Reasoning About Temporal Relations: The Tractable Subalgebras of Allen's Interval Algebra

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Allen's interval algebra is one of the best established formalisms for temporal reasoning. This paper is the final step in the classification of complexity in this algebra. Reasoning in the full Allen's algebra is known to be NP-complete, but eighteen tractable subalgebras have so far been identified; we show here that these are the only possible tractable subalgebras. In other words, we show that Allen's algebra contains exactly eighteen maximal tractable subalgebras, and reasoning in any fragment not entirely contained in one of these subalgebras is NP-complete. We obtain this result by giving a new uniform description of the known maximal tractable subalgebras and then systematically using an algebraic technique for identifying maximal subalgebras with a given property.

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1. INTRODUCTION

Reasoning about temporal constraints is an important task in many areas of computer science and elsewhere, including scheduling [Papadimitriou and Yannakakis 1979], natural language processing [Song and Cohen 1988], planning [Allen 1991], database theory [Koubarakis 1997], technical diagnosis [Nökel 1991], circuit design [Ward and Halstead 1990], archaeology [Kendall 1969; Golumbic and Shamir 1993], and behavioral psychology [Coombs and Smith 1973]; similar problems have also been studied in genetics [Benzer 1959]. Several frameworks for formalizing this type of problem have been suggested (see [Schwalb and Vila 1998] for a survey); for instance, the *point algebra* [van Beek and Cohen 1990] (for expressing relations between time points), the *point-interval algebra* [Vilain 1982] (for expressing relations between time points and intervals) and the famous *Allen's interval algebra* [Allen 1983] for expressing qualitative relations between time intervals.

Reasoning within certain restricted fragments of Allen's algebra (with some additional restrictions on the overall structure of instances) is equivalent to some well-known problems such as the *interval graph* recognition problem and the *interval order* recognition problem (see [Pe'er and Shamir 1997]) which play an important role in molecular biology [Golumbic et al. 1994; Karp 1993], namely in the construction of a physical mapping of DNA.

Allen's algebra has also become the kernel of some other formalisms, where it is extended with different types of metric or qualitative constraints [Angelsmark and Jonsson 2000; Anger and Rodriguez 1994; Drakengren and Jonsson 1997a; Krokhin et al. 2002; Krokhin and Jonsson 2002; Meiri 1996]. Finally, we mention that Allen's algebra and some of its extensions are closely related to a number of interval-based temporal logics used for real-time system specification (see [Bellini et al. 2000]).

Throughout this paper we shall assume that $\mathrm{P} \neq \mathrm{NP}$. The basic satisfiability problem in Allen's algebra is NP-complete [Vilain et al. 1989], so it is unlikely that efficient algorithms exist for reasoning in the full algebra. This computational difficulty has motivated the study of algorithms and complexity in *fragments* of the algebra, e.g., [Belfer and Golumbic 1990; Drakengren and Jonsson 1997a; 1997b; 1998; Gerevini and Schubert 1993; Golumbic and Shamir 1993; Ligozat 1996; 1998; Meiri 1996; Nebel and Bürckert 1995; Pe'er and Shamir 1997; van Beek 1992; van Beek and Cohen 1990; Vilain et al. 1989][1], and the subsequent search for effective heuristics based on *tractable* fragments, e.g. [Ladkin and Reinefeld 1997; Nebel 1997b; van Beek and Manchak 1996]. Nebel and Bürckert [1995] presented the 'ORD-Horn' algebra, the first example of a maximal tractable subclass of Allen's algebra. Since then, research in this direction has focused on identifying *maximal* tractable fragments, i.e., fragments which cannot be extended without losing tractability. Eighteen maximal tractable fragments of the algebra have so far been identified [Drakengren and Jonsson 1997a; 1997b; Ligozat 1998; Nebel and Bürckert 1995].

In this paper we complete the analysis of complexity within Allen's algebra by showing that these eighteen are the *only* forms of tractability within Allen's algebra.

[1] In [Belfer and Golumbic 1990; Golumbic and Shamir 1993; Pe'er and Shamir 1997], some additional restriction on the overall structure of problems is assumed.

Our main result (Theorem 2.3) states that Allen's algebra contains exactly eighteen maximal tractable subalgebras and that reasoning within any subset not included in one of these is NP-complete.

In complexity theory, it is well known that if P $\neq$ NP then there exist infinitely many complexity classes between P and NP. In view of this, there has been considerable interest in so-called *dichotomy* theorems, which state that for one or another important NP-complete problem, all of the natural subproblems are either tractable or NP-complete (see, e.g., [Bulatov 2002; Creignou et al. 2001; Feder and Vardi 1998; Hell and Nešetřil 1990; Schaefer 1978] [2]). The main result obtained in this paper can also be considered as a new example of a dichotomy theorem.

A complete classification of complexity within a certain large part of Allen's algebra was previously obtained in [Drakengren and Jonsson 1998]. We state this result later as Proposition 5.1. This result was achieved by a lengthy computer-assisted exhaustive search (as were most similar results, e.g. [Jonsson and Drakengren 1997; Jonsson et al. 1999]). However, it was noted in [Drakengren and Jonsson 1998] that, for further progress, theoretical studies of the structure of Allen's algebra would be necessary, since using the method from that paper to obtain a complete analysis of complexity in Allen's algebra would require dealing with more than 10^{50} individual cases, which is clearly not feasible.

There have been some previous theoretical investigations of the structure of Allen's algebra, (see, e.g., [Hirsch 1996; 1997; Ladkin and Maddux 1994]); however these studies considered relation algebras in the sense defined by Tarski [Tarski 1941]. A Tarski relation algebra allows more operations on relations than originally specified by Allen [Allen 1983], and most subalgebras of Allen's algebra are not Tarski relation algebras. In fact, none of the maximal tractable subalgebras of Allen's interval algebra is a Tarski relation algebra, so methods based on that approach are not appropriate for obtaining a complete classification of complexity.

The first novel element in our approach is a new uniform description for all of the maximal tractable subalgebras of Allen's algebra which have already been identified (Table III). The second novel element is that we fully exploit the algebraic properties of Allen's algebra by importing a technique from general algebra. This technique has been used in many other contexts to obtain a description of maximal subalgebras of a given algebra with a given property (e.g., [Szendrei 1995; Yang 2000]). A similar algebraic approach was used for one fragment of Allen's algebra in [Ligozat 1998]. Here, for the first time, we systematically apply this algebraic technique to obtain a classification for all possible fragments of Allen's algebra.

The paper is organized as follows: in Section 2 we give the basic definitions of Allen's algebra, present the known maximal tractable subalgebras in the new form, and state our main result. In Section 3 we apply this result to classify the complexity of more general problems involving Allen's algebra extended with certain forms of metric information. In Section 4 we discuss the general idea of the proof, the algebraic technique we use for obtaining results of this type, and the techniques we use to prove NP-completeness of fragments of Allen's algebra. Sections 5 and 6 contain the proof of the new classification result—Section 5 considers the sub-

[2]The problem of satisfiability from propositional logic [Creignou et al. 2001; Schaefer 1978] should not be confused with the problem of satisfiability of temporal constraints discussed in this paper.

Basic relation		Example	Endpoints
x precedes y	p	xxx	$x^+ < y^-$
y preceded by x	p^{-1}	yyy	
x meets y	m	xxxx	$x^+ = y^-$
y met-by x	m^{-1}	yyyy	
x overlaps y	o	xxxx	$x^- < y^- < x^+$,
y overl.-by x	o^{-1}	yyyy	$x^+ < y^+$
x during y	d	xxx	$x^- > y^-$,
y includes x	d^{-1}	yyyyyyy	$x^+ < y^+$
x starts y	s	xxx	$x^- = y^-$,
y started by x	s^{-1}	yyyyyyy	$x^+ < y^+$
x finishes y	f	xxx	$x^+ = y^+$,
y finished by x	f^{-1}	yyyyyyy	$x^- > y^-$
x equals y	$\equiv$	xxxx	$x^- = y^-$,
		yyyy	$x^+ = y^+$

Table I. The thirteen basic relations. The endpoint relations $x^- < x^+$ and $y^- < y^+$ that are valid for all relations have been omitted.

algebras of Allen's algebra that contain non-trivial basic relations and Section 6 contains the proof for all other subalgebras. A number of NP-completeness results used in Section 6 are collected in the Appendix.

2. ALLEN'S INTERVAL ALGEBRA

Allen's interval algebra [Allen 1983] is based on the notion of *relations between intervals*. An interval x is represented as a pair $[x^-, x^+]$ of real numbers with $x^- < x^+$, denoting the left and right endpoints of the interval, respectively. The relations between intervals are the $2^{13} = 8192$ possible unions[3] of the 13 *basic interval relations*, which are shown in Table I. Note that the basic relations are jointly exhaustive and pairwise disjoint in the sense that any two given intervals are related by exactly one basic relation. For the sake of brevity, relations between intervals will be written as collections of basic relations. So, for instance, we write (pmf^{-1}) instead of $\mathsf{p} \cup \mathsf{m} \cup \mathsf{f}^{-1}$. Allen's algebra $\mathcal{A}$ consists of the 8192 possible relations between intervals together with the operations *converse* $\cdot^{-1}$*, intersection* $\cap$*, and composition* $\circ$, which are defined as follows:

$$\begin{aligned} \forall x,y:\ xr^{-1}y &\Leftrightarrow yrx \\ \forall x,y:\ x(r\cap s)y &\Leftrightarrow xry \;\&\; xsy \\ \forall x,y:\ x(r\circ s)y &\Leftrightarrow \exists z:\ (xrz \;\&\; zsy) \end{aligned}$$

It follows that the converse of $r = (b_1 \dots b_n)$ is equal to $(b_1^{-1} \dots b_n^{-1})$. The intersection of two relations can be expressed as the usual set-theoretic intersection. Since the basic relations are pairwise disjoint, the intersection of two relations $r_1, r_2 \in \mathcal{A}$ consists of the basic relations that are present in both r_1 and r_2. Using the definition of composition, it can be shown that

$$(b_1 \dots b_n) \circ (b'_1 \dots b'_m) = \bigcup \{b_i \circ b'_j \mid 1 \le i \le n,\ 1 \le j \le m\}.$$

[3]Including the empty relation.

$\circ$	$\equiv$	p	p^{-1}	m	m^{-1}	o	o^{-1}	d	d^{-1}	s	s^{-1}	f	f^{-1}
$\equiv$	$\equiv$	p	p^{-1}	m	m^{-1}	o	o^{-1}	d	d^{-1}	s	s^{-1}	f	f^{-1}
p	p	p	$\top$	p	ρ	p	ρ	ρ	p	p	p	ρ	p
p^{-1}	p^{-1}	$\top$	p^{-1}	λ^{-1}	p^{-1}	λ^{-1}	p^{-1}	λ^{-1}	p^{-1}	λ^{-1}	p^{-1}	p^{-1}	p^{-1}
m	m	p	ρ^{-1}	p	θ	p	β	β	p	m	m	β	p
m^{-1}	m^{-1}	λ	p^{-1}	σ	p^{-1}	γ^{-1}	p^{-1}	γ^{-1}	p^{-1}	γ^{-1}	p^{-1}	m^{-1}	m^{-1}
o	o	p	ρ^{-1}	p	β^{-1}	α	ν	β	λ	o	γ	β	α
o^{-1}	o^{-1}	λ	p^{-1}	γ	p^{-1}	ν	α^{-1}	γ^{-1}	ρ^{-1}	γ^{-1}	α^{-1}	o^{-1}	β^{-1}
d	d	p	p^{-1}	p	p^{-1}	ρ	λ^{-1}	d	$\top$	d	λ^{-1}	d	ρ
d^{-1}	d^{-1}	λ	ρ^{-1}	γ	β^{-1}	γ	β^{-1}	ν	d^{-1}	γ	d^{-1}	β^{-1}	d^{-1}
s	s	p	p^{-1}	p	m^{-1}	α	γ^{-1}	d	λ	s	σ	d	α
s^{-1}	s^{-1}	λ	p^{-1}	γ	m^{-1}	γ	o^{-1}	γ^{-1}	d^{-1}	σ	s^{-1}	o^{-1}	d^{-1}
f	f	p	p^{-1}	m	p^{-1}	β	α^{-1}	d	ρ^{-1}	d	α^{-1}	f	θ
f^{-1}	f^{-1}	p	ρ^{-1}	m	β^{-1}	o	β^{-1}	β	d^{-1}	o	d^{-1}	θ	f^{-1}

$$\alpha = (\mathsf{pmo}) \qquad \beta = (\mathsf{ods}) \qquad \gamma = (\mathsf{od}^{-1}\mathsf{f}^{-1}) \qquad \sigma = (\equiv \mathsf{ss}^{-1}) \qquad \theta = (\equiv \mathsf{ff}^{-1})$$
$$\rho = (\mathsf{pmods}) \qquad \lambda = (\mathsf{pmod}^{-1}\mathsf{f}^{-1}) \qquad \nu = (\equiv \mathsf{oo}^{-1}\mathsf{dd}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1})$$
$$\top = (\equiv \mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{oo}^{-1}\mathsf{dd}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1})$$

Table II. Composition table for the basic relations in Allen's algebra

Hence the composition of two relations $r_1, r_2 \in \mathcal{A}$ is determined by the compositions of the basic relations they contain. The compositions of all possible pairs of basic relations are given in Table II, and by using this table one can verify all the algebraic calculations in the forthcoming sections.

