP-complete. Since structure $\mathcal{P}_P$ contains all relations of the form $\{p\}$, $p \in P$, it has only idempotent polymorphisms. By Theorem 2, it is sufficient to prove that $\mathcal{P}_P$ has no Taylor polymorphism. Assume, for contradiction, that it has a Taylor polymorphism h. It is straightforward to check that if $r : \mathcal{P} \to \mathcal{Q}$ is a retraction then $(r \circ h)|_{\mathcal{Q}}$ is a Taylor polymorphism of $\mathcal{Q}$. Let $\mathcal{Q}$ be one of the five posets mentioned in Lemma 1. Then $\mathcal{Q}$ has a subposet $\mathcal{Q}'$ (isomorphic to $\mathbf{2}+\mathbf{2}$) witnessing that $\mathcal{Q} \notin \mathbb{A}$. Using the fact that the polymorphisms are monotone operations one can check easily that the restriction of every idempotent polymorphism of $\mathcal{Q}$ to $\mathcal{Q}'$ is a polymorphism of $\mathcal{Q}'$. So $(r \circ h)|_{\mathcal{Q}'}$ is a Taylor polymorphism of $\mathcal{Q}'$. By [6], any idempotent polymorphism of $\mathbf{2}+\mathbf{2}$ is a projection onto some variable. Clearly, a Taylor operation is not a projection onto any variable, a contradiction.

Suppose now that $\mathcal{P} \in \mathbb{A}$. The case of a disconnected $\mathcal{P}$ is treated exactly as in Theorem 4, so it is enough to consider connected posets. From now on, we assume that $\mathcal{P}$ is connected. We will use the same vocabulary τ_1 as in Section 3 (only writing $x \leq y$ instead of $R(x,y)$) and find a τ_1-sentence in $\mathbf{FO}+\mathbf{STC}$ defining $\neg\mathrm{PoRet}(\mathcal{P})$. By applying Theorem 3, this will prove the required result.

Let $\Phi = \exists x(\bigvee \Theta_{A,B}(x))$ where the disjunction is taken over all pairs (A,B) of antichains in $\mathcal{P}$ such that $\mathcal{P} \not\models \exists x \Theta_{A,B}(x)$. For every $H \subseteq P$ such that H is non-empty and disconnected, and is of the form $\mathcal{P}_{\Theta_{A',B'}}$ for some A', B', define predicate ψ_H by the rule

$$\psi_H = \mathrm{STC}_{x,y}[(\bigvee_{\mathcal{P}_{\Theta_{A,B}} \subseteq H} \Theta_{A,B}(x)) \wedge (\bigvee_{\mathcal{P}_{\Theta_{A,B}} \subseteq H} \Theta_{A,B}(y)) \wedge (x \leq y)].$$

Let $\Psi = \bigvee \psi_H(c_{p_1}, c_{p_2})$ where the disjunction is taken over all tuples (H, p_1, p_2) such that H is as above, and p_1 and p_2 belong to different connected components of H. It is easy to see that any retraction from $\mathcal{Q}$ onto $\mathcal{P}$ must map $\mathcal{Q}_{\Theta_{A,B}}$ to $\mathcal{P}_{\Theta_{A,B}}$. If $\mathcal{Q} \models \Phi$ then $\mathcal{Q}$ has an element that cannot be mapped anywhere by a retraction, due to the rule above. So Φ describes a sufficient condition for non-existence of a retraction. Sentence Ψ is another sufficient condition saying that, in $\mathcal{Q}$, there is a path of elements that are forced to be mapped to some disconnected H and that connect two elements from different connected components of H.

Sentence $\Phi \vee \Psi$ belongs to $\mathbf{FO}+\mathbf{STC}$. We shall prove that $\mathcal{Q} \models \Phi \vee \Psi$ if and only if $\mathcal{Q}$ is in $\neg\mathrm{PoRet}(\mathcal{P})$. As explained above, the fact that $\mathcal{Q} \models \Phi \vee \Psi$ implies that $\mathcal{Q}$ is in $\neg\mathrm{PoRet}(\mathcal{P})$. Now fix an arbitrary poset $\mathcal{Q}$ which extends $\mathcal{P}$ and such that $\mathcal{Q} \not\models \Phi \vee \Psi$. We show that there is a retraction $r : \mathcal{Q} \to \mathcal{P}$. In order to define r we need some constructions and notation.

In the following we write $\leq$ to denote $\leq^{\mathcal{Q}}$, this will cause no confusion. Take $q \in Q$, and let L_q be the set of maximal elements of $\{a \in P \mid a \leq q\}$, and U_q the set of minimal elements of $\{b \in P \mid b \geq q\}$. Let $H_q = \mathcal{P}_{\Theta_{L_q,U_q}}$.

It is not hard to show that if $H_q = \mathcal{P}_{\Theta_{L_q,U_q}}$ is not connected then there exists no set of the form $\mathcal{P}_{\Theta_{A,B}}$ such that H_q is properly contained in $\mathcal{P}_{\Theta_{A,B}}$ and the elements from different connected components of H_q are still disconnected in $\mathcal{P}_{\Theta_{A,B}}$. We know by construction of Ψ that there is at most one connected component K_q of H_q such that $\psi_{H_q}(p,q)$ is true for some $p \in K_q$. For every disconnected H_q, let us fix an arbitrary element $z(H_q)$ such that $z(H_q) = \overline{w}(K)$ for some connected component K of H_q. Note that $z(H_q)$ actually depends on the set H_q, not on q. If U_q and L_q are both non-empty, denote by X_q the set of of all $x \in P$ such that x is minimal with the property that

1. $\mathcal{P}_{\Theta_{L_q,U_q \cup \{x\}}}$ is non-empty, and
2. $U_q \cup \{x\}$ is an antichain, and
3. $\sup(x,u)$ does not exist for some $u \in U_q$.

Denote $\mathcal{P}_{\Theta_{L_q,U_q \cup X_q}}$ by H'_q. Throughout the proof we use the following property which can be easily derived from definitions: for every $x \in X_q$, either $\mathcal{P}_{\Theta_{L_q,U_q \cup \{x\}}} = \mathcal{P}_{\Theta_{L_q,U_q}}$ or $\sup(x,u)$ does not exist for all $u \in U_q$. We need the following three claims in order to define the retraction.

Claim 1. If $H_q \neq \emptyset$ then $H'_q \neq \emptyset$.
We show that every minimal element of H_q belongs to H'_q. Suppose, for contradiction,

that there is a minimal element $a \in H_q$ and some $x \in X_q$ such that $a \not\leq x$. Then $\{a, b, x, u\}$ where b is some minimal element of H_q such that $b \leq x$, and u is an arbitrary element in U_q, form an N-poset.

Claim 2. If H_q is not connected then $H_q = H'_q$.
Indeed, both U_q and L_q contain more than one element. Take an arbitrary $x \in X_q$. If x is above one maximal element of $\mathcal{P}_{\Theta_{L_q,U_q}}$ then it is above all elements of $\mathcal{P}_{\Theta_{L_q,U_q}}$ (or else one can easily find an N-poset), and then $\mathcal{P}_{\Theta_{L_q,U_q}} = \mathcal{P}_{\Theta_{L_q,U_q\cup\{x\}}}$. By the choice of x, there is a in $\mathcal{P}_{\Theta_{L_q,U_q\cup\{x\}}}$. Take a maximal element b in $\mathcal{P}_{\Theta_{L_q,U_q}}$ such that $a \not\leq b$, and any element $u \in U_q$. One can check that $\{a, b, x, u\}$ is an N-poset.