The problem of *satisfiability* ($\mathcal{A}$-SAT) for a set of interval variables with specified relations between them is that of deciding whether there exists an assignment of intervals on the real line for the interval variables, such that all of the relations between the intervals are satisfied. This is defined as follows.

Definition 2.1. Let $X \subseteq \mathcal{A}$ be a set of interval relations. An instance I of $\mathcal{A}$-SAT*(X)* is a set, V, of variables and a set of constraints of the form xry where $x, y \in V$ and $r \in X$. The question is whether I is *satisfiable*, i.e., whether there exists a function, f, from V to the set of all intervals such that $f(x)\, r\, f(y)$ holds for every constraint xry in I. Any such function f is called a *model* of I.

EXAMPLE 2.2. *1) The instance* $\{x(\mathsf{m})y, y(\mathsf{m})z, x(\mathsf{m})z\}$ *is not satisfiable because the first two constraints imply that interval* x *must precede interval* z *which contradicts the third constraint.*

2) The instance $I = \{x(\mathsf{mo})y, y(\mathsf{df}^{-1})z, x(\mathsf{m}^{-1}\mathsf{s})z\}$ *is satisfiable. The function* f *given by* $f(x) = [0, 2]$*,* $f(y) = [1, 3]$*, and* $f(z) = [0, 4]$ *is a model of* I*.*

An instance of $\mathcal{A}$-SAT(X) can also be represented, in an obvious way, as a labelled digraph, where the nodes are the variables from V, and the labelled arcs correspond to the constraints. This way of representing instances can sometimes be more transparent.

If there exists a polynomial-time algorithm solving all instances of $\mathcal{A}$-SAT(X) then we say that X is tractable. On the other hand, if $\mathcal{A}$-SAT(X) is NP-complete then we say that X is NP-complete. Since the problem $\mathcal{A}$-SAT$(\mathcal{A})$ is NP-complete [Vilain et al. 1989], there arises the question of identifying the tractable subsets of $\mathcal{A}$.

Subsets of $\mathcal{A}$ that are closed under the operations of intersection, converse and composition are said to be *subalgebras*. For a given subset X of $\mathcal{A}$, the smallest subalgebra containing X is called the subalgebra *generated* by X and is denoted by $\langle X\rangle$. It is easy to see that $\langle X\rangle$ is obtained from X by adding all relations that can be obtained from the relations in X by using the three operations of $\mathcal{A}$.

It is known [Nebel and Bürckert 1995], and easy to prove, that, for every $X \subseteq \mathcal{A}$, the problem $\mathcal{A}$-SAT$(\langle X\rangle)$ is polynomially equivalent to $\mathcal{A}$-SAT(X). Therefore, to classify the complexity of all subsets of $\mathcal{A}$ it is only necessary to consider *subalgebras* of $\mathcal{A}$. Obviously, adding relations to a subalgebra can only increase the complexity of the corresponding satisfiability problem. Thus, since $\mathcal{A}$ is finite, the problem of describing tractability in $\mathcal{A}$ can be reduced to the problem of describing the *maximal* tractable subalgebras in $\mathcal{A}$, that is, subalgebras that cannot be extended without losing tractability.

The known maximal tractable subalgebras [Drakengren and Jonsson 1997a; 1997b; Nebel and Bürckert 1995] are presented in Table III. In this table, and in our proofs below, we use the symbol $\pm$, which should be interpreted as follows. A condition involving $\pm$ means the conjunction of two conditions: one corresponding to $+$ and one corresponding to $-$. For example, condition $(\mathsf{o})^{\pm1} \subseteq r \Leftrightarrow (\mathsf{d})^{\pm1} \subseteq r$ means that both $(\mathsf{o}) \subseteq r \Leftrightarrow (\mathsf{d}) \subseteq r$ and $(\mathsf{o}^{-1}) \subseteq r \Leftrightarrow (\mathsf{d}^{-1}) \subseteq r$ hold. The main advantage of using the $\pm$ symbol is conciseness: in any subalgebra of $\mathcal{A}$, the '+' and the '−' conditions are satisfied (or not satisfied) simultaneously, and, therefore only one of them needs to be verified.

In order to improve readability, the names of some of the subalgebras in Table III are changed from those used in earlier presentations, in the following way. Let $r_1 = (\mathsf{p}^{-1}\mathsf{m}^{-1}\mathsf{o}^{-1}\mathsf{dsf})$, $r_2 = (\mathsf{pmod}^{-1}\mathsf{sf}^{-1})$, $r_3 = (\mathsf{pmodsf})$, and $r_4 = (\mathsf{pmodsf}^{-1})$. Then, the subalgebras $\mathcal{A}_i$, $1 \le i \le 4$, from Table III correspond to the algebras $A(r_i, \mathsf{s})$, $1 \le i \le 4$, introduced in [Drakengren and Jonsson 1997b], while the subalgebras $\mathcal{B}_i$, $1 \le i \le 4$, from Table III correspond to $A(r_4^{-1}, \mathsf{f})$, $A(r_3, \mathsf{f})$, $A(r_1, \mathsf{f})$, and $A(r_2^{-1}, \mathsf{f})$ [Drakengren and Jonsson 1997b].

In previous papers, the subalgebras from Table III were defined in other ways. However, in all cases except for $\mathcal{H}$, it is very straightforward to verify that our definitions are equivalent to the original ones. The subalgebra $\mathcal{H}$ was originally defined as the 'ORD-Horn algebra' [Nebel and Bürckert 1995], but has also been characterized as the algebra of 'pre-convex' relations [Ligozat 1998]. Using the latter description it is not hard to show that our definition of $\mathcal{H}$ is equivalent.

We are now ready to state our main theorem.

THEOREM 2.3. *Any subset of Allen's algebra is either NP-complete or included in one of the eighteen tractable subalgebras listed in Table III.*

The proof of this theorem is given in Sections 5 and 6.

As one immediate application of Theorem 2.3, consider the problem of reasoning

$$\begin{aligned}
\mathcal{S}_{\mathsf{p}} &= \{r \mid r \cap (\mathsf{pmod}^{-1}\mathsf{f}^{-1})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{p})^{\pm 1} \subseteq r\}\\
\mathcal{S}_{\mathsf{d}} &= \{r \mid r \cap (\mathsf{pmod}^{-1}\mathsf{f}^{-1})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{d}^{-1})^{\pm 1} \subseteq r\}\\
\mathcal{S}_{\mathsf{o}} &= \{r \mid r \cap (\mathsf{pmod}^{-1}\mathsf{f}^{-1})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{o})^{\pm 1} \subseteq r\}\\
\mathcal{A}_1 &= \{r \mid r \cap (\mathsf{pmod}^{-1}\mathsf{f}^{-1})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{s}^{-1})^{\pm 1} \subseteq r\}\\
\mathcal{A}_2 &= \{r \mid r \cap (\mathsf{pmod}^{-1}\mathsf{f}^{-1})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{s})^{\pm 1} \subseteq r\}\\
\mathcal{A}_3 &= \{r \mid r \cap (\mathsf{pmodf})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{s})^{\pm 1} \subseteq r\}\\
\mathcal{A}_4 &= \{r \mid r \cap (\mathsf{pmodf}^{-1})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{s})^{\pm 1} \subseteq r\}
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}_{\mathsf{p}} &= \{r \mid r \cap (\mathsf{pmods})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{p})^{\pm 1} \subseteq r\}\\
\mathcal{E}_{\mathsf{d}} &= \{r \mid r \cap (\mathsf{pmods})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{d})^{\pm 1} \subseteq r\}\\
\mathcal{E}_{\mathsf{o}} &= \{r \mid r \cap (\mathsf{pmods})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{o})^{\pm 1} \subseteq r\}\\
\mathcal{B}_1 &= \{r \mid r \cap (\mathsf{pmods})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{f}^{-1})^{\pm 1} \subseteq r\}\\
\mathcal{B}_2 &= \{r \mid r \cap (\mathsf{pmods})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{f})^{\pm 1} \subseteq r\}\\
\mathcal{B}_3 &= \{r \mid r \cap (\mathsf{pmod}^{-1}\mathsf{s}^{-1})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{f}^{-1})^{\pm 1} \subseteq r\}\\
\mathcal{B}_4 &= \{r \mid r \cap (\mathsf{pmod}^{-1}\mathsf{s})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{f}^{-1})^{\pm 1} \subseteq r\}
\end{aligned}$$

$$\mathcal{E}^* = \left\{ r \;\middle|\; \begin{array}{l} 1)\ r \cap (\mathsf{pmod})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{s})^{\pm 1} \subseteq r, \text{ and} \\ 2)\ r \cap (\mathsf{ff}^{-1}) \neq \emptyset \Rightarrow (\equiv) \subseteq r \end{array} \right\}$$

$$\mathcal{S}^* = \left\{ r \;\middle|\; \begin{array}{l} 1)\ r \cap (\mathsf{pmod}^{-1})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{f}^{-1})^{\pm 1} \subseteq r, \text{ and} \\ 2)\ r \cap (\mathsf{ss}^{-1}) \neq \emptyset \Rightarrow (\equiv) \subseteq r \end{array} \right\}$$

$$\mathcal{H} = \left\{ r \;\middle|\; \begin{array}{l} 1)\ r \cap (\mathsf{os})^{\pm 1} \neq \emptyset \;\&\; r \cap (\mathsf{o}^{-1}\mathsf{f})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{d})^{\pm 1} \subseteq r, \text{ and} \\ 2)\ r \cap (\mathsf{ds})^{\pm 1} \neq \emptyset \;\&\; r \cap (\mathsf{d}^{-1}\mathsf{f}^{-1})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{o})^{\pm 1} \subseteq r, \text{ and} \\ 3)\ r \cap (\mathsf{pm})^{\pm 1} \neq \emptyset \;\&\; r \not\subseteq (\mathsf{pm})^{\pm 1} \Rightarrow (\mathsf{o})^{\pm 1} \subseteq r \end{array} \right\}$$

$$\mathcal{A}_{\equiv} = \{r \mid r \neq \emptyset \Rightarrow (\equiv) \subseteq r\}$$

Table III. The 18 maximal tractable subalgebras of Allen's algebra.

with a single relation r, that is, the problem $\mathcal{A}$-SAT($\{r\}$). It follows from Theorem 2.3 that $\mathcal{A}$-SAT($\{r\}$) is NP-complete if and only if r either satisfies $r \cap r^{-1} = (\mathsf{mm}^{-1})$ or is a relation with $r \cap r^{-1} = \emptyset$ and such that neither r nor r^{-1} is contained in one of $(\mathsf{pmod}^{-1}\mathsf{sf}^{-1})$, $(\mathsf{pmod}^{-1}\mathsf{s}^{-1}\mathsf{f}^{-1})$, (pmodsf) and (pmodsf^{-1}). Using this characterisation it is easy to check that there are precisely 667 individual temporal relations r such that $\mathcal{A}$-SAT($\{r\}$) is NP-complete.

3. ALLEN'S INTERVAL ALGEBRA EXTENDED WITH METRIC INFORMATION

In this section we give some applications of Theorem 2.3 to more general temporal reasoning problems. Namely, we consider Allen's algebra combined with some forms of *disjunctive linear constraints*, a well-known framework which subsumes many different types of temporal reasoning problems. Some examples of these problems, including scheduling, planning, and indefinite temporal constraint databases, can be found in [Jonsson and Bäckström 1998; Koubarakis 2001; Stergiou and Koubarakis 2000] (see also [Cohen et al. 2000] for more discussion of tractable disjunctive constraints).

Definition 3.1. Let $V = \{x_1, \ldots, x_n\}$ be a set of real-valued variables, and α, β linear polynomials (that is, polynomials of degree one) over V with rational coefficients. A *linear relation* over V is an expression of the form $\alpha R \beta$, where $R \in \{<, \leq, =, \neq, \geq, >\}$.

A *disjunctive linear relation* (DLR) over V is a disjunction of a nonempty finite set of linear relations. A DLR is said to be *Horn* if at most one of its disjuncts is not of the form $\alpha \neq \beta$.

The problem of *satisfiability* for finite sets D of DLRs, denoted DLRSAT(D), is that of checking whether there exists an assignment f of variables in V to real numbers such that all DLRs in D are satisfied. Such an f is said to be a *model* of D. The satisfiability problem for finite sets H of Horn DLRs is denoted HORNDLRSAT(H).

EXAMPLE 3.2.

$$x + 2y \leq 3z + 42.3$$

is a linear relation,

$$(x + 2y \leq 3z + 42.3) \vee (x + z < 4y - 8) \vee (x > \frac{3}{12})$$

is a disjunctive linear relation, and

$$(x + 2y \leq 3z + 42.3) \vee (x + z \neq 4y - 8) \vee (x \neq \frac{3}{12})$$

is a Horn disjunctive linear relation.

PROPOSITION 3.3 [JONSSON AND BÄCKSTRÖM 1998; KOUBARAKIS 2001]. *The problem* DLRSAT *is NP-complete and the problem* HORNDLRSAT *is solvable in polynomial time.*

We can now define the general interval satisfiability problem with metric information.

Definition 3.4. Let I be an instance of $\mathcal{A}$-SAT(X) over a set V of variables and let H be a finite set of DLRs over the set $\{v^+, v^- \mid v \in V\}$ of variables, v^- representing starting points and v^+ ending points of variables $v \in V$.

An instance of the problem of *interval satisfiability with metric information* for a set X of interval relations, denoted $\mathcal{A}^m$-SAT(X), is a pair $Q = (I, H)$.

If f is a model for I, and $v \in V$, let $f(v^-)$ and $f(v^+)$ denote the starting point and the ending point of the interval $f(v)$, respectively.

An instance Q is said to be *satisfiable* if there exists a model f of I such that the DLRs in H are satisfied, with values for all v^- and v^+ given by $f(v^-)$ and $f(v^+)$, respectively.

Obviously, the $\mathcal{A}^m$-SAT(X) problem is NP-complete for all choices of X since every relation in $\mathcal{A}$ can be expressed in terms of DLRs. We let $\mathcal{A}^h$-SAT(X) denote the problem $\mathcal{A}^m$-SAT(X) restricted to metric constraints consisting of Horn DLRs only.

THEOREM 3.5. *The problem $\mathcal{A}^h$-SAT(X) is tractable if and only if $X \subseteq \mathcal{H}$. Otherwise $\mathcal{A}^h$-SAT(X) is NP-complete.*

PROOF. $\mathcal{A}^h$-SAT$(\mathcal{H})$ is a tractable problem [Jonsson and Bäckström 1998]. The interval constraint $a(\mathsf{m})b$ is equivalent to the metric constraint $a^+ = b^-$ so we can assume that $(\mathsf{m}) \in X$ and the result follows immediately from Theorem 2.3 because $\mathcal{H}$ is the only maximal tractable subalgebra containing (m). □

We can obtain another classification result if we further restrict the possible metric constraints. Define $\mathcal{A}^h_s$-SAT(X) to be the $\mathcal{A}^h$-SAT(X) problem where the metric constraints H are restricted in the following way: H may contain only the variables v^-, i.e., it may only relate starting points of intervals. The problem $\mathcal{A}^h_e$-SAT(X) is defined symmetrically by exchanging starting and ending points.

THEOREM 3.6. *1) $\mathcal{A}^h_s$-SAT(X) is tractable if and only if X is contained in one of the algebras $\mathcal{H}$, $\mathcal{S}_\mathsf{p}$, $\mathcal{S}_\mathsf{o}$, $\mathcal{S}_\mathsf{d}$ or $\mathcal{S}^*$. Otherwise $\mathcal{A}^h_s$-SAT(X) is NP-complete.*

2) $\mathcal{A}^h_e$-SAT(X) is tractable if and only if X is contained in one of the algebras $\mathcal{H}$, $\mathcal{E}_\mathsf{p}$, $\mathcal{E}_\mathsf{o}$, $\mathcal{E}_\mathsf{d}$ or $\mathcal{E}^*$. Otherwise $\mathcal{A}^h_e$-SAT(X) is NP-complete.

PROOF. We prove only part 1); part 2) is similar. If X is contained in one of the five algebras listed in 1) then tractability of $\mathcal{A}^h_s$-SAT(X) follows from [Drakengren and Jonsson 1997a]. The interval constraints $a(\equiv \mathsf{ss}^{-1})b$ and $a(\mathsf{pmod}^{-1}\mathsf{f}^{-1})b$ are equivalent to the metric constraints $a^- = b^-$ and $a^- < b^-$, respectively. Thus, we can assume that $\{(\equiv \mathsf{ss}^{-1}), (\mathsf{pmod}^{-1}\mathsf{f}^{-1})\} \subseteq X$. It follows from Theorem 2.3 that if $\mathcal{A}$-SAT(X) is tractable then X is contained in one of the five algebras listed in 1), and that otherwise this problem is NP-complete. □

4. PROOF TECHNIQUES

In this section we describe the general structure of the proof of Theorem 2.3, the algebraic techniques used in this paper, and the methods used for proving NP-completeness results.

4.1 Outline of the Algebraic Approach

In contrast to earlier approaches [Drakengren and Jonsson 1998; Jonsson and Drakengren 1997; Jonsson et al. 1999] we do not make any use of computer-assisted exhaustive search. Instead, we develop an analytical method which breaks the proof down into a collection of simple cases, and makes extensive use of the operations defined in the algebra (converse, intersection and composition). This approach is commonly used in general algebra to identify those substructures of a given structure that have a property P which is *hereditary*, that is, if some substructure possesses P then so does any smaller substructure contained in it. Note that the tractability of a subalgebra is an example of such a hereditary property in Allen's algebra. For examples of a similar approach in other algebraic contexts see [Szendrei 1995; Wilson 1999; Yang 2000].

To prove Theorem 2.3, we need to show that any fragment of $\mathcal{A}$ that is not entirely contained in one of the 18 known maximal tractable algebras is NP-complete. The major simplification we use has already been indicated: it is sufficient to consider only those sets, $\mathcal{S}$, which are *subalgebras* of Allen's algebra. Therefore it is sufficient to prove that every subalgebra that is not NP-complete must be contained in one of the 18 maximal tractable subalgebras.

The general idea of the proof is simple: we divide the set of all possible subalgebras of $\mathcal{A}$ into 6 cases. The first case contains all subalgebras which are already classified; these are covered by Proposition 5.1, adapted from [Drakengren and Jonsson 1998]. For each of the remaining five cases we first obtain a certain (relatively) small number of NP-complete fragments of $\mathcal{A}$. We then consider an arbitrary subalgebra $\mathcal{S}$ which satisfies the assumptions of the case, and contains none of these NP-complete fragments. Using the operations of $\mathcal{A}$, and the new description of the 18 maximal tractable subalgebras given in Table III, we obtain increasingly detailed information about $\mathcal{S}$ until we can eventually prove that it must be contained in one of these 18 maximal tractable subalgebras.

Now let us discuss the more technical details of the algebraic part of our method. First note that we always allow pairs of variables in instances to be unrelated, and hence we can assume without loss of generality that the subalgebra $\mathcal{S}$ we are considering contains the relation $\top$ (the union of all basic relations). Now for each basic relation b of $\mathcal{A}$, we will write r_b to denote the least relation $r \in \mathcal{S}$ such that $(b) \subseteq r$, i.e., the intersection of all $r \in \mathcal{S}$ with this property. Note that for any basic relation b, the relation r_b is well-defined and non-empty[4].

We use these relations r_b as our main tool to show that a subalgebra $\mathcal{S}$ is contained in one or another maximal tractable subalgebra. For example, suppose we have established that the relation (d) is contained in r_{o}. It follows that any relation $r \in \mathcal{S}$ such that $(\mathsf{o}) \subseteq r$ also satisfies $(\mathsf{d}) \subseteq r$. (To see this, note that if there is $r_1 \in \mathcal{S}$ such that $(\mathsf{o}) \subseteq r$, but $(\mathsf{d}) \not\subseteq r$, then $(\mathsf{o}) \subseteq r_1 \cap r_{\mathsf{o}}$ and $r_1 \cap r_{\mathsf{o}}$ is strictly contained in r_{o}, which contradicts the minimality of r_{o}.) By a similar argument, if we have established that (d) is contained in all of r_{p}, r_{m}, r_{o}, and r_{s}, then we can conclude that any relation $r \in \mathcal{S}$ such that $r \cap (\mathsf{pmods}) \neq \emptyset$ also satisfies $(\mathsf{d}) \subseteq r$. From this it follows immediately that $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{d}}$, by definition of $\mathcal{E}_{\mathsf{d}}$ (Table III). Many of the algebraic proofs in this paper follow this general scheme of collecting information about the relations r_b and then using Table III to conclude that $\mathcal{S}$ is contained in one or another maximal tractable subalgebra.

Throughout the proofs we also use the obvious fact that if $r_1 \subseteq r_2$ then, for any r, we have $r \circ r_1 \subseteq r \circ r_2$ and $r_1 \circ r \subseteq r_2 \circ r$.

4.2 Techniques for Proving NP-completeness

We will use several generic methods to establish NP-completeness for various small fragments of $\mathcal{A}$.

4.2.1 *The Betweenness Technique.* The first technique is based on the technical Lemma 4.1 below. The main feature of this technique is that, in order to prove NP-completeness of some fixed set of relations, X, one only needs to check the satisfiability of three small (5-variable and 9-constraint) instances of $\mathcal{A}$-SAT(X). We use these instances as 'gadgets' to simulate, in polynomial time, the NP-complete

[4]Obviously, the relations r_b depend on $\mathcal{S}$; however $\mathcal{S}$ will always be clear from the context.

problem BETWEENNESS[5] [Garey and Johnson 1979], which is defined as follows:

INSTANCE: A finite set A, a collection T of ordered triples (a, b, c) of distinct elements from A.
QUESTION: Is there a total ordering $<$ on A such that for each $(a, b, c) \in T$, we have either $a < b < c$ or $c < b < a$?

The particular instances of $\mathcal{A}$-SAT that we use to simulate this problem are defined as follows. For any given relations $R, R_1, R_2 \in \mathcal{A}$ we define $\Gamma(a, b, c, x, y)$ to be the following problem instance over the variables $\{a, b, c, x, y\}$:

$$\{xR_1a, xR_1b, xR_2c, yR_2a, yR_1b, yR_1c\}.$$

We then define the instances

$$\begin{aligned}\Gamma_1 &= \Gamma(a, b, c, x, y) \cup \{aRb, bRc, aRc\},\\ \Gamma_2 &= \Gamma(a, b, c, x, y) \cup \{bRa, bRc, aR \cup R^{-1}c\},\\ \Gamma_3 &= \Gamma(a, b, c, x, y) \cup \{aRb, cRb, aR \cup R^{-1}c\}.\end{aligned}$$

The problem instance Γ_1 is illustrated in Figure 1.

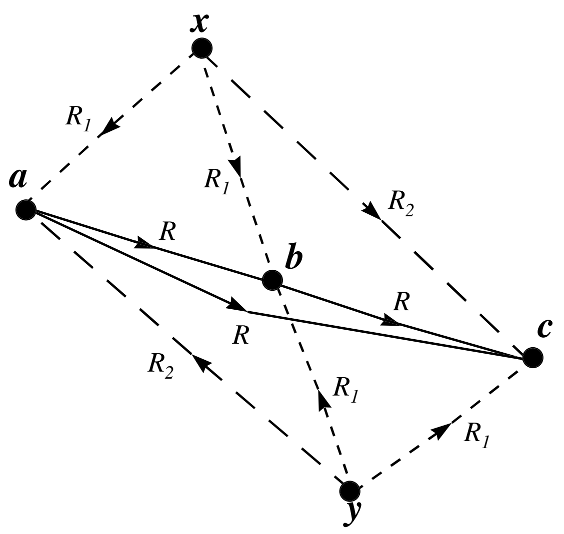

Fig. 1. The problem instance Γ_1 used in Lemma 4.1

LEMMA 4.1. *Let $R \in \{(\mathsf{p}), (\mathsf{o}), (\mathsf{d}), (\mathsf{s}), (\mathsf{f})\}$, $R_1, R_2 \in \mathcal{A}$, and let Γ_1, Γ_2, Γ_3 be as above. If Γ_1 is satisfiable while Γ_2 and Γ_3 are not, then $\{R \cup R^{-1}, R_1, R_2\}$ is NP-complete.*

[5]This problem is also known as the TOTAL ORDERING PROBLEM [Opatrný 1979].

PROOF. We describe a polynomial-time reduction from BETWEENNESS. Let (A,T) be an arbitrary instance of BETWEENNESS and construct an instance I of $\mathcal{A}$-SAT$(\{R \cup R^{-1}, R_1, R_2\})$ as follows:

(1) for each pair of distinct elements $a, b \in A$, add the constraint $aR \cup R^{-1}b$ to I;

(2) for each triple $(a,b,c) \in T$, introduce two fresh variables, x and y, and add $\Gamma(a,b,c,x,y)$ to I.

We will henceforth refer to the variables in I that correspond to the set A as 'basic' variables and the other variables as 'auxiliary' variables.

Assume that I has a model f. Then, due to the constraints added in step (1), the intervals $f(a), a \in A$, are pairwise distinct. Moreover, the relation R induces a total order on the set $\{f(a) \mid a \in A\}$. Suppose now that there is a triple $(a,b,c) \in T$ such that the model f satisfies $f(b)\,R\,f(a)\,R\,f(c)$. Then the instance Γ_2, with the auxiliary variables x and y introduced in step (2) for the triple (a,b,c), is satisfiable, a contradiction. With the help of Γ_2 and Γ_3, we can analogously rule out all orderings of $f(a), f(b), f(c)$ except $f(a)\,R\,f(b)\,R\,f(c)$ and $f(c)\,R\,f(b)\,R\,f(a)$. Hence there is a solution to the instance (A,T): for all $a, b \in A$, set $a < b$ if and only if $f(a)\,R\,f(b)$.