Claim 3. If H_q is connected then so is H'_q.
It is shown above that all minimal elements of $\mathcal{P}_{\Theta_{L_q,U_q}}$ belong to $\mathcal{P}_{\Theta_{L_q,U_q\cup X_q}}$. If there is only one minimal element then it ensures connectivity. Otherwise, for every pair a, b of distinct minimal elements, fix their minimal common upper bound (in $\mathcal{P}_{\Theta_{L_q,U_q}}$) $v_{a,b}$. If every $v_{a,b}$ is below all $x \in X_q$ then we get connectivity. Assume, on the contrary, that there exist $v_{a,b} \not\leq x$. Then $\{a, b, v_{a,b}, x\}$ is a 4-crown, and there is no $\inf(a, b)$ (since a, b are minimal in $\mathcal{P}_{\Theta_{L_q,U_q}}$). If there exists $u = \sup(x, v_{a,b})$ then take any maximal element w in $\mathcal{P}_{\Theta_{L_q,U_q}}$ such that $\sup(w, x)$ does not exist and show that $\{x, v_{a,b}, u, w\}$ is an N-poset. Indeed, w and u are incomparable. Obviously, $u \not\leq w$, and we have $w \not\leq u$, since otherwise $u = \sup(w, x)$. Thus, $u = \sup(x, v_{a,b})$ does not exist, and we obtain a contradiction with $\mathcal{P} \in \mathbb{A}$. Claim 3 is proved.

Now we are ready to define the mapping r. Let

$$r(q) = \begin{cases} \overline{w}(\mathcal{P}), & \text{if } L_q = U_q = \emptyset, \\ \overline{w}(H_q), & \text{if } L_q \neq \emptyset,\ U_q = \emptyset, \text{ and } H_q \text{ is connected}, \\ \underline{w}(H_q), & \text{if } L_q = \emptyset,\ U_q \neq \emptyset, \text{ and } H_q \text{ is connected}, \\ \overline{w}(H'_q), & \text{if } L_q \neq \emptyset,\ U_q \neq \emptyset, \text{ and } H_q \text{ is connected}, \\ \overline{w}(K_q), & \text{if } L_q \neq \emptyset,\ H_q \text{ is disconnected, and } K_q \text{ is defined}, \\ \underline{w}(K_q), & \text{if } L_q = \emptyset,\ H_q \text{ is disconnected, and } K_q \text{ is defined}, \\ z(H_q), & \text{if } H_q \text{ is disconnected and } K_q \text{ is not defined}. \end{cases}$$

Using Lemmas 2 and 3 it can be verified that r is well-defined, and that it is identity on P. One can prove by a combinatorial case-by-case analysis that r is indeed a homomorphism. □

In fact, we can say more about the complexity of PoRet($\mathcal{P}$) for the posets mentioned in Theorem 5.

Proposition 1 *If $\mathcal{P} \in \mathbb{A}$ and $\mathcal{P}$ has no NU polymorphisms then* PoRet($\mathcal{P}$) *is* **SL***-complete (under many-one logarithmic space reductions).*

Proof. We know by Theorem 5 that PoRet($\mathcal{P}$) is in **SL**. It remains to prove it is **SL**-hard. We reduce the standard **SL**-complete problem UNDIRECTED NON-s,t-CONNECTIVITY [1, 35] to PoRet($\mathcal{P}$). Recall that an instance of UNDIRECTED NON-s,t-CONNECTIVITY is given by an undirected graph G with two designated vertices s and t in it, and the question is whether s and t are not connected by a path in G.

Suppose that $\mathcal{P}$ is connected, we will discuss the disconnected case later. By the result of Zádori [45] mentioned after Lemma 1, and by Theorem 5 and Lemma 1, $\mathcal{P}$ has a retraction onto one of $\mathbf{1}+\mathbf{2}+\mathbf{2}$, $\mathbf{2}+\mathbf{2}+\mathbf{1}$, and $\mathbf{1}+\mathbf{2}+\mathbf{2}+\mathbf{2}+\mathbf{1}$.

Assume first that $\mathcal{P}$ has a retraction onto $\mathbf{2}+\mathbf{2}+\mathbf{1}$, the case of $\mathbf{1}+\mathbf{2}+\mathbf{2}$ is dual. Let a, a' and b, b' denote the bottom and middle elements in $\mathbf{2}+\mathbf{2}+\mathbf{1}$. Consider the subset X of $\mathcal{P}$ consisting of all elements which are below both b and b'. Clearly a and a' must sit in different components of X since $\mathcal{P}$ retracts onto $\mathbf{2}+\mathbf{2}+\mathbf{1}$ and X must be mapped to $\{a, a'\}$.

Let G be a graph with designated vertices s and t. Construct an instance $\mathcal{Q}$ for PoRet($\mathcal{P}$) as follows. First construct poset $\mathcal{Q}'$, its elements the vertices of G *and* the edges of G; a vertex is smaller than an edge if it is incident with it. Now construct $\mathcal{Q}$ by gluing poset $\mathcal{Q}'$ inside $\mathcal{P}$ in such a way that s and t are identified with a and a', and all elements of $\mathcal{Q}'$ are below both b and b'. *No* other comparabilities are added, except for those implied by transitivity.

We claim that $\mathcal{Q}$ retracts onto $\mathcal{P}$ if and only if there is no path between s and t in G. Clearly if there is a path in G then there is a path in $\mathcal{Q}'$, and hence a path between a and a' lying completely below both b and b', so no retraction is possible. Otherwise, the retraction is clear: simply send all the new vertices either to a or to a' according to which they are connected in $\mathcal{Q}'$.

The case where $\mathcal{P}$ retracts onto $\mathbf{1}+\mathbf{2}+\mathbf{2}+\mathbf{2}+\mathbf{1}$ is similar: denote the antichains by c, c' (bottom), a, a' (middle) and b, b' (top). Let X be the set of all x in $\mathcal{P}$ that are above both c and c' and below both b and b'. Now argue as above.

Clearly, the above transformation can be performed in logarithmic space. Reading an edge from the read-only tape one can write out two new vertex-edge comparabilities and all comparabilities implied by transitivity, that is those involving one new element (corresponding to an edge or a vertex of G) and one element from $\mathcal{P}$.

Finally, if $\mathcal{P}$ is disconnected, then **SL**-hardness follows as above (in fact, no other assumption on $\mathcal{P}$ is necessary). Simply identify s and t with two elements form different components of $\mathcal{P}$, and use the same argument as above. □

6 Conclusion

It is an open problem whether the presence of an NU polymorphism leads to solvabilty of a CSP($\mathcal{B}$) in non-deterministic logarithmic space for general structures. In this paper we solved the problem positively in the case of order constraints and also proved that this algebraic condition is not necessary. Hence, if the general problem above has a positive answer, then the algebraic condition characterizing CSPs in **NL** (if exists) is of a more general form than an NU polymorphism.