We assume now that there exists a total order $<$ on A that has the required property and show how to construct a model f of I. For all $a, b \in A$, set $f(a)\,R\,f(b)$ if and only if $a < b$. Clearly, this satisfies all constraints added in step (1).

By assumption, for each triple $(a,b,c) \in T$, the instance Γ_1 is satisfiable, that is, it has some model g. Then, since R is a basic relation, we know precisely how the intervals $g(a)$, $g(b)$, and $g(c)$ are related. It now follows that the model g can be adjusted by moving the intervals $g(a)$, $g(b)$, $g(c)$, $g(x)$, and $g(y)$ along the real line and stretching or shrinking them (but without changing the relations between them) so that the new model assigns $f(a)$, $f(b)$, and $f(c)$ to a, b, and c, respectively. The fact that $\Gamma_1 = \Gamma(a,b,c,x,y) \cup \{aRb, bRc, aRc\}$ is satisfiable and the symmetry of Γ imply that $\Gamma(a,b,c,x,y) \cup \{cRb, bRa, cRa\}$ is satisfiable (by exchanging the value for a by the value for c and vice versa, and doing the same for x and y). Thus, for every triple $(a,b,c) \in T$, we can find values for the auxiliary variables x and y so that all sets of constraints of the form $\Gamma(a,b,c,x,y)$ are satisfied at the same time, and hence there exists a model of I. $\square$

In order to use Lemma 4.1 to prove NP-completeness of some fixed set of relations, one only needs to check the satisfiability of three small instances of $\mathcal{A}$-SAT. One straightforward way to do this is to use B. Nebel's CSP solver [Nebel 1997a], which is a computer program for checking satisfiablity of an instance of $\mathcal{A}$-SAT.

For example, if we set $R = (\mathsf{o})$, $R_1 = (\mathsf{d})$, $R_2 = (\mathsf{oo}^{-1})$, then it is easy to check that Γ_1 is satisfiable, but Γ_2 and Γ_3 are not; hence, by Lemma 4.1, the set $\{(\mathsf{d}), (\mathsf{oo}^{-1})\}$ is NP-complete. In Figure 2, we illustrate the construction used in the proof for this case by showing how the auxiliary variables x and y can be given consistent values in the two 'allowed' cases (corresponding to $f(a) < f(b) < f(c)$ and $f(c) < f(b) < f(a)$). The reader is encouraged to verify that x and y cannot be given consistent values for the remaining four orderings.

4.2.2 *The Derivation Technique.* The second method we use to establish NP-completeness is based on the notion of *derivation*. The idea is that if one can

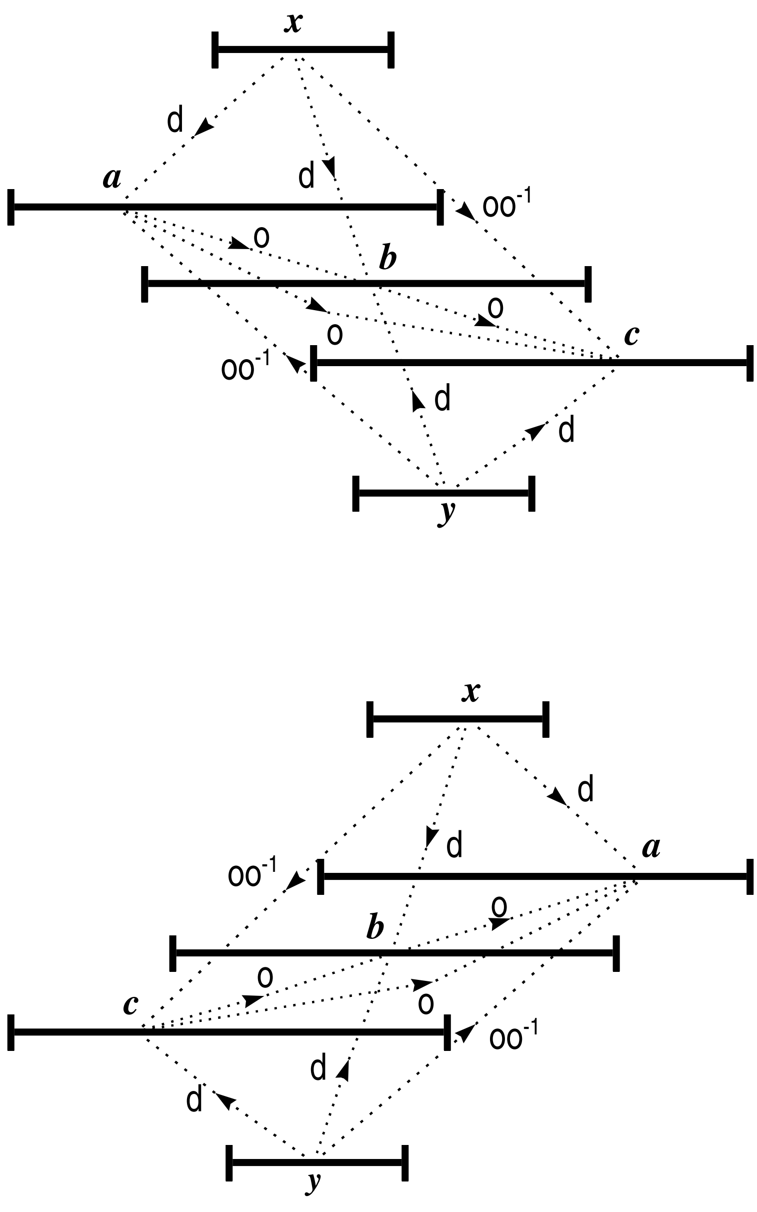

Fig. 2. Example of using Lemma 4.1.

'simulate' every relation in some known NP-complete fragment of $\mathcal{A}$ by using 'gadgets' built from relations in some $X \subseteq \mathcal{A}$ then the fragment X is also NP-complete.

Suppose $X \subseteq \mathcal{A}$ and I is an instance of $\mathcal{A}$-SAT(X) containing the variables x and y. Further, let $r \in \mathcal{A}$ be the relation defined as follows: r is the set containing all basic relations b such that the instance obtained from I by adding the constraint xby is satisfiable. In this case, we say that r is *derived* from X. Intuitively, r can be simulated by relations in X via the 'gadget' I.

For example, consider the following set of constraints:

$$a(\mathsf{dd}^{-1})b, x(\mathsf{ss}^{-1})a, x(\mathsf{ff}^{-1})b, y(\mathsf{ff}^{-1})a, y(\mathsf{ss}^{-1})b.$$

One can easily check that the derived relation between x and y is (oo^{-1}). Note that (oo^{-1}) does not belong to the subalgebra generated by (dd^{-1}), (ss^{-1}), and (ff^{-1}).

It should be noted that if the instance $I_1 = I \cup \{xr'y\}$ is satisfiable, then, for any two given intervals i_1, j_1 such that $i_1 r' j_1$, there is a model f of I_1 such that $f(x) = i_1$ and $f(y) = j_1$. This can be established as follows: since I_1 is satisfiable, it has a model g. Denote $g(x)$ by i_2 and $g(y)$ by j_2; then $i_2 r' j_2$. There exists a continuous monotone injective mapping φ of the real line such that φ takes i_2 to i_1 and j_2 to j_1. Obviously, φ maps intervals to intervals, and it does not change the qualitative relations between intervals. Therefore, combining φ and g we obtain the required model f.

Now it can easily be checked that adding a derived relation r to X does not change the complexity of $\mathcal{A}$-SAT(X) because, in any instance, any constraint involving r can be replaced by the set of constraints in I (introducing fresh variables when needed), and this can be done in polynomial time.

Generally, one can derive more relations from a given $X \subseteq \mathcal{A}$ than one can generate using the three operations of Allen's algebra, that is, any relation generated from X can also be derived from X. This follows from the example above and the facts that any relation obtained by multiple derivations can also be obtained by a single derivation, and that relations r^{-1}, $r_1 \cap r_2$, and $r_1 \circ r_2$, between x and y are derived from the instances $\{yrx\}$, $\{xr_1y, xr_2y\}$, and $\{xr_1z, zr_2y\}$, respectively. However, derivation is essentially harder to manage in general, while the operations of Allen's algebra give us the advantage of employing algebraic techniques. Therefore we use derivations only in NP-completeness proofs. Note that derivations can also be calculated using B. Nebel's CSP solver [Nebel 1997a].

4.2.3 *The Duality Technique.* Our last proof technique is a principle of *duality*, which is based on the natural symmetry of the real line and will be used to simplify many of the forthcoming proofs. We make use of a function reverse which is defined on the basic relations of $\mathcal{A}$ by the following table:

b	$\equiv$	p	p^{-1}	m	m^{-1}	o	o^{-1}	d	d^{-1}	s	s^{-1}	f	f^{-1}
reverse(b)	$\equiv$	p^{-1}	p	m^{-1}	m	o^{-1}	o	d	d^{-1}	f	f^{-1}	s	s^{-1}

and is defined for all other elements of $\mathcal{A}$ by setting $\mathsf{reverse}(r) = \bigcup_{b \in r} \mathsf{reverse}(b)$.

Let I be any instance of $\mathcal{A}$-SAT, and let I' be obtained from I by replacing every r with $\mathsf{reverse}(r)$. It is easy to check that I has a model f if and only if I' has a model f' given by

$$f'(x_i) = [-f(x_i^+), -f(x_i^-)].$$

In other words, f' is obtained from f by redirecting the real line and leaving all intervals (as geometric objects) in their places. This observation leads to the following lemma.

LEMMA 4.2. *Let $\mathcal{R} = \{r_1, \ldots, r_n\} \subseteq \mathcal{A}$ and $\mathcal{R}' = \{r'_1, \ldots, r'_n\} \subseteq \mathcal{A}$ be such that, for all $1 \le k \le n$, $r'_k = \mathsf{reverse}(r_k)$. Then $\mathcal{R}$ is tractable (NP-complete) if and only if $\mathcal{R}'$ is tractable (NP-complete).*

As an example of the use of Lemma 4.2, note that a proof of NP-completness for, say, $\{(\mathsf{ods}^{-1})\}$, immediately yields a proof of NP-completeness for $\{(\mathsf{o}^{-1}\mathsf{df}^{-1})\}$.

5. SUBALGEBRAS WITH NON-TRIVIAL BASIC RELATIONS

This section and the next contain the proof of Theorem 2.3.

For a subalgebra $\mathcal{S}$ of $\mathcal{A}$, we denote by $bas(\mathcal{S})$ the set of basic relations in $\mathcal{S}$. We can assume without loss of generality that $\mathcal{S}$ contains the relation $(\equiv)$, since it is easy to show that $\mathcal{S}$ and $\mathcal{S} \cup \{(\equiv)\}$ have the same complexity (up to polynomial-time equivalence). This implies that the size of $bas(\mathcal{S})$ is odd, since $\mathcal{S}$ is closed under converse.

The following proposition is proved in [Drakengren and Jonsson 1998].

PROPOSITION 5.1.
1) Let $\mathcal{S}$ be a subalgebra of $\mathcal{A}$ with $|bas(\mathcal{S})| > 3$. Then $\mathcal{S}$ is tractable if it is contained in one of the following 7 algebras: $\mathcal{S}_\mathsf{p}$, $\mathcal{S}_\mathsf{d}$, $\mathcal{S}_\mathsf{o}$, $\mathcal{E}_\mathsf{p}$, $\mathcal{E}_\mathsf{d}$, $\mathcal{E}_\mathsf{o}$, and $\mathcal{H}$. Otherwise $\mathcal{S}$ is NP-complete.
2) Let $\mathcal{S}$ be a subalgebra of $\mathcal{A}$ such that $(\mathsf{m}) \in \mathcal{S}$ or $(\mathsf{p}) \in \mathcal{S}$. Then $\mathcal{S}$ is tractable if $\mathcal{S} \subseteq \mathcal{S}_\mathsf{p}$, or $\mathcal{S} \subseteq \mathcal{E}_\mathsf{p}$, or $\mathcal{S} \subseteq \mathcal{H}$. Otherwise $\mathcal{S}$ is NP-complete.
3) Let $\mathcal{S}$ be a subalgebra of $\mathcal{A}$ such that $(\mathsf{pp}^{-1}) \in \mathcal{S}$ or $(\mathsf{pp}^{-1}\mathsf{mm}^{-1}) \in \mathcal{S}$. Then $\mathcal{S}$ is tractable if $\mathcal{S} \subseteq \mathcal{S}_\mathsf{p}$ or $\mathcal{S} \subseteq \mathcal{E}_\mathsf{p}$. Otherwise $\mathcal{S}$ is NP-complete.

We shall say that a relation is *non-trivial* if it is not equal to the empty relation or the relation $(\equiv)$. The result to be shown in this section is the following:

PROPOSITION 5.2. *Let $\mathcal{S}$ be a subalgebra of $\mathcal{A}$ which contains a non-trivial basic relation. Then $\mathcal{S}$ is tractable if it is contained in one of the 18 algebras listed in Table III. Otherwise $\mathcal{S}$ is NP-complete.*

Note that if $\mathcal{S}$ contains a non-trivial basic relation, then $\mathcal{S} \not\subseteq \mathcal{A}_\equiv$.

By combining Proposition 5.1(1) and the observation preceding it, it suffices to consider only the case $|bas(\mathcal{S})| = 3$. By Proposition 5.1(2), it suffices to consider the cases where $bas(\mathcal{S})$ is one of the following sets: $\{\equiv, \mathsf{d}, \mathsf{d}^{-1}\}$, $\{\equiv, \mathsf{o}, \mathsf{o}^{-1}\}$, $\{\equiv, \mathsf{s}, \mathsf{s}^{-1}\}$, and $\{\equiv, \mathsf{f}, \mathsf{f}^{-1}\}$. We do this in Subsections 5.1–5.3.

Given a relation r, we write r^* to denote the relation $r \cap r^{-1}$. Evidently, every subalgebra of $\mathcal{A}$ is closed under the operation $\cdot^*$ (of taking the symmetric part of a relation). By $sym(\mathcal{S})$ we denote the set $\{r \in \mathcal{S} \mid r^* = r\}$.

5.1 The case $bas(\mathcal{S}) = \{\equiv, \mathsf{d}, \mathsf{d}^{-1}\}$

In this subsection, we will show that if $\mathcal{S}$ is a subalgebra with $bas(\mathcal{S}) = \{\equiv, \mathsf{d}, \mathsf{d}^{-1}\}$, then $\mathcal{S} \subseteq \mathcal{S}_\mathsf{d}$, $\mathcal{S} \subseteq \mathcal{E}_\mathsf{d}$ or $\mathcal{S}$ is NP-complete.

To obtain this result we shall assume throughout this subsection that $\mathcal{S}$ is a subalgebra of $\mathcal{A}$ satisfying the following assumptions:

Assumption 1. $bas(\mathcal{S}) = \{\equiv, \mathsf{d}, \mathsf{d}^{-1}\}$.

Assumption 2. $\mathcal{S}$ is not NP-complete.

Using these assumptions we obtain increasingly detailed information about $\mathcal{S}$ in Lemmas 5.4-5.12, until we are able to show that in all cases $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{d}}$ or $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{d}}$. These lemmas rely on the following NP-completeness result.

LEMMA 5.3. *The subsets* $\{(\mathsf{d}), (\mathsf{oo}^{-1})\}$ *and* $\{(\mathsf{d}^{-1}), (\mathsf{pp}^{-1})\}$ *of* $\mathcal{A}$ *are NP-complete.*

PROOF. Apply Lemma 4.1 with $R = (\mathsf{o})$, $R_1 = (\mathsf{d})$, $R_2 = (\mathsf{oo}^{-1})$, or with $R = (\mathsf{p})$, $R_1 = (\mathsf{d}^{-1})$, $R_2 = (\mathsf{pp}^{-1})$, respectively. $\square$

Before we give the proofs, we note that

$$\nu = (\equiv \mathsf{oo}^{-1}\mathsf{dd}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1}) = (\mathsf{d}^{-1}) \circ (\mathsf{d}) \in \mathcal{S},$$

(see Table II).

LEMMA 5.4. *With the assumptions above,*

$$sym(\mathcal{S}) \subseteq \{r^* | (\mathsf{dd}^{-1}) \subseteq r\} \cup \{r^* | r \subseteq (\equiv \mathsf{ss}^{-1})\} \cup \{r^* | r \subseteq (\equiv \mathsf{ff}^{-1})\}.$$

PROOF. We will show that if $sym(\mathcal{S})$ is not included in the above set then $\mathcal{S}$ is NP-complete, which contradicts Assumption 2.

Suppose first that $(\equiv \mathsf{ss}^{-1}\mathsf{ff}^{-1})$ or $(\mathsf{ss}^{-1}\mathsf{ff}^{-1})$ belongs to $sym(\mathcal{S})$. Then

$$r = (\equiv \mathsf{ss}^{-1}\mathsf{ff}^{-1}) \circ (\mathsf{d}) = (\mathsf{ss}^{-1}\mathsf{ff}^{-1}) \circ (\mathsf{d}) = (\mathsf{oo}^{-1}\mathsf{dsf}) \in \mathcal{S},$$

so we have $(\mathsf{oo}^{-1}) = r^* \in \mathcal{S}$, which implies that $\mathcal{S}$ is NP-complete, by Lemma 5.3.

Suppose now there is $r^* \in sym(\mathcal{S})$ such that $r^* \subseteq (\equiv \mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{oo}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1})$ and $r^* \not\subseteq (\equiv \mathsf{ss}^{-1}\mathsf{ff}^{-1})$. Consider $r_1 = (\mathsf{d}) \circ r^* \subseteq (\mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{oo}^{-1}\mathsf{dsf})$. It is easy to check that r_1^* is non-empty and $r_1^* \subseteq (\mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{oo}^{-1})$. If $(\mathsf{oo}^{-1}) \subseteq r_1^*$, then $r_1^* \cap \nu = (\mathsf{oo}^{-1}) \in \mathcal{S}$, hence $\mathcal{S}$ is NP-complete, by Lemma 5.3. Otherwise $r_1^* \subseteq (\mathsf{pp}^{-1}\mathsf{mm}^{-1})$, so $(\mathsf{d}) \circ r_1^* = (\mathsf{pp}^{-1}) \in \mathcal{S}$, which again implies that $\mathcal{S}$ is NP-complete, by Lemma 5.3. $\square$

LEMMA 5.5. *With the assumptions above,*

$$\mathcal{S} \subseteq \{r | r \cap (\mathsf{dd}^{-1}) \neq \emptyset\} \cup \{r | r \subseteq (\equiv \mathsf{ss}^{-1})\} \cup \{r | r \subseteq (\equiv \mathsf{ff}^{-1})\}.$$

PROOF. Assume for contradiction that $\mathcal{S}$ contains a relation r with $r \cap (\mathsf{dd}^{-1})$ empty, $r \not\subseteq (\equiv \mathsf{ss}^{-1})$ and $r \not\subseteq (\equiv \mathsf{ff}^{-1})$.

Among such relations, choose r to be minimal with respect to inclusion. Then, since $\nu \in \mathcal{S}$, we have either $r \subseteq (\equiv \mathsf{oo}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1})$, or $r \subseteq (\equiv \mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{ss}^{-1})$, or $r \subseteq (\equiv \mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{ff}^{-1})$.

<u>Case 1.</u> $r \subseteq (\equiv \mathsf{oo}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1})$

Assume first that $(\mathsf{o}) \subseteq r$ (the argument for (o^{-1}) is dual by using Lemma 4.2). Consider $r_1 = (\mathsf{d}^{-1}) \circ r \subseteq (\mathsf{oo}^{-1}\mathsf{d}^{-1}\mathsf{s}^{-1}\mathsf{f}^{-1})$. By the minimality of r, $r_1 \cap r = r$ or empty, but $(\mathsf{o}) \subseteq r$, so $(\mathsf{o}) \subseteq r_1$, so $r_1 \cap r$ is not empty and hence $r \subseteq (\mathsf{oo}^{-1}\mathsf{s}^{-1}\mathsf{f}^{-1})$. Now consider $r_2 = (\mathsf{d}) \circ r \subseteq (\mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{oo}^{-1}\mathsf{dsf})$. By a similar argument we get $r \subseteq (\mathsf{oo}^{-1}\mathsf{sf})$. Combining these two results gives $r \subseteq (\mathsf{oo}^{-1})$, and then by Lemma 5.4 we get $r = (\mathsf{o})$, which contradicts Assumption 1.

Hence, we must have $r \subseteq (\equiv \mathsf{ss}^{-1}\mathsf{ff}^{-1})$. If $(\mathsf{sf}) \subseteq r$ then $((\mathsf{d}^{-1}) \circ r)^* = (\mathsf{oo}^{-1}) \in \mathcal{S}$, which contradicts Lemma 5.4. If $r \in \{(\mathsf{sf}^{-1}), (\equiv \mathsf{sf}^{-1})\}$ then $r \cap ((\mathsf{d}) \circ r) = (\mathsf{s}) \in \mathcal{S}$, which contradicts Assumption 1. Hence, case 1 is impossible.
<u>Case 2</u>. $r \subseteq (\equiv \mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{ss}^{-1})$.
If $r \cap (\mathsf{pp}^{-1}) \neq \emptyset$ then $r_1 = r \cap ((\mathsf{d}) \circ r) \subseteq (\mathsf{pp}^{-1}\mathsf{m}^{-1})$. Furthermore, $(\mathsf{d}) \circ r_1$ is a nonempty subrelation of (pp^{-1}), which contradicts Assumption 1 or Lemma 3. If $(\mathsf{mm}^{-1}) \subseteq r$ then $((\mathsf{d}) \circ r)^* = (\mathsf{pp}^{-1})$, a contradiction again. It remains to consider the case $(\mathsf{m}) \subseteq r \subseteq (\equiv \mathsf{mss}^{-1})$. If $(\mathsf{s}^{-1}) \subseteq r$ then, again, $((\mathsf{d}) \circ r)^* = (\mathsf{pp}^{-1}) \in \mathcal{S}$. Otherwise $r_2 = r \circ r$ satisfies $(\mathsf{p}) \subseteq r_2 \subseteq (\equiv \mathsf{pms})$, and then $(\mathsf{p}) = r_2 \cap ((\mathsf{d}) \circ r_2) \in \mathcal{S}$, which contradicts Assumption 1.
<u>Case 3</u>. $r \subseteq (\equiv \mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{ff}^{-1})$.
Dual to Case 2. □

LEMMA 5.6. *With the assumptions above, if* $(\mathsf{pmods}) \in \mathcal{S}$ *then* $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{d}}$*; if* $(\mathsf{pmod}^{-1}\mathsf{f}^{-1}) \in \mathcal{S}$ *then* $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{d}}$.

PROOF. We prove only the first statement, the second one is dual.
First note that if $r_{\mathsf{s}} \subseteq (\equiv \mathsf{ss}^{-1})$ then if $(\mathsf{pmods}) \in \mathcal{S}$ we have $r_{\mathsf{s}} = (\mathsf{s})$, which contradicts Assumption 1. Furthermore, each of the relations $r_{\mathsf{p}}, r_{\mathsf{m}}, r_{\mathsf{o}}, r_{\mathsf{s}}$ must be contained in (pmods) so, by Lemma 5.5, each of them contains (d). This implies that for every $r \in \mathcal{S}$, we have $(\mathsf{d}) \subseteq r$ whenever $r \cap (\mathsf{pmods})$ is non-empty. This condition precisely means that $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{d}}$. □

LEMMA 5.7. *With the assumptions above, if $\mathcal{S}$ contains a non-trivial relation r such that* $r \subseteq (\equiv \mathsf{ss}^{-1})$*, then* $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{d}}$*; if $\mathcal{S}$ contains a non-trivial relation r such that* $r \subseteq (\equiv \mathsf{ff}^{-1})$*, then* $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{d}}$.

PROOF. The two cases are dual so we consider only the first one.
Observe that $r \circ r^{-1} = (\equiv \mathsf{ss}^{-1})$ so we can assume that $r = (\equiv \mathsf{ss}^{-1})$. We have $(\mathsf{d}) \circ (\equiv \mathsf{ss}^{-1}) = (\mathsf{p}^{-1}\mathsf{m}^{-1}\mathsf{o}^{-1}\mathsf{df})$, so the inverse relation $(\mathsf{pmod}^{-1}\mathsf{f}^{-1}) \in \mathcal{S}$ and the result follows from Lemma 5.6. □

In view of Lemma 5.7 and Lemma 5.5, it is sufficient to consider only cases such that for any non-trivial $r \in \mathcal{S}$, $r \cap (\mathsf{dd}^{-1})$ is non-empty.

LEMMA 5.8. *With the assumptions above, if* $(\mathsf{dd}^{-1}) \not\subseteq r_{\mathsf{o}}$ *then we have* $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{d}}$ *or* $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{d}}$.