Motivated by the problem, Dalmau asked [8] whether structures with an NU polymorphism have bounded path duality. At present, we do not know whether the structures $\mathcal{P}_P$ from the two classes described in this paper have bounded path duality. It follows from results of [9, 28] that if $\mathcal{P}_P$ has an NU polymorphism then $\neg$CSP($\mathcal{P}_P$) is definable in $(1,2)$-Datalog, and the same can be shown for structures $\mathcal{P}_P$ with $\mathcal{P} \in \mathbb{A}$. There may be some way of using special properties of posets for transforming such $(1,2)$-Datalog programs into linear Datalog programs.

It is easy to see that the posets from our two classes are TC-feasible (see definition in Section 3). It was shown in [37] that the class of TC-feasible posets is closed under isomorphism, dual posets, disjoint union, direct products, and retractions. Using these operations one can easily construct TC-feasible posets lying outside the two classes. For example, direct product of $\mathbf{2}+\mathbf{2}+\mathbf{1}$ and its dual poset neither has an NU polymorphism nor is series-parallel. In this way one can find further order constraint satisfaction problems in **NL**. Theorem 1 provides another way of obtaining further CSPs in **NL** because, clearly, the equality relation $=_P$ can be expressed in $\mathcal{P}_P$.

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This appendix contains proofs omitted from the main body of the paper due to space constraints. To be read at the discretion of the referee.

7 Appendix: Proofs

The following remark is straightforward.

Remark 1 *1) If $\mathcal{Q}_i$ retracts onto $\mathcal{P}_i$ $(i = 1, 2)$ then $\mathcal{Q}_1 + \mathcal{Q}_2$ retracts onto $\mathcal{P}_1 + \mathcal{P}_2$.*

2) Every poset retracts onto every $\mathbf{1}$ it contains, and every disconnected poset retracts onto some $\mathbf{2}$.

3) Let $x, y \in \mathcal{P}$ be incomparable. Then $\mathcal{P}$ retracts onto $\mathbf{1} + \{x, y\}$ if and only if x and y have a common upper bound, but no common lower bound in $\mathcal{P}$. The dual property holds as well.

4) If $Q_1 \subseteq Q_2 \subseteq Q_3$, Q_1 has an infimum (supremum) in $\mathcal{Q}_3$, and $\mathcal{Q}_3$ retracts onto $\mathcal{Q}_2$ then Q_1 has an infimum (supremum) in $\mathcal{Q}_2$.

Proof of Lemma 1. By Remark 1, if a poset $\mathcal{P}$ retracts onto one of the posets in the list then, by the remark above, it contains a crown for which the three conditions from the definition of $\mathbb{A}$ fail. Now we prove the converse by induction on $|P|$. If $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ then $\mathcal{P}_1$ or $\mathcal{P}_2$ contains the offending crown so we are done by induction. Otherwise we have that $\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2$.

<u>Claim 1.</u> Let $\mathcal{P}$ be a series-parallel poset. Let x and y be distinct maximal elements of $\mathcal{P}$. Then:

1. $\mathcal{P}$ retracts onto $\{x, y\}$ if and only if $\{x, y\}$ has no lower bound in $\mathcal{P}$;
2. $\mathcal{P}$ retracts onto $\mathbf{1} + \mathbf{2} + \{x, y\}$ or $\mathbf{2} + \{x, y\}$ if and only if $\{x, y\}$ has a lower bound in $\mathcal{P}$ but no greatest upper bound;
3. if $\{x, y\}$ has a lower bound in $\mathcal{P}$ then $\mathcal{P}$ retracts onto $\mathbf{1} + \{x, y\}$.

Parts (1) and (3) are obvious. (2) If $\mathcal{P}$ retracts onto $\mathbf{1} + \mathbf{2} + \{x, y\}$ or $\mathbf{2} + \{x, y\}$ then clearly $\{x, y\}$ has a lower bound in $\mathcal{P}$ and by the remark above it has no infimum.

We prove the converse on induction on $|P|$. Clearly the result holds for small posets. Suppose first that $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$. Then result follows immediately by induction. So suppose that $\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2$. Then $\{x, y\} \subseteq P_2$. If $\{x, y\}$ has a lower bound in $\mathcal{P}$ but not infimum, it has no infimum in $\mathcal{P}_2$. If $\{x, y\}$ has a lower bound in $\mathcal{P}_2$ then either $\mathcal{P}_2$ retracts onto $\mathbf{1} + \mathbf{2} + \{x, y\}$ and so does $\mathcal{P}$, or $\mathcal{P}_2$ retracts onto $\mathbf{2} + \{x, y\}$ and then $\mathcal{P}$ retracts onto $\mathbf{1} + \mathbf{2} + \{x, y\}$. Otherwise $\mathcal{P}_2$ retracts onto $\{x, y\}$, and $\mathcal{P}_1$ has at least 2 maximal elements. If these elements have a common lower bound then $\mathcal{P}_1$ retracts onto $\mathbf{1} + \mathbf{2}$ and hence $\mathcal{P}$ retracts onto $\mathbf{1} + \mathbf{2} + \{x, y\}$; otherwise $\mathcal{P}_1$ retracts onto $\mathbf{2}$ and $\mathcal{P}$ retracts onto $\mathbf{2} + \{x, y\}$. Claim 1 is proved.

Suppose first that $\mathcal{P}_1$ contains an offending crown. Then this crown is offending in $\mathcal{P}_1$ so by induction hypothesis we may assume that $\mathcal{P}_1$ is one of the five posets in the list. Note that if $\mathcal{P}_1$ is not $\mathbf{2} + \mathbf{2}$ nor $\mathbf{1} + \mathbf{2} + \mathbf{2} + \mathbf{2}$ then it is easy to show using the remark above that the result follows. Now suppose that $\mathcal{P}_1 = \mathbf{1} + \mathbf{2} + \mathbf{2} + \mathbf{2}$. Then the offending crown must consist of the top four elements, which means that $\mathcal{P}_2$ must have at least 2 minimal elements. By Claim 1 and its dual $\mathcal{P}_2$ retracts onto $\mathbf{2} + \mathbf{1}$ or $\mathbf{2}$ so $\mathcal{P}$ retracts onto $\mathbf{1} + \mathbf{2} + \mathbf{2} + \mathbf{2} + \mathbf{2}$ (which retracts onto $\mathbf{1} + \mathbf{2} + \mathbf{2} + \mathbf{2}$) or onto $\mathbf{1} + \mathbf{2} + \mathbf{2} + \mathbf{2} + \mathbf{2} + \mathbf{1}$. If $\mathcal{P}_2$ contains an offending crown the proof is dual. So we may now suppose that $\mathcal{P}_1$ has 2 maximal elements with no infimum and $\mathcal{P}_2$ has 2 minimal elements with no supremum. By the Claim and its dual, $\mathcal{P}_1$ retracts onto $\mathbf{2}$ or $\mathbf{1} + \mathbf{2} + \mathbf{2}$, and $\mathcal{P}_2$ retracts onto $\mathbf{2}$ or $\mathbf{2} + \mathbf{2} + \mathbf{1}$. Combining the four cases yields the desired result. $\square$

Proof of Lemma 2. Since $\mathcal{P}$ is series-parallel, it follows that $\mathcal{P}$ is the sum of two posets. In particular, every maximal element in $\mathcal{P}$ is above any minimal one. If $\mathcal{P}$ has the greatest element then it is comparable with all other elements, so $\overline{w}(\mathcal{P})$ and $\underline{w}(\mathcal{P})$ exist.