PROOF. Suppose first that $r_{\mathsf{o}} \cap (\mathsf{dd}^{-1}) = (\mathsf{d})$. Then, by Lemma 5.4, we have $(\mathsf{oo}^{-1}) \not\subseteq r_{\mathsf{o}}$. By the minimality of r_{o}, $r_{\mathsf{o}} \cap \nu = r_{\mathsf{o}}$, so we have

$$r_1 = (\mathsf{od}) \subseteq r_{\mathsf{o}} \subseteq (\equiv \mathsf{odss}^{-1}\mathsf{ff}^{-1}) = r_2.$$

Consequently, $r_1 \circ (\mathsf{d}) \subseteq r_{\mathsf{o}} \circ (\mathsf{d}) \subseteq r_2 \circ (\mathsf{d})$, that is, $(\mathsf{ods}) \subseteq r_{\mathsf{o}} \circ (\mathsf{d}) \subseteq (\mathsf{oo}^{-1}\mathsf{dsf})$. Then, by minimality of r_{o}, we have $(\mathsf{od}) \subseteq r_{\mathsf{o}} \subseteq (\mathsf{odsf})$. Finally,

$$(\mathsf{pmods}) = (\mathsf{d}) \circ (\mathsf{od}) \subseteq (\mathsf{d}) \circ r_{\mathsf{o}} \subseteq (\mathsf{d}) \circ (\mathsf{odsf}) = (\mathsf{pmods}),$$

that is, $(\mathsf{pmods}) = (\mathsf{d}) \circ r_{\mathsf{o}} \in \mathcal{S}$, which implies, by Lemma 5.6, that $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{d}}$.
Dual calculations show that if $r_{\mathsf{o}} \cap (\mathsf{dd}^{-1}) = (\mathsf{d}^{-1})$ then $(\mathsf{pmod}^{-1}\mathsf{f}^{-1}) \in \mathcal{S}$, and hence, by Lemma 5.6, $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{d}}$. □

LEMMA 5.9. *With the assumptions above, if we have* $r_{\mathsf{s}} \cap (\mathsf{dd}^{-1}) = (\mathsf{d}^{-1})$ *or* $r_{\mathsf{f}} \cap (\mathsf{dd}^{-1}) = (\mathsf{d}^{-1})$*, then* $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{d}}$ *or* $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{d}}$.

PROOF. The two cases are dual so we prove only the first one.

If $r_{\mathsf{s}} \cap (\mathsf{dd}^{-1}) = (\mathsf{d}^{-1})$, then, by Lemma 5.4, we have $(\mathsf{ss}^{-1}) \not\subseteq r_{\mathsf{s}}$. By the minimality of r_{s}, $r_{\mathsf{s}} \cap \nu = r_{\mathsf{s}}$, so we have $(\mathsf{d}^{-1}\mathsf{s}) \subseteq r_{\mathsf{s}} \subseteq (\equiv \mathsf{oo}^{-1}\mathsf{d}^{-1}\mathsf{sff}^{-1})$.

Suppose $r_{\mathsf{s}} \cap (\mathsf{oo}^{-1})$ is non-empty. Then $(\mathsf{dd}^{-1}) \not\subseteq r_{\mathsf{o}}$ so $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{d}}$ or $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{d}}$, by Lemma 5.8. Suppose to the contrary that $r_{\mathsf{s}} \cap (\mathsf{oo}^{-1}) = \emptyset$. Then we have

$$(\mathsf{d}^{-1}\mathsf{s}) \subseteq r_{\mathsf{s}} \subseteq (\equiv \mathsf{d}^{-1}\mathsf{sff}^{-1}) \text{ and}$$
$$(\mathsf{od}^{-1}\mathsf{f}^{-1}) = (\mathsf{d}^{-1}) \circ (\mathsf{d}^{-1}\mathsf{s}) \subseteq (\mathsf{d}^{-1}) \circ r_{\mathsf{s}} \subseteq (\mathsf{d}^{-1}) \circ (\equiv \mathsf{d}^{-1}\mathsf{sff}^{-1}) = (\mathsf{od}^{-1}\mathsf{s}^{-1}\mathsf{f}^{-1}).$$

Therefore we have $(\mathsf{od}^{-1}\mathsf{f}^{-1}) \in \mathcal{S}$ or $(\mathsf{od}^{-1}\mathsf{s}^{-1}\mathsf{f}^{-1}) \in \mathcal{S}$, which implies $(\mathsf{dd}^{-1}) \not\subseteq r_{\mathsf{o}}$, so the result follows from Lemma 5.8. □

LEMMA 5.10. *With the assumptions above, if* $(\mathsf{dd}^{-1}) \not\subseteq r_{\mathsf{p}}$ *then* $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{d}}$ *or* $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{d}}$.

PROOF. We consider only the case $r_{\mathsf{p}} \cap (\mathsf{dd}^{-1}) = (\mathsf{d})$, since the other case, $r_{\mathsf{p}} \cap (\mathsf{dd}^{-1}) = (\mathsf{d}^{-1})$, is dual.

By Lemma 5.4, we have $(\mathsf{pp}^{-1}) \not\subseteq r_{\mathsf{p}}$. So, we have

$$(\mathsf{pd}) \subseteq r_{\mathsf{p}} \subseteq (\equiv \mathsf{pmm}^{-1}\mathsf{oo}^{-1}\mathsf{dss}^{-1}\mathsf{ff}^{-1}).$$

If $r_1 = r_{\mathsf{p}} \cap (\mathsf{oo}^{-1}\mathsf{s}^{-1}\mathsf{f}^{-1})$ is non-empty then using Lemmas 5.8 and 5.9 we easily get the required result; for example, $r_{\mathsf{p}} = (\mathsf{pod})$ implies $(\mathsf{dd}^{-1}) \not\subseteq r_{\mathsf{o}}$ and $r_{\mathsf{p}} = (\mathsf{pds}^{-1})$ implies $r_{\mathsf{s}} \cap (\mathsf{dd}^{-1}) = (\mathsf{d}^{-1})$. So we may assume that $(\mathsf{pd}) \subseteq r_{\mathsf{p}} \subseteq (\equiv \mathsf{pmm}^{-1}\mathsf{dsf})$. Then

$$(\mathsf{pd}) = (\mathsf{d}) \circ (\mathsf{pd}) \subseteq (\mathsf{d}) \circ r_{\mathsf{p}} \subseteq (\mathsf{d}) \circ (\equiv \mathsf{pmm}^{-1}\mathsf{dsf}) = (\mathsf{pp}^{-1}\mathsf{d}).$$

Since $(\mathsf{pp}^{-1}\mathsf{d}) \notin \mathcal{S}$ (otherwise its symmetric part (pp^{-1}) belongs to $\mathcal{S}$ which contradicts Lemma 5.4), we get $(\mathsf{pd}) = (\mathsf{d}) \circ r_{\mathsf{p}} \in \mathcal{S}$. This implies that $(\mathsf{pmods}) = (\mathsf{pd}) \circ (\mathsf{pd}) \in \mathcal{S}$ and hence $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{d}}$, by Lemma 5.6. □

LEMMA 5.11. *With the assumptions above, if* $(\mathsf{dd}^{-1}) \not\subseteq r_{\mathsf{m}}$ *then* $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{d}}$ *or* $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{d}}$.

PROOF. We consider only the case $r_{\mathsf{m}} \cap (\mathsf{dd}^{-1}) = (\mathsf{d})$, since the other case, $r_{\mathsf{m}} \cap (\mathsf{dd}^{-1}) = (\mathsf{d}^{-1})$, is dual.

As in the proof of the previous lemma, if $r_{\mathsf{m}} \cap (\mathsf{pp}^{-1}\mathsf{oo}^{-1}\mathsf{s}^{-1}\mathsf{f}^{-1})$ is non-empty then we get the required result by Lemmas 5.8, 5.9 and 5.10. So we may assume that $(\mathsf{md}) \subseteq r_{\mathsf{m}} \subseteq (\equiv \mathsf{mdsf})$. Then

$$(\mathsf{pd}) = (\mathsf{d}) \circ (\mathsf{md}) \subseteq (\mathsf{d}) \circ r_{\mathsf{m}} \subseteq (\mathsf{d}) \circ (\equiv \mathsf{mdsf}) = (\mathsf{pd}).$$

Thus, $(\mathsf{pd}) \in \mathcal{S}$. This implies that $(\mathsf{pmods}) = (\mathsf{pd}) \circ (\mathsf{pd}) \in \mathcal{S}$ and hence $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{d}}$, by Lemma 5.6. □

LEMMA 5.12. *With the assumptions above,* $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{d}}$ *or* $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{d}}$.

PROOF. If a subalgebra $\mathcal{S}$ satisfies none of the conditions of Lemmas 5.7-5.11 then, by Lemma 5.5, (d) is contained in all of the minimal relations r_{p}, r_{m}, r_{o}, and r_{s}. Therefore, every $r \in \mathcal{S}$ satisfies $r \cap (\mathsf{pmods}) \neq \emptyset \Rightarrow (\mathsf{d}) \subseteq r$, which precisely means that $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{d}}$. □

5.2 The case $bas(\mathcal{S}) = \{\equiv, \mathsf{o}, \mathsf{o}^{-1}\}$

In this subsection, we will show that if $\mathcal{S}$ is a subalgebra with $bas(\mathcal{S}) = \{\equiv, \mathsf{o}, \mathsf{o}^{-1}\}$, then $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{o}}$, $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{o}}$ or $\mathcal{S}$ is NP-complete.

To obtain this result we shall assume throughout this subsection that $\mathcal{S}$ is a subalgebra of $\mathcal{A}$ satisfying the following assumptions:

Assumption 1. $bas(\mathcal{S}) = \{\equiv, \mathsf{o}, \mathsf{o}^{-1}\}$.

Assumption 2. $\mathcal{S}$ is not NP-complete.

Using these assumptions we obtain increasingly detailed information about $\mathcal{S}$ in Lemmas 5.14-5.20, until we are able to show that in all cases we have $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{o}}$ or $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{o}}$. (The structure of this proof is quite similar to the proof of the case when $bas(\mathcal{S}) = \{\equiv, \mathsf{d}, \mathsf{d}^{-1}\}$, above.) These lemmas rely on the following NP-completeness result.

LEMMA 5.13. *The subset* $\{(\mathsf{o}), (\mathsf{dd}^{-1})\}$ *of* $\mathcal{A}$ *is NP-complete.*

PROOF. Apply Lemma 4.1 with $R = (\mathsf{d})$, $R_1 = (\mathsf{o})$, $R_2 = (\mathsf{dd}^{-1})$. □

In the proofs below, we will make frequent use of the fact that $\nu = (\mathsf{o}) \circ (\mathsf{o}^{-1}) \in \mathcal{S}$. Note also that $(\mathsf{pmo}) = (\mathsf{o}) \circ (\mathsf{o}) \in \mathcal{S}$ and the relation (pm) does not belong to $\mathcal{S}$ since, otherwise, $(\mathsf{p}) = (\mathsf{pm}) \circ (\mathsf{pm}) \in \mathcal{S}$, which contradicts Assumption 1. Therefore $r_{\mathsf{p}} = r_{\mathsf{m}} = (\mathsf{pmo})$ which implies that, for every $r \in \mathcal{S}$, if $r \cap (\mathsf{pmo}) \neq \emptyset$ then $(\mathsf{o}) \subseteq r$.

LEMMA 5.14. *With the assumptions above,*

$$sym(\mathcal{S}) \subseteq \{r^* | (\mathsf{oo}^{-1}) \subseteq r\} \cup \{r^* | r \subseteq (\equiv \mathsf{ss}^{-1}) \cup \{r^* | r \subseteq (\equiv \mathsf{ff}^{-1})\}.$$

PROOF. We will show that if $sym(\mathcal{S})$ is not included in the above set then $\mathcal{S}$ is NP-complete, which contradicts Assumption 2.

Suppose first that $(\equiv \mathsf{ss}^{-1}\mathsf{ff}^{-1})$ or $(\mathsf{ss}^{-1}\mathsf{ff}^{-1})$ belongs to $sym(\mathcal{S})$. Then

$$r = (\mathsf{o}) \circ (\equiv \mathsf{ss}^{-1}\mathsf{ff}^{-1}) = (\mathsf{o}) \circ (\mathsf{ss}^{-1}\mathsf{ff}^{-1}) = (\mathsf{pmodd}^{-1}\mathsf{sf}^{-1}) \in \mathcal{S},$$

so we have $(\mathsf{dd}^{-1}) = r^* \in \mathcal{S}$, which implies that $\mathcal{S}$ is NP-complete, by Lemma 5.13.

Suppose now that there exists $r^* \in sym(\mathcal{S})$ such that

$$r^* \subseteq (\equiv \mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{dd}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1}) \text{ and } r^* \not\subseteq (\equiv \mathsf{ss}^{-1}\mathsf{ff}^{-1}).$$

If $r_1 = r^* \cap (\mathsf{pp}^{-1}\mathsf{mm}^{-1})$ is non-empty then $r' = r^* \cap (\mathsf{pmo}) \subseteq (\mathsf{pm})$ implying that $(\mathsf{p}) = r' \circ r'$ belongs to $\mathcal{S}$, which contradicts Assumption 1. Therefore $(\mathsf{dd}^{-1}) \subseteq r^*$ since $r^* \not\subseteq (\equiv \mathsf{ss}^{-1}\mathsf{ff}^{-1})$. Now, it is easy to check that if $r_2 = ((\mathsf{o}) \circ r^*) \cap ((\mathsf{o}^{-1}) \circ r^*)$ then $r_2^* = (\mathsf{dd}^{-1}) \in \mathcal{S}$, which implies that $\mathcal{S}$ is NP-complete, by Lemma 5.13. □

LEMMA 5.15. *With the assumptions above,*

$$\mathcal{S} \subseteq \{r | r \cap (\mathsf{oo}^{-1}) \neq \emptyset\} \cup \{r | r \subseteq (\equiv \mathsf{ss}^{-1})\} \cup \{r | r \subseteq (\equiv \mathsf{ff}^{-1})\}.$$

PROOF. Assume for contradiction that $\mathcal{S}$ contains a relation r with $r \cap (\mathsf{oo}^{-1})$ empty, $r \not\subseteq (\equiv \mathsf{ss}^{-1})$ and $r \not\subseteq (\equiv \mathsf{ff}^{-1})$.

Among such relations, choose r to be minimal with respect to inclusion. Then, since, as noted above, $r \cap (\mathsf{pmo}) \neq \emptyset$ implies $(\mathsf{o}) \subseteq r$, we have $r \subseteq (\equiv \mathsf{dd}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1})$.

Assume first that $(\mathsf{d}) \subseteq r$ (the argument for (d^{-1}) is dual). Consider relation $r_1 = (\mathsf{o}) \circ r \subseteq (\mathsf{pmodd}^{-1}\mathsf{sf}^{-1})$. By the minimality of r, $r_1 \cap r = r$ or empty, but $(\mathsf{d}) \subseteq r$, so $(\mathsf{d}) \subseteq r_1$, so $r_1 \cap r$ is not empty and hence $r \subseteq (\mathsf{dd^{-1}sf^{-1}})$. Now consider $r_2 = (\mathsf{o}^{-1}) \circ r \subseteq (\mathsf{p^{-1}m^{-1}o^{-1}dd^{-1}s^{-1}f})$. By a similar argument we get $r \subseteq (\mathsf{dd^{-1}s^{-1}f})$. Combining these two results gives $r \subseteq (\mathsf{dd}^{-1})$, and then by Lemma 5.14 we get $r = (\mathsf{d})$, which contradicts Assumption 1.

Hence, we must have $r \subseteq (\equiv \mathsf{ss^{-1}ff^{-1}})$. If $(\mathsf{s^{-1}f}) \subseteq r$ then $((\mathsf{o}) \circ r)^* = (\mathsf{dd}^{-1}) \in \mathcal{S}$, which contradicts Lemma 5.14. If $r = (\equiv \mathsf{sf})$ or $r = (\mathsf{sf})$ then $r \cap ((\mathsf{o}) \circ r) = (\mathsf{s}) \in \mathcal{S}$, which contradicts Assumption 1. □

The proofs of the following two lemmas are omitted since they are very similar to the proofs of Lemmas 5.6 and 5.7, respectively.

LEMMA 5.16. *With the assumptions above, if* $(\mathsf{pmods}) \in \mathcal{S}$ *then* $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{o}}$*; and if* $(\mathsf{pmod^{-1}f^{-1}}) \in \mathcal{S}$ *then* $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{o}}$.

LEMMA 5.17. *With the assumptions above, if* $\mathcal{S}$ *contains a non-trivial relation* r *such that* $r \subseteq (\equiv \mathsf{ss}^{-1})$*, then* $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{o}}$*; if* $\mathcal{S}$ *contains a non-trivial relation* r *such that* $r \subseteq (\equiv \mathsf{ff}^{-1})$*, then* $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{o}}$.

In view of Lemma 5.17 and Lemma 5.15, it is sufficient to consider only cases such that for any non-trivial $r \in \mathcal{S}$, $r \cap (\mathsf{oo}^{-1})$ is non-empty.

LEMMA 5.18. *With the assumptions above, if* $(\mathsf{oo}^{-1}) \not\subseteq r_{\mathsf{d}}$ *then* $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{o}}$ *or* $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{o}}$.

PROOF. Suppose first that $r_{\mathsf{d}} \cap (\mathsf{oo}^{-1}) = (\mathsf{o})$. Then $(\mathsf{dd}^{-1}) \not\subseteq r_{\mathsf{d}}$, since the opposite would contradict Lemma 5.14. By the minimality of r_{d}, $r_{\mathsf{d}} \cap \nu = r_{\mathsf{d}}$ so $r_1 = (\mathsf{od}) \subseteq r_{\mathsf{d}} \subseteq (\equiv \mathsf{odss^{-1}ff^{-1}}) = r_2$. Consequently, $(\mathsf{o}) \circ r_1 \subseteq (\mathsf{o}) \circ r_{\mathsf{d}} \subseteq (\mathsf{o}) \circ r_2$, that is, $(\mathsf{pmods}) \subseteq (\mathsf{o}) \circ r_{\mathsf{d}} \subseteq (\mathsf{pmodd^{-1}sf^{-1}})$. Then, by minimality of r_{d}, we have $(\mathsf{od}) \subseteq r_{\mathsf{d}} \subseteq (\mathsf{odsf}^{-1})$. Finally,

$$(\mathsf{pmods}) = (\mathsf{o}) \circ (\mathsf{od}) \subseteq (\mathsf{o}) \circ r_{\mathsf{d}} \subseteq (\mathsf{o}) \circ (\mathsf{odsf}^{-1}) = (\mathsf{pmods}),$$

that is, $(\mathsf{pmods}) = (\mathsf{o}) \circ r_{\mathsf{d}} \in \mathcal{S}$, which implies, by Lemma 5.16, that $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{o}}$.

Dual calculations show that if $r_{\mathsf{d}} \cap (\mathsf{oo}^{-1}) = (\mathsf{o}^{-1})$ then $(\mathsf{pmod^{-1}f^{-1}}) \in \mathcal{S}$, and hence $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{o}}$, by Lemma 5.16. □

LEMMA 5.19. *If* $r_{\mathsf{s}} \cap (\mathsf{oo}^{-1}) = (\mathsf{o}^{-1})$ *then* $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{o}}$ *or* $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{o}}$.

PROOF. We have $(\mathsf{ss}^{-1}) \not\subseteq r_{\mathsf{d}}$, since the opposite would contradict Lemma 5.14. By the minimality of r_{d}, $r_{\mathsf{d}} \cap \nu = r_{\mathsf{d}}$ so $(\mathsf{o^{-1}s}) \subseteq r_{\mathsf{s}} \subseteq (\equiv \mathsf{o^{-1}dd^{-1}sff^{-1}})$.

Suppose first that $r_{\mathsf{s}} \cap (\mathsf{dd}^{-1})$ is non-empty. Then, $(\mathsf{oo}^{-1}) \not\subseteq r_{\mathsf{d}}$ and $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{o}}$ or $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{o}}$ by Lemma 5.18.

Suppose to the contrary that $r_{\mathsf{s}} \cap (\mathsf{dd}^{-1}) = \emptyset$. Then $(\mathsf{o^{-1}s}) \subseteq r_{\mathsf{s}} \subseteq (\equiv \mathsf{o^{-1}sff^{-1}})$ and

$$(\mathsf{o}^{-1}) \circ (\mathsf{o^{-1}s}) \subseteq (\mathsf{o}^{-1}) \circ r_{\mathsf{s}} \subseteq (\mathsf{o}^{-1}) \circ (\equiv \mathsf{o^{-1}sff^{-1}}),$$

that is,

$$(\mathsf{p^{-1}m^{-1}o^{-1}df}) \subseteq (\mathsf{o}^{-1}) \circ r_{\mathsf{s}} \subseteq (\mathsf{p^{-1}m^{-1}o^{-1}dd^{-1}s^{-1}f}).$$

Hence $(\mathsf{oo}^{-1}) \not\subseteq r_{\mathsf{d}}$, so the result follows by Lemma 5.18. □

LEMMA 5.20. *With the assumptions above,* $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{o}}$ *or* $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{o}}$.

PROOF. If $\mathcal{S}$ satisfies none of the conditions of Lemmas 5.17-5.19, then, by Lemma 5.15, (o) is contained in both r_{d} and r_{s}. Since, as we noted in the beginning of this subsection, (o) is also contained in both r_{p} and r_{m}, we conclude that $\mathcal{S}$ is contained in $\mathcal{E}_{\mathsf{o}}$. $\square$

5.3 The case $bas(\mathcal{S}) = \{\equiv, \mathsf{s}, \mathsf{s}^{-1}\}$ or $bas(\mathcal{S}) = \{\equiv, \mathsf{f}, \mathsf{f}^{-1}\}$

In this subsection, we will show that if $bas(\mathcal{S}) = \{\equiv, \mathsf{s}, \mathsf{s}^{-1}\}$, then either $\mathcal{S}$ is NP-complete or $\mathcal{S}$ is contained in one of the subalgebras $\mathcal{H}$, $\mathcal{S}_{\mathsf{d}}$, $\mathcal{S}_{\mathsf{o}}$, $\mathcal{S}_{\mathsf{p}}$, $\mathcal{E}^*$, or in one of $\mathcal{A}_i$, $1 \leq i \leq 4$. By using the obvious symmetry between the relations (s) and (f), it immediately follows that if $bas(\mathcal{S}) = \{\equiv, \mathsf{f}, \mathsf{f}^{-1}\}$, then either $\mathcal{S}$ is NP-complete or contained in one of $\mathcal{H}$, $\mathcal{E}_{\mathsf{d}}$, $\mathcal{E}_{\mathsf{o}}$, $\mathcal{E}_{\mathsf{p}}$, $\mathcal{S}^*$ or $\mathcal{B}_i$, $1 \leq i \leq 4$.

To obtain this result we shall assume throughout this subsection that $\mathcal{S}$ is a subalgebra of $\mathcal{A}$ satisfying the following assumptions:

Assumption 1. $bas(\mathcal{S}) = \{\equiv, \mathsf{s}, \mathsf{s}^{-1}\}$.

Assumption 2. $\mathcal{S}$ is not NP-complete.

Using these assumptions we obtain increasingly detailed information about $\mathcal{S}$ in Lemmas 5.22-5.29, until we are able to obtain the result. These lemmas rely on the following NP-completeness result.

LEMMA 5.21. *The subset $\{r\}$ of $\mathcal{A}$ is NP-complete whenever*

$$(\mathsf{ods}^{-1}) \subseteq r \subseteq (\mathsf{pmods}^{-1}\mathsf{f}^{-1}).$$

PROOF. Let r_3 be the union of all basic relations except for $\equiv$ and s^{-1}, and consider the instance $\Gamma_4 = \{xra, xrb, yrb, ary, bra\}$ over the variables x, y, a, b. In the cases when $r = (\mathsf{ods}^{-1})$ or $r = (\mathsf{pmods}^{-1}\mathsf{f}^{-1})$, it can be shown that $\Gamma_4 \cup \{xr'y\}$ is satisfiable for every basic relation $r' \subseteq r_3$ but not satisfiable for any other choice of r'. It follows that, for every r such that $(\mathsf{ods}^{-1}) \subseteq r \subseteq (\mathsf{pmods}^{-1}\mathsf{f}^{-1})$, we can derive r_3 from r. Further, we can derive the relation $r_4 = r \cap r_3$ satisfying $(\mathsf{od}) \subseteq r_4 \subseteq (\mathsf{pmodf}^{-1})$, and the relation $r_5 = r_4 \circ r_4$. It is easy to check that

$$(\mathsf{pmods}) \subseteq r_5 \subseteq (\mathsf{pmodsf}^{-1})$$

which implies that $r_5 \cap r^{-1} = (\mathsf{s})$. Furthermore, $(\mathsf{s}) \circ (\mathsf{s}^{-1}) = (\equiv \mathsf{ss}^{-1})$ and $r \circ r$ is the disequality relation, so the relation (ss^{-1}) can be obtained from the relation r. If $R = (\mathsf{s})$, $R_1 = r^{-1}$ and $R_2 = r$, then these relations satisfy the conditions of Lemma 4.1 so $\{(\mathsf{ss}^{-1}), r^{-1}, r\}$ is NP-complete. Since all of these relations can be derived from the single relation r, it follows that $\{r\}$ is NP-complete. $\square$

LEMMA 5.22. *With the assumptions above, if $\mathcal{S}$ contains the relation (od), then every $r \in \mathcal{S}$ satisfies condition 3) of $\mathcal{H}$.*

PROOF. Arbitrarily choose $r \in \mathcal{S}$. Since $(\mathsf{od}) \in \mathcal{S}$, it follows that $r_{\mathsf{d}} = r_{\mathsf{o}} = (\mathsf{od})$ and $(\mathsf{o})^{\pm 1} \subseteq r \Leftrightarrow (\mathsf{d})^{\pm 1} \subseteq r$. Furthermore, $r_1 = (\mathsf{s}) \circ (\mathsf{od}) = (\mathsf{pmod}) \in \mathcal{S}$ so $r_{\mathsf{p}} \subseteq r_1$ and $r_{\mathsf{m}} \subseteq r_1$. Since $(\mathsf{pm}) \circ (\mathsf{pm}) = (\mathsf{p})$, we have $r_{\mathsf{p}} \not\subseteq (\mathsf{pm})$ and $r_{\mathsf{m}} \not\subseteq (\mathsf{pm})$, and therefore $r_{\mathsf{p}} \cap (\mathsf{od}) \neq \emptyset$ and $r_{\mathsf{m}} \cap (\mathsf{od}) \neq \emptyset$. Now it follows that $(\mathsf{od}) \subseteq r_{\mathsf{p}}$ and $(\mathsf{od}) \subseteq r_{\mathsf{m}}$. Thus, if $r \cap (\mathsf{pm}) \neq \emptyset$, then $(\mathsf{od}) \subseteq r$ which means that r satisfies condition 3) of $\mathcal{H}$. $\square$