Suppose $\mathcal{P}$ has at least two maximal elements and let V denote the set of all maximal common lower bounds of the set ,of all maximal elements of $\mathcal{P}$. If $V = \{p\}$ then $p = \overline{w}(\mathcal{P})$. Indeed, if there is $a \in P$ incomparable with p then there exists a maximal element of m such that $a \not\leq m$. Take any maximal element m' such that $a < m'$. Then $\{a, p, m, m'\}$ is an N-poset, a contradiction. Suppose $|V| \geq 2$. If the elements of V have a unique greatest common lower bound, that is $\inf V$, then one can show that it is $\overline{w}(\mathcal{P})$. Assume $\inf V$ does not exist. If $V_1, V_2 \subseteq V$, $V_1 \cap V_2 \neq \emptyset$, and $\inf V_1$ and $\inf V_2$ would both exist then they would be comparable, because $\mathcal{P}$ is N-free. Moreover, we would have $\inf(\inf V_1, \inf V_2) = \inf(V_1 \cup V_2)$. Since $\inf V$ does not exist, one can find $v_i, v_j \in V$ such that $\inf(v_i, v_j)$ does not exist. Consider $W = \{x \in P \mid x \geq v_i, v_j\}$. Note that this set contains all maximal elements of $\mathcal{P}$. The poset W cannot be connected, since otherwise any of its minimal elements would be below all maximal elements of $\mathcal{P}$ which is impossible by the choice of V. Now pick two minimal elements w_k, w_l from different connected components of W. It is easy to see that $\{v_i, v_j, w_k, w_l\}$ form a 4-crown that contradicts the assumption that $\mathcal{P} \in \mathbb{A}$.

Since there exists an element in $\mathcal{P}$ comparable with all other elements, $\underline{w}(P)$ exists as well. $\square$

Lemma 3 follows immediately from defintions, as an offending 4-crown in $\mathcal{P}_{\Theta_{A,B}}$ would be such in $\mathcal{P}$.

Proof of Theorem 5. Suppose now that $\mathcal{P} \in \mathbb{A}$, and $\mathcal{P}$ is connected. We use the same notation as in the main body of the paper. We only need to show that r is a homomorphism. Take $q < q'$ in $\mathcal{Q}$ and prove that $r(q) \leq r(q')$. We distinguish seven cases depending on L_q, U_q, $L_{q'}$, and $U_{q'}$. Throughout the proof we use the following facts without explicit reference: 1) $\mathcal{Q}_{\Theta_{A,B}}$ is non-empty whenever $\mathcal{P}_{\Theta_{A,B}}$ is non-empty (this follows from the fact that $\mathcal{Q} \not\models \Phi$) and 2) if $p_1, p_2 \in P$ and there is a path from p_1 to p_2 in $\mathcal{Q}_{\Theta_{A,B}}$ then there is a path from p_1 to p_2 in $\mathcal{P}_{\Theta_{A,B}}$ (this follows from the fact that $\mathcal{Q} \not\models \Psi$). Moreover, in the second case p_1 and p_2 either are comparable or have a common upper or lower bound, since $\mathcal{P}$ is N-free.

From now on, inf and sup stand for $\inf_{\mathcal{P}}$ and $\sup_{\mathcal{P}}$.

<u>Case 1.</u> $L_q = U_q = \emptyset$ or $L_{q'} = U_{q'} = \emptyset$.
Suppose that $L_q = U_q = \emptyset$, that is $r(q) = \overline{w}(\mathcal{P})$. It follows that $U_{q'} = \emptyset$. Assume that $L_{q'} \neq \emptyset$ (otherwise trivially $r(q) = r(q')$). If $\mathcal{P}_{\Theta_{L_{q'},\emptyset}}$ is not connected then it is easy to see that $r(q)$ is below every element in $\mathcal{P}_{\Theta_{L_{q'},\emptyset}}$. If $\mathcal{P}_{\Theta_{L_{q'},\emptyset}}$ is connected then $r(q) \leq r(q')$ because $r(q)$ is comparable with every element in $\mathcal{P}$ and $r(q) > r(q')$ would contradict the choice of $r(q')$.

<u>Case 2.</u> $L_q = U_{q'} = \emptyset$.
We have $r(q) \leq \underline{w}(\mathcal{P}) \leq \overline{w}(\mathcal{P}) \leq r(q')$ (similarly to Case 1).

<u>Case 3.</u> $L_q \neq \emptyset$, $U_q = \emptyset$.
It follows that $L_{q'} \neq \emptyset$ and $U_{q'} = \emptyset$.

<u>Subcase 3.1.</u> $\mathcal{P}_{\Theta_{L_{q'},\emptyset}}$ is not connected.
We have $\mathcal{P}_{\Theta_{L_{q'},\emptyset}} \subseteq \mathcal{P}_{\Theta_{L_q,\emptyset}}$.

Indeed, since $q' \in \mathcal{Q}_{\Theta_{L_q \cup L_{q'},\emptyset}}$, we know that $\mathcal{P}_{\Theta_{L_q \cup L_{q'},\emptyset}}$ is non-empty. Then there exists a maximal element $u \in \mathcal{P}_{\Theta_{L_{q'},\emptyset}}$ such that $u \in \mathcal{P}_{\Theta_{L_q,\emptyset}}$. Then every maximal element v from any connected component K of $\mathcal{P}_{\Theta_{L_{q'},\emptyset}}$ such that $u \notin K$ must belong to $\mathcal{P}_{\Theta_{L_q,\emptyset}}$, or otherwise one can easily find an N-poset there. Now if there is $x \in \mathcal{P}_{\Theta_{L_{q'},\emptyset}}$ such that

$a \not\leq x$ for some $a \in L_q$ then take any maximal element v from a connected component of $\mathcal{P}_{\Theta_{L_{q'},\emptyset}}$ not containing x, and take any element $b \in L_{q'}$. It is easy to see that $\{a, b, x, v\}$ is an N-poset.

If every element $x \in \mathcal{P}_{\Theta_{L_{q'},\emptyset}}$ is connected with exactly the same elements in $\mathcal{P}_{\Theta_{L_q,\emptyset}}$ as it is in $\mathcal{P}_{\Theta_{L_{q'},\emptyset}}$ then $\mathcal{P}_{\Theta_{L_q,\emptyset}} = \mathcal{P}_{\Theta_{L_{q'},\emptyset}}$. Indeed, otherwise one can choose $a < u$ such that $a \notin \mathcal{P}_{\Theta_{L_{q'},\emptyset}}$ but $u \in \mathcal{P}_{\Theta_{L_{q'},\emptyset}}$. Take any element v from a connected component of $\mathcal{P}_{\Theta_{L_{q'},\emptyset}}$ not containing u and any element $b \in L_{q'}$ such that $b \not\leq a$. Then $\{a, b, u, v\}$ form an N-poset, a contradiction. So, in this case $r(q) = r(q')$ because q and q' are connected in $\mathcal{Q}_{\Theta_{L_{q'},\emptyset}} = \mathcal{Q}_{\Theta_{L_q,\emptyset}}$.

The argument from the previous paragraph also shows that if $\mathcal{P}_{\Theta_{L_q,\emptyset}}$ strictly contains $\mathcal{P}_{\Theta_{L_{q'},\emptyset}}$ then there exist a and u as described above and, moreover, every such a must lie below each element from $\mathcal{P}_{\Theta_{L_{q'},\emptyset}}$ (or else we can find an N-poset). So we now assume that $\mathcal{P}_{\Theta_{L_{q'},\emptyset}}$ is contained in one connected component of $\mathcal{P}_{\Theta_{L_q,\emptyset}}$. Now it is easy that if $\mathcal{P}_{\Theta_{L_q,\emptyset}}$ is connected then $r(q) \leq r(q')$. If $\mathcal{P}_{\Theta_{L_q,\emptyset}}$ has at least two components then every element of $L_{q'}$ is above every element of L_q, or else there is an N-poset (two incomparable elements from $L_{q'}$ and L_q, and two maximal elements u, v such that $u \in \mathcal{P}_{\Theta_{L_{q'},\emptyset}}$ and v is from any other component of $\mathcal{P}_{\Theta_{L_q,\emptyset}}$). Therefore, $L_{q'}$, $\mathcal{P}_{\Theta_{L_{q'},\emptyset}}$, $r(q)$ and $r(q')$ are all in the same connected component K of $\mathcal{P}_{\Theta_{L_q,\emptyset}}$. Since $r(q)$ is then below all elements of $\mathcal{P}_{\Theta_{L_{q'},\emptyset}}$, we have $r(q) \leq r(q')$.

<u>Subcase 3.2.</u> $\mathcal{P}_{\Theta_{L_{q'},\emptyset}}$ is connected.
Every maximal element of $\mathcal{P}_{\Theta_{L_{q'},\emptyset}}$ is a maximal element of $\mathcal{P}_{\Theta_{L_q,\emptyset}}$ or otherwise one can easily find an N-poset.

If $\mathcal{P}_{\Theta_{L_q,\emptyset}}$ is connected then $r(q) \leq r(q')$ because of the above inclusion. If $\mathcal{P}_{\Theta_{L_q,\emptyset}}$ has at least two components then the argument is the same as in the previous subcase.

<u>**Case 4.**</u> $U_{q'} \neq \emptyset$, $L_{q'} = \emptyset$.
Dual to Case 3.

From now on, we assume that $U_q \neq \emptyset$ and $L_{q'} \neq \emptyset$, and at least one of L_q, $U_{q'}$ is non-empty.

Obviously, if $u \leq l$ for some $u \in U_q$ and $l \in L_{q'}$ then $r(q) \leq r(q')$. Assume that $l_1 \not< u_1$, $l_1 < u_2$ for some $l_1 \in L_{q'}$ and $u_1, u_2 \in U_q$. Then $l_1 \not< r(q)$. If $r(q)$ is incomparable with l_1 then $\{r(q), l_1, u_1, u_2\}$ is an N-poset, a contradiction. So $r(q) \leq l_1 \leq r(q')$. The argument is similar if there are $l_1, l_2 \in L_{q'}$ and $u \in U_q$ such that $l_1 < u$ and $l_2 \not< u$.

In what follows we will always assume that either $l < u$ for all $l \in L_{q'}$, $u \in U_q$, or all such l and u are incomparable.

<u>**Case 5.**</u> $L_q = \emptyset$ and $U_{q'} \neq \emptyset$.

<u>Subcase 5.1</u> H_q is disconnected, and $l < u$ for all $l \in L_{q'}$, $u \in U_q$.
Suppose there are $l'_1, l'_2 \in L_{q'}$ belonging to two different connected components of H_q. If $r(q') \geq u$ for some element in $u \in U_q$ then $r(q) \leq r(q')$. If not then, since H_q is disconnected, there is $u \in U_q$ such that $r(q')$ and u are incomparable. Let l'_1 not belong to the connected component of H_q containing $r(q)$. Then $r(q) \leq r(q')$, since otherwise $\{r(q), l'_1, u, r(q')\}$ is an N-poset.

So let $L_{q'}$ be contained in one connected component of H_q. Then, for every $a \in H_{q'}$, either $a \in H_q$, or $a > b$ for every $b \in H_q$, since otherwise $\{u, a, l, x\}$ is an N-poset, where $l \in L_{q'}$ is such that $l < a$, $u \in U_q$ is such that u and a are incomparable, and x is a suitable maximal element of H_q.

Assume first that $H'_{q'} \subseteq H_q$. Then, for any $l \in L_{q'}$, $\psi_{H_q}(q, l)$ is true, and therefore both $r(q)$ and $r(q')$ belong to the same connected component K_q of H_q, and $r(q) = \underline{w}(K_q)$. Then $r(q)$ and $r(q')$ are comparable. Moreover, by the definition of U_q, $r(q) \in H_{q'}$. If

$r(q) \in H'_{q'}$ then $r(q) \leq r(q')$, as $r(q)$ and $r(q')$ are both comparable with all elements of $H'_{q'}$. Assume that $r(q) > r(q')$. Then $r(q) \not< x$ for some $x \in X_{q'}$. Let $A = \{a \mid a < r(q)$ and $r(q') \not\leq a\}$. This set is non-empty, since otherwise $r(q)$ could not be above $r(q')$. Let A' be the set of all maximal elements of A. If some $a \in A'$ satisfies $a \not\leq x$ for some $x \in X_q$ then $\{a, r(q'), r(q), x\}$ is an N-poset, a contradiction. If there exists $\inf(r(q'), a)$ for all $a \in A'$ then there exists $c = \inf(A' \cup \{r(q')\})$, and c would contradict the choice of $r(q)$ because it is easy to show that c is comparable with all elements in H_q. So $\inf(r(q'), a)$ does not exist for some $a \in A'$. Then $\{a, r(q'), r(q), x\}$ is a crown contradicting $\mathcal{P} \in \mathbb{A}$ because the existence of $\sup(r(q), x)$ would imply the existence of $\sup(x, u)$ for every $u \in U_q$ which, in turn, would contradict the definition of $x \in X_q$. So if $H_{q'} \subseteq H_q$ then $r(q) \leq r(q')$.

Assume now that there is $a \in H'_{q'}$ such that $a > b$ for every $b \in H_q$. (Note that then $H_{q'}$ is connected or every $a \in H_{q'}$ is above every $b \in H_q$.) Consider minimal such elements. If there is only one such element a then $r(q) \leq a \leq r(q')$. If there are more than two such elements $a_1, \ldots, a_n$ then every pair (a_i, a_j) must have a supremum (or else $\{a_i, a_j, y, z\}$, where y, z are from different connected components of H_q, forms a crown that contradicts $\mathcal{P} \in \mathbb{A}$). Then there exists $c = \sup(\{a_1, \ldots, a_n\})$. It is easy to check that c is comparable with all elements in $H'_{q'}$. Hence $r(q) \leq c \leq r(q')$.

<u>Subcase 5.2</u> H_q is disconnected, and l and u are incomparable for all $l \in L_{q'}$, $u \in U_q$. If there are $l \in L_{q'}$ and $a \in H_q$ such that $l > a$ then $l > b$ for every b from any connected component of H_q not containing a, or otherwise $\{u, l, a, b\}$ is an N-poset for any $u \in U_q$. So in this case $r(q) < l \leq r(q')$. We may assume for the rest of this subcase that $a \not\leq l$ for all $l \in L_{q'}$ and $a \in H_q$.

One can easily show that $u \leq u'$ for every $u \in U_q$ and $u' \in U_{q'} \cup X_{q'}$ (or else one can find an N-poset).

For every $l \in L_{q'}$ there exists $b_l \in H_q$ such that $l < b_l$ and $u < b_l$ for some $u \in U_q$, since there is a path from l to u in $\mathcal{Q}_{\Theta_{\emptyset, U_{q'}}}$. Then $b_l > u$ for each $u \in U_q$, or otherwise one can easily find an N-poset.

Take arbitrary $u_1, u_2 \in U_q$. Since $\mathcal{P} \in \mathbb{A}$, there exists $u = \sup(u_1, u_2)$. We know that $u \leq u'$ for every $u \in U_q$ and $u' \in U_{q'} \cup X_{q'}$. If $u > l$ for some $l \in L_{q'}$ then one can take $b_l = u$, and, in fact $u = \sup U_q$. We show that u is comparable with all elements of $H'_{q'}$. Indeed, if u is incomparable with some $a \in H'_{q'}$ then, for each $u_i \in U_q$ either $\{u, a, u_i, l\}$ is an N-poset or $a > u_i$. In the latter case, a must be comparable with $u = \sup U_q$. Now we have $r(q) < u \leq r(q')$ regardless whether $u \in H'_{q'}$ or not.

We may assume now that $l \not\leq u$ for each $l \in L_{q'}$.

Assume that $H'_{q'}$ is disconnected. If there is $x \in H_{q'}$ such that $u \not\leq x$ then there exists $c \in H_{q'}$ such that $c > x$ and $c > u$, and then one can always find a maximal element $y \in H_{q'}$ such that $\{x, u, c, y\}$ is an N-poset. If $u \leq x$ for all $x \in H_{q'}$ then we have $r(q) \leq r(q')$.

Assume that $H'_{q'}$ is connected. We know that $r(q') \not\leq u$. Assume that $r(q')$ and u are incomparable and derive a contradiction. Since $r(q')$ and u are connected by a path in $\mathcal{Q}_{\Theta_{\emptyset, U_{q'}}}$, they have a common bound in $\mathcal{P}_{\Theta_{\emptyset, U_{q'}}}$. It is easy to show that if they have a common lower bound then one can prove as in the first paragraph of this subcase that $r(q) \leq r(q')$. So assume that $r(q')$ and u have no common lower bound, and there is $b \in H_{q'}$ such that $u < b$ and $r(q') < b$. Let B be the set of all minimal b with this property. It is not hard to verify that there cannot be only one such minimal element. Any $b \in B$ must belong to $H'_{q'}$. Indeed, if $b \not\leq x$ for some $x \in X_{q'}$ then $\{b, x, u, r(q')\}$ form a crown contradicting the assumption that $\mathcal{P} \in \mathbb{A}$. If $\sup(b_i, b_j)$ exists for any pair $b_i, b_j \in B$ then there exists $\sup B$. It is easy to check this element belongs to $H'_{q'}$, and it is comparable with every element of this set. This is a contradiction with the choice of $r(q')$. So $\sup(b_i, b_j)$ does not exist for some $b_i, b_j \in B$. Then $\{u, r(q'), b_i, b_j\}$ is a crown contradicting the assumption that $\mathcal{P} \in \mathbb{A}$. Thus we have $r(q) \leq u \leq r(q')$.

Subcase 5.3 H_q is connected, and $l < u$ for all $l \in L_{q'}$, $u \in U_q$.
In this case $r(q) = \underline{w}(H_q)$. We assume that $r(q) \in H_{q'}$, since otherwise $r(q) \le l \le r(q')$ for some $l \in L_{q'}$.

Assume that $H'_{q'}$ is connected, and $r(q)$ is incomparable with some $x \in X_{q'}$, that is, $r(q) \notin H'_{q'}$. Note that $\sup(x, r(q))$ does not exist. If $r(q)$ and $r(q')$ are incomparable then, for any $u' \in U_{q'}$, $\{r(q), r(q'), u', x\}$ is an N-poset. Assume, for contradiction that $r(q) > r(q')$. Let A be the set of all a minimal with the following property: $a < r(q)$ and $a \not> r(q')$. If $\inf(a, r(q'))$ exists for each $a \in A$ then there exists $\sup(A \cup \{r(q')\}$, and this element is comparable with all elements in H_q, a contradiction with the choice of $r(q)$. So $\inf(a(r(q'))$ does not exist for some $a \in A$. Then $\{a, r(q'), r(q), x\}$ is a crown contradicting $\mathcal{P} \in \mathbb{A}$. Consequently, if $H'_{q'}$ is connected then $r(q) \in H'_{q'}$. Then, as $r(q') = \overline{w}(H'_{q'})$, $r(q)$ and $r(q')$ are comparable. Assume, for contradiction, that $r(q) > r(q')$. Let B be the set of all elements b minimal with the following property: $b < r(q)$ and $b \not> r(q')$. Simlarly, let C be the set of all elements $c \in H'_{q'}$ minimal with the following property: $c > r(q')$ and $c \not< r(q)$. By definitions of $r(q)$ and $r(q')$, both B and C are non-empty. Arguing as above, one can show that there are $b \in B$ and $c \in C$ such that neither $\inf(b, r(q'))$ nor $\sup(c, r(q))$ exist. Then $\{b, r(q'), c, r(q)\}$ is a crown contradicting that $\mathcal{P} \in \mathbb{A}$.

Assume now that $H_{q'}$ is disconnected. Note that $L_{q'}$ has at least two elements. Let b be a minimal element from the connected component of $H_{q'}$ containing $r(q)$ and a a minimal element from any other connected component. In H_q, consider a maximal (with respect to inclusion) antichain D such that $L_{q'} \subseteq D$. It is easy to show that, for every $d \in D$, both $d < a$ and $d < b$ holds, or else $\mathcal{P} \notin \mathbb{A}$. The standard argument shows that there must be $d_1, d_2 \in D$ such that $\inf(d_1, d_2)$ does not exist. By the choice of a and b, $\sup(a, b)$ does not exist either. Then $\{a, b, d_1, d_2\}$ is a crown contradicting that $\mathcal{P} \in \mathbb{A}$.

Subcase 5.4 H_q is connected, and l and u are incomparable for all $l \in L_{q'}$, $u \in U_q$.
Assume that $H_{q'}$ is disconnected. If $r(q) \le l$ for some $l \in L_{q'}$ then $r(q) \le r(q')$. Assume now that $r(q)$ is incomparable with each $l \in L_{q'}$. Since, in $\mathcal{Q}_{\Theta_{\emptyset, U_{q'}}}$, there is a path from $r(q)$ to any such l, we see that $r(q)$ and l have a common bound in $\mathcal{P}_{\Theta_{\emptyset, U_{q'}}}$. If, for some $l \in L_{q'}$, there is b such that $b > l$ and $b > r(q)$ then using disconnectedness of $H_{q'}$ it is easy to prove that $r(q) < a$ for every $a \in H_{q'}$, in particular, $r(q) < r(q')$. Assume now that, for each $l \in L_{q'}$, $r(q)$ and l have only lower bounds in common. It follows, in particular, that $\sup(l, r(q))$ exist for no $l \in L_{q'}$. Fix an arbitrary $l \in L_{q'}$ and consider the set A consisting of all maximal common lower bounds of $r(q)$ and l. Extend A to a maximal antichain D in H_q. Arguing as in the previous subcase, one can derive a contradiction with $\mathcal{P} \in \mathbb{A}$.

Assume now that $H'_{q'}$ is connected. We know that $r(q') \not\le r(q)$. Assume that these elements are incomparable and derive a contradiction. If $r(q) \not< x$ for some $x \in X_{q'}$ then $\{r(q), r(q'), x, u'\}$ is an N-poset for every $u' \in U_{q'}$. So we have $r(q) \in \mathcal{P}_{\Theta_{\emptyset, U_{q'} \cup X_{q'}}}$. Arguing as above, one can prove that if $\inf(r(q), r(q'))$ exists then it must be $r(q)$. Similarly, if $\sup(r(q), r(q'))$ exists then it must be $r(q')$. So by our assumption neither $\inf(r(q), r(q'))$ nor $\sup(r(q), r(q'))$ exists. We know that $r(q)$ and $r(q')$ have a common upper or lower bound in $\mathcal{P}_{\Theta_{\emptyset, U_{q'}}}$, since they are connected by a path in $\mathcal{Q}_{\Theta_{\emptyset, U_{q'}}}$. Assume that none of the common bounds belongs to $\mathcal{P}_{\Theta_{\emptyset, U_{q'} \cup X_{q'}}}$. Then all their common bounds are upper. Take a minimal upper bound $c \in \mathcal{P}_{\Theta_{\emptyset, U_{q'}}}$. There is $x \in X_{q'}$ such that c and x are incomparable. Then, by construction of $X_{q'}$, $\sup(c, x)$ does not exist, and $\{r(q), r(q'), x, c\}$ form a crown contradicting $\mathcal{P} \in \mathbb{A}$. So $r(q)$ and $r(q')$ do have common bounds in $\mathcal{P}_{\Theta_{\emptyset, U_{q'} \cup X_{q'}}}$. Now depending on whether they have lower or upper bounds, one can find, using the "antichain" method above, a crown with the top or the bottom elements $r(q)$ and $r(q')$ such that this crown contradicts the assumption $\mathcal{P} \in \mathbb{A}$.

Case 6. $L_q \ne \emptyset$ and $U_{q'} = \emptyset$.

Subcase 6.1 $H_{q'}$ is disconnected, and $l < u$ for all $l \in L_{q'}$, $u \in U_q$.
The argument dual to the one from the first two paragraphs of Subcase 5.1 shows that it

is enough to consider the case when U_q is contained in one connected component of $H_{q'}$, and, for every $a \in H_q$, either $a \in H_{q'}$ or $a < b$ for every $b \in H_{q'}$.

Assume first that $H'_q \subseteq H_{q'}$. Then, for any $u \in U_q$, $\psi_{H_{q'}}(u, q')$ is true, and therefore $r(q)$ and $r(q')$ are in the same connected component $K_{q'}$ of $H_{q'}$, and $r(q') = \overline{w}(K_{q'})$. Then $r(q)$ and $r(q')$ are comparable. If $r(q') \not\leq u$ for each $u \in U_q$ then, since $r(q') = \overline{w}(K_{q'})$, we have $r(q') > u$ for some $u \in U_q$, and $r(q) < r(q')$. So let $r(q') \in H_q$. Then H_q is connected and $r(q) = \overline{w}(H'_q)$. Assume, for contradiction, that $r(q) > r(q')$. Let A be a maximal (with respect to inclusion) antichain in $K_{q'}$ such that $U_q \subseteq A$, and let $B = \{b \in A \mid r(q) \leq b\}$. If $A = B$ then it is easy to show that $r(q)$ is comaparable with each element in $K_{q'}$, a contradiction with the choice of $r(q')$. Assume that $A \neq B$. If, for every $x \in A \setminus B$ and every $u \in U_q$, there exists $\sup(x, u)$ then there exists $c = \sup(U_q \cup (A \setminus B))$. If $b \leq c$ for every $b \in B$ then it is easy to show that c is comparable with each element in $K_{q'}$, a contradiction with the choice of $r(q')$. If $b \not\leq c$ for some $b \in B$ then, for any $x \in A \setminus B$, $\{x, r(q), b, c\}$ is an N-poset. Hence there are $x \in A \setminus B$ and $u \in U_q$ such that $\sup(x, u)$ does not exist. Then $x \in X_q$. Since $r(q) \not\leq x$, we obtain a contradiction with the choice of $r(q)$. Thus if $H'_q \subseteq H_{q'}$ then $r(q) \leq r(q')$.

Assume now that there is $a \in H'_q$ such that $b < a$ for every every $b \in H_{q'}$. If all elements in H'_q are such then, obviously, $r(q) < r(q')$. If not then H'_q is connected and then $r(q) = \overline{w}(H'_q)$. Note that there exist $l \in L_q$ and $l' \in L_{q'}$ such that $l < l'$. Take any element x from a connected component of $H_{q'}$ not containing U_q. The element l ensures that $x \in X_q$. Thus we have $r(q) \leq x$, and $r(q)$ must be below all elements in $H_{q'}$, in particular, $r(q) < r(q')$.

<u>Subcase 6.2</u> $H_{q'}$ is disconnected, and l and u are incomparable for all $l \in L_{q'}$, $u \in U_q$. Take arbitrary l and u. Since l and u are connected by a path in $\mathcal{Q}_{\Theta_{L_q,\emptyset}}$, they have a common bound in $\mathcal{P}_{\Theta_{L_q,\emptyset}}$. Assume that some l and u have a common upper bound $b \in \mathcal{P}_{\Theta_{L_q,\emptyset}}$. Let $l' \not< b$ for some $l' \in L_{q'}$. Then either $b < a$ for every $a \in H_{q'}$ (and then $r(q) \leq r(q')$) or there is a from a suitable connected component of $H_{q'}$ such that $\{l, l', a, b\}$ is an N-poset, a contradiction. Assume now that $l' < b$ for all $l' \in L_{q'}$. Again, either $u < a$ for every $a \in H_{q'}$ (and then $r(q) \leq r(q')$) or one can find an N-poset in $\mathcal{P}$, a contradiction.

Assume for the rest of this subcase that no $l \in L_{q'}$ and $u \in U_q$ have a common upper bound. Fix $l \in L_{q'}$. For any $u \in U_q$, there is a common lower bound $d_u \in \mathcal{P}_{\Theta_{L_q,\emptyset}}$ of l and u. Choose d_u to be minimal in $\mathcal{P}_{\Theta_{L_q,\emptyset}}$. If $d_u \not< u_1$ for some $u_1 \in U_q$ then $\{d_{u_1}, d_u, u_1, l\}$ is an N-poset. Hence $d_u \in H_q$, and, moreover, $\mathcal{P}_{\Theta_{L_q,H_q\cup\{l\}}}$ is non-empty. If H_q is connected then, by the definition of X_q, there is $x \in X_q$ such that $x \leq l$, and then $r(q) \leq x \leq l \leq r(q')$. If H_q is disconnected then $a < l$ for every $a \in H_q$, and then $r(q) < l < r(q')$, or else one can choose $a \in H_q$ so that $\{a, d_u, u, l\}$ is an N-poset.

<u>Subcase 6.3</u> $H_{q'}$ is connected, and $l < u$ for all $l \in L_{q'}$, $u \in U_q$.
In this case $r(q') = \overline{w}(H_{q'})$. We assume that $r(q') \in H_q$, since otherwise $r(q) \leq u \leq r(q')$ for some $u \in U_q$.

Let H_q be connected. If $r(q)$ and $r(q')$ are incomparable then there is $x \in X_q$ such that $r(q') \not< x$, and $\{r(q), r(q'), x, u\}$ is an N-poset for any $u \in U_q$. So $r(q)$ and $r(q')$ are comparable. Assume, for contradiction, that $r(q) > r(q')$. The argument similar to the one from the second paragraph of the previous subcase shows that this leads to a contradiction with the choice of $r(q')$

Let H_q be disconnected. It is easy to see that $r(q)$ and $r(q')$ belong to the same connected component of H_q. Then the proof is similar to (the dual of) the last paragraph of subcase 5.3.

<u>Subcase 6.4</u> $H_{q'}$ is connected, and l and u are incomparable for all $l \in L_{q'}$, $u \in U_q$.
We have $r(q') = \overline{w}(H_{q'})$, and we may assume that $r(q')$ is in comparable with u for each $u \in U_q$, since otherwise we immediately have $r(q) < r(q')$.

Take $l \in L_{q'}$ and $u \in U_q$. Since u and l are connected by a path in $\mathcal{Q}_{\Theta_{L_q,\emptyset}}$, they have a common bound in $\mathcal{P}_{\Theta_{L_q,\emptyset}}$.

Assume they have a common upper bound b. Then either $a < l$ for every $a \in H_q$, and then $r(q) < r(q')$, or $u_1 < b$ for every $u_1 \in U_q$. Similarly, either $u < c$ for every $c \in H_{q'}$, and then $r(q) < r(q')$, or $l_1 < b$ for every $l_1 \in L_{q'}$. Assume that b is above each element in $U_q \cup L_{q'}$. Then b is comparable with $r(q')$. If $b \leq r(q')$ then, of course, $r(q) < r(q')$. So let $r(q') < b$ for every such b. Let B be the set of all minimal elements of $\{b \mid r(q') < b \text{ and } u < b \text{ for all } u \in U_q\}$. It is easy to check that every a such that $r(q') \leq a$ is comparable with some element of B. If there exist $\sup(b_1, b_2)$ for all b_1, b_2 in B then there exists $\sup B$, and this element contradicts the choice of $r(q')$. So $\sup(b_1, b_2)$ does not exist for some b_1, b_2 in B. Since $\mathcal{P} \in \mathbb{A}$, there is $\inf(r(q'), u)$ for every $u \in U_q$. As in subcase 6.2, we can choose $d_u \leq \inf(r(q'), u)$ so that $d_u \in H_q$ and then derive that $r(q) < r(q')$.

Case 7. $L_q \neq \emptyset$ and $U_{q'} \neq \emptyset$.

Subcase 7.1 The inequality $l < u$ holds for all $l \in L_{q'}$, $u \in U_q$.
Assume, for contradiction, that $r(q)$ and $r(q')$ are incomparable. Then, since $\mathcal{P}$ is N-free, for every $u \in U_q$ and $l' \in L_{q'}$, we have $r(q) > l'$ or $r(q') < u$. For the same reason, if $r(q) > l'$ for some $l' \in L_{q'}$ then $r(q) > l'$ for each $l' \in L_{q'}$, and if $r(q') < u$ for some $u \in U_q$ then $r(q') < u$ for each $u \in U_q$. Hence, $r(q) \in H_{q'}$ or $r(q') \in H_q$, or both. It is easy to verify that $r(q) \in H_{q'}$ implies $r(q) \in H'_{q'}$ and $r(q') \in H_q$ implies $r(q') \in H'_q$.

If $u \leq r(q')$ then, obviously, $r(q) \leq r(q')$. Assume that $r(q')$ is incomparable with some $u \in U_q$. Then $r(q')$ is incomparable with each $u \in U_q$ and we have $r(q) \in H'_{q'}$. Since $\mathcal{P}$ is N-free, it follows that $u < u'$ for each $u \in U_q$, $u' \in U_{q'}$, that is $U_q \subseteq H_{q'}$, and, moreover, $U_q \subseteq H'_{q'}$. Since, by definition, $r(q')$ is comparable with all elements from its connected component of $H'_{q'}$, we conclude that $H_{q'}$ is disconnected, and the elements $r(q)$ and $r(q')$ belong to different connected components of $H'_{q'}$. Moreover, $\sup(u, r(q'))$ does not exist for any $u \in U_q$. There exists $a \in H_q$ such that $a \notin H_{q'}$, since otherwise, by definition, $r(q)$ and $r(q')$ would belong to the same connected component of $H_{q'}$. Then $l' \not\leq a$ for some $l' \in L_{q'}$. If $a \leq l'$ then $a \in \mathcal{P}_{\Theta_{L_q, U_q \cup \{r(q')\}}}$ and $r(q) \leq x \leq r(q')$ for some $x \in X_q$, a contradiction with our assumption that $r(q)$ and $r(q')$ are incomparable. If a and l' incomparable then it is easy to show that $l' \in \mathcal{P}_{\Theta_{L_q, U_q \cup \{r(q')\}}}$, so $r(q) \leq r(q')$, a contradiction again. We conclude that $r(q')$ cannot be incomparable with each $u \in U_q$, and so $r(q') \in H'_q$. Similarly, one can show that $r(q) \in H'_{q'}$.

Then both H_q and $H_{q'}$ are disconnected, and $r(q)$ and $r(q')$ belong to different connected components of both H_q and $H_{q'}$, since otherwise $r(q)$ and $r(q')$ would be comparable. It follows that $\mathcal{P}_{\Theta_{L_{q'}, U_q}}$ is non-empty and disconnected, and, of course, $\mathcal{P}_{\Theta_{L_{q'}, U_q}} \subseteq H_q \cap H_{q'}$. As we have already noticed when defining H_q, such an inclusion implies $\mathcal{P}_{\Theta_{L_{q'}, U_q}} = H_q = H_{q'}$. But then, by definition, $r(q)$ and $r(q')$ would belong to the same connected component of this set, and so would be comparable. This is a contradiction with our assumption that $r(q)$ and $r(q')$ are incomparable.

Assume now, for contradiction, that $r(q) > r(q')$. In this case, we again have $r(q') \in H_q$ and $r(q) \in H_{q'}$. If is easy to show that if $x' \in X_{q'}$ then either $x' \in X_q$ or $u < x'$ for some $u \in U_q$. In both cases $r(q)$ must be below x', and so $r(q) \in H'_{q'}$. We have $r(q') \in H'_q$, as $r(q') < r(q)$. Now choose A and B as in the second paragraph of subcase 6.1, and repeat the argument from that paragraph to derive a contradiction.

Subcase 7.2 $H_{q'}$ is disconnected, and l and u are incomparable for all $l \in L_{q'}$, $u \in U_q$. Note that $l < l'$ for every $l \in L_q$, $l' \in L_{q'}$, and $u < u'$ for every $u \in U_q$, $u' \in U_{q'}$ because the poset P is N-free.

Take arbitrary l and u. Since l and u are connected by a path in $\mathcal{Q}_{\Theta_{L_q, H_{q'}}}$, they have a common bound in $\mathcal{P}_{\Theta_{L_q, H_{q'}}}$. The rest of the proof for this subcase is identical to subcase 6.2.

Subcase 7.3 $H_{q'}$ is disconnected, and l and u are incomparable for all $l \in L_{q'}$, $u \in U_q$. Similar to subcase 6.4.

Theorem 5 is proved. $\square$