LEMMA 5.23. *With the assumptions above, if $\mathcal{S}$ contains a non-trivial relation r' with $r' \subseteq (\equiv \mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{ff}^{-1})$ then $\mathcal{S}$ is included in one of $\mathcal{H}$, $\mathcal{E}^*$, or $\mathcal{S}_\mathsf{p}$.*

PROOF. Case 1. $r' = (\mathsf{ff}^{-1})$
Since $(\mathsf{ff}^{-1}) \circ (\mathsf{s}) = (\mathsf{od}) \in \mathcal{S}$, it follows that any $r \in \mathcal{S}$ satisfies condition 3) of $\mathcal{H}$ by Lemma 5.22. Note also that now, for every $r \in \mathcal{S}$, we have

$$(\mathsf{o})^{\pm 1} \subseteq r \Leftrightarrow (\mathsf{d})^{\pm 1} \subseteq r \text{ and } (\mathsf{f}) \subseteq r \Leftrightarrow (\mathsf{f}^{-1}) \subseteq r. \tag{1}$$

Suppose that $\mathcal{S} \not\subseteq \mathcal{H}$, i.e. some $r \in \mathcal{S}$ fails to satisfy condition 1) or condition 2) of $\mathcal{H}$. Then, using the conditions (1) from the previous paragraph, it is not hard to check that the relation r can be chosen so that

$$(\mathsf{s}^{-1}\mathsf{ff}^{-1}) \subseteq r \subseteq (\equiv \mathsf{pmodss}^{-1}\mathsf{ff}^{-1}) \text{ or } (\mathsf{ods}^{-1}) \subseteq r \subseteq (\equiv \mathsf{pmodss}^{-1}\mathsf{ff}^{-1}).$$

In both cases, multiplying the relations by (s) from the left we get

$$(\equiv \mathsf{pmodss}^{-1}) \subseteq (\mathsf{s}) \circ r \subseteq (\equiv \mathsf{pmodss}^{-1}).$$

Therefore $(\equiv \mathsf{pmodss}^{-1}) \in \mathcal{S}$, and

$$(\mathsf{pmods}^{-1}) = (\equiv \mathsf{pmodss}^{-1}) \cap ((\equiv \mathsf{pmodss}^{-1})^{-1} \circ (\mathsf{od})^{-1}) \in \mathcal{S},$$

so $\mathcal{S}$ is NP-complete by Lemma 5.21, which contradicts Assumption 2.
Case 2. $r' \subseteq (\equiv \mathsf{ff}^{-1})$.
Multiplying r' and its inverse we get $(\equiv \mathsf{ff}^{-1})$, so we may assume that $r' = (\equiv \mathsf{ff}^{-1})$. If some relation $r_2 \in \mathcal{S}$ fails to satisfy condition 2) of $\mathcal{E}^*$ then $r_2 \cap r'$ is either $(\mathsf{f})^{\pm 1}$, which is impossible, or (ff^{-1}) going back to Case 1. Suppose now that each $r \in \mathcal{S}$ satisfies condition 2) of $\mathcal{E}^*$.

We have $r' \circ (\mathsf{s}) = (\mathsf{ods}) \in \mathcal{S}$. If $(\mathsf{od}) \in \mathcal{S}$ then $r' \cap ((\mathsf{s}^{-1}) \circ (\mathsf{od})) = (\mathsf{ff}^{-1}) \in \mathcal{S}$, which implies $\mathcal{S} \subseteq \mathcal{H}$ by Case 1. Suppose $(\mathsf{od}) \notin \mathcal{S}$. Then, for every $r \in \mathcal{S}$, we have $r \cap (\mathsf{od})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{s})^{\pm 1} \subseteq r$. Assume a relation $r_3 \in \mathcal{S}$ does not satisfy condition 1) of $\mathcal{E}^*$, that is, $r_3 \cap (\mathsf{pmod}) \neq \emptyset$ and $(\mathsf{s}) \not\subseteq r_3$. Since $(\mathsf{pmods}) = (\mathsf{s}) \circ r' \in \mathcal{S}$, we have $r_4 = r_3 \cap (\mathsf{pmods}) \in \mathcal{S}$ and $r_4 \subseteq (\mathsf{pm})$. This implies $r_4 \circ r_4 = (\mathsf{p})$, which contradicts Assumption 1. Therefore the relations in $\mathcal{S}$ must satisfy both conditions of $\mathcal{E}^*$, that is, $\mathcal{S} \subseteq \mathcal{E}^*$.
Case 3. $r' \subseteq (\equiv \mathsf{mm}^{-1}\mathsf{ff}^{-1})$ and $r' \cap (\mathsf{mm}^{-1}) \neq \emptyset$.
Without loss of generality we may assume that $(\mathsf{m}) \subseteq r$. Then we have

$$(\mathsf{m}) \subseteq r_1 = (r' \cap (r' \circ (\mathsf{s}^{-1})) \subseteq (\mathsf{mm}^{-1})$$

so $(\mathsf{m}) = r_1 \cap (r_1 \circ (\mathsf{s}^{-1})) \in \mathcal{S}$, which contradicts Assumption 1.
Case 4. $r' \cap (\mathsf{pp}^{-1}) \neq \emptyset$.
Assume without loss of generality that $(\mathsf{p}) \subseteq r'$. Then

$$(\mathsf{p}) \subseteq (\mathsf{s}) \circ r' \subseteq (\mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{ods}).$$

Further,

$$(\mathsf{p}) \subseteq r_5 = r' \cap ((\mathsf{s}) \circ r') \subseteq (\mathsf{pp}^{-1}\mathsf{mm}^{-1})$$

and

$$(\mathsf{p}) \subseteq r_6 = r_5 \circ (\mathsf{s}^{-1}) \subseteq (\mathsf{pp}^{-1}\mathsf{m}).$$

If $(\mathsf{p}^{-1}) \not\subseteq r_6$ then $r_6 \circ r_6 = (\mathsf{p}) \in \mathcal{S}$, which contradicts Assumption 1. Otherwise $(\mathsf{pp}^{-1}) = r_6^* \in \mathcal{S}$. Then $(\mathsf{p}) \subseteq r \Leftrightarrow (\mathsf{p}^{-1}) \subseteq r$ holds for every $r \in \mathcal{S}$.

We have $(\mathsf{s}^{-1}) \circ (\mathsf{pp}^{-1}) = (\mathsf{pp}^{-1}\mathsf{mod}^{-1}\mathsf{f}^{-1}) \in \mathcal{S}$. For every non-empty r_7 with $r_7 \subseteq (\mathsf{mod}^{-1}\mathsf{f}^{-1})$, we have $((\mathsf{s}) \circ r_7) \cap (\mathsf{pp}^{-1}) = (\mathsf{p})$. Therefore no such r_7 belongs to $\mathcal{S}$. We conclude that, for any $r \in \mathcal{S}$, if $r \cap (\mathsf{mod}^{-1}\mathsf{f}^{-1}) \neq \emptyset$ then $(\mathsf{p}) \subseteq r$, which means that $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{p}}$. □

In view of Lemma 5.23, it is now sufficient to consider cases where the following additional property holds:

Assumption 3. For every non-trivial $r \in \mathcal{S}$, we have $r \cap (\mathsf{oo}^{-1}\mathsf{dd}^{-1}\mathsf{ss}^{-1}) \neq \emptyset$.

Lemma 5.24. *With the assumptions above, if* $r_{\mathsf{d}} \cap (\mathsf{oo}^{-1}\mathsf{dd}^{-1}\mathsf{ss}^{-1}) = (\mathsf{d})$ *or* $(\mathsf{dd}^{-1}) \subseteq r_{\mathsf{d}}$ *then* $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{d}}$.

Proof. Case 1. $r_{\mathsf{d}} \cap (\mathsf{oo}^{-1}\mathsf{dd}^{-1}\mathsf{ss}^{-1}) = (\mathsf{d})$.
By assumption, the relation r_{d} satisfies condition $(\mathsf{d}) \subseteq r_{\mathsf{d}} \subseteq (\equiv \mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{dff}^{-1})$. Let r_1 be calculated as $r_{\mathsf{d}} \cap ((\mathsf{s}) \circ r_{\mathsf{d}})$. Then we have $(\mathsf{d}) \subseteq r_1 \subseteq (\mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{d})$. By minimality of r_{d} we get $(\mathsf{d}) \subseteq r_{\mathsf{d}} \subseteq (\mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{d})$. Calculating r_1 again, we get $(\mathsf{d}) \subseteq r_{\mathsf{d}} \subseteq (\mathsf{pp}^{-1}\mathsf{m}^{-1}\mathsf{d})$. By Assumption 3, we have $(\mathsf{pp}^{-1}) \notin \mathcal{S}$. Furthermore, since we have $(\mathsf{m}^{-1}\mathsf{d}) \subseteq r_{\mathsf{d}} \circ (\mathsf{s})$ only if $(\mathsf{p}^{-1}) \subseteq r_{\mathsf{d}}$, we may assume that r_{d} is either (pd), or $(\mathsf{p}^{-1}\mathsf{d})$, or $(\mathsf{p}^{-1}\mathsf{m}^{-1}\mathsf{d})$. In the first case we get $(\mathsf{pd}) \cap ((\mathsf{p}^{-1}\mathsf{d}^{-1}) \circ (\mathsf{s})) = (\mathsf{d})$, which contradicts Assumption 1. Let $(\mathsf{p}^{-1}\mathsf{d}) \subseteq r_{\mathsf{d}} \subseteq (\mathsf{p}^{-1}\mathsf{m}^{-1}\mathsf{d})$. Then we have $(\mathsf{p}^{-1}\mathsf{m}^{-1}\mathsf{o}^{-1}\mathsf{df}) = r_{\mathsf{d}} \circ (\mathsf{s}) \in \mathcal{S}$. Suppose $\mathcal{S}$ contains a non-empty subrelation r_2 of $(\mathsf{p}^{-1}\mathsf{m}^{-1}\mathsf{o}^{-1}\mathsf{f})$. Then $r_3 = r_{\mathsf{d}} \cap (r_2 \circ (\mathsf{s}^{-1}))$ is a non-empty subrelation of $(\mathsf{p}^{-1}\mathsf{m}^{-1})$ implying that $(\mathsf{p}^{-1}) = r_3 \circ r_3 \in \mathcal{S}$, which contradicts Assumption 1. Therefore , for every $r \in \mathcal{S}$, we have $r \cap (\mathsf{p}^{-1}\mathsf{m}^{-1}\mathsf{o}^{-1}\mathsf{f}) \neq \emptyset \Rightarrow (\mathsf{d}) \subseteq r$, which means that $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{d}}$.
Case 2. $(\mathsf{dd}^{-1}) \subseteq r_{\mathsf{d}}$.
Suppose a relation $r_4 \in \mathcal{S}$ satisfies $(\mathsf{dd}^{-1}) \not\subseteq r_4$. Then it is not hard to verify that if $r_4 \not\subseteq (\equiv \mathsf{ss}^{-1})$ then either $((\mathsf{s}^{-1}) \circ r_4) \cap r_{\mathsf{d}}$ or $(r_4 \circ (\mathsf{s})) \cap r_{\mathsf{d}}$ contains exactly one of (d) and (d^{-1}) which contradicts the minimality of r_{d}. Thus, for every $r \in \mathcal{S}$ such that $r \not\subseteq (\equiv \mathsf{ss}^{-1})$, we have $(\mathsf{dd}^{-1}) \subseteq r$. This implies that $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{d}}$. □

Lemma 5.25. *With the assumptions above, if* $r_{\mathsf{o}} \cap (\mathsf{oo}^{-1}\mathsf{dd}^{-1}\mathsf{ss}^{-1}) = (\mathsf{o})$ *or* $(\mathsf{oo}^{-1}) \subseteq r_{\mathsf{o}}$ *then* $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{o}}$.

Proof. Similar to the previous lemma. □

Lemma 5.26. *With the assumptions above, if* $r_{\mathsf{d}} \cap (\mathsf{oo}^{-1}\mathsf{dd}^{-1}\mathsf{ss}^{-1}) = (\mathsf{o}^{-1}\mathsf{d})$ *and* $r_{\mathsf{d}} = r_{\mathsf{o}^{-1}}$*, then* $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{d}}$.

Proof. We can obtain $(\mathsf{o}^{-1}\mathsf{d}) \subseteq r_{\mathsf{d}} \subseteq (\mathsf{pp}^{-1}\mathsf{m}^{-1}\mathsf{o}^{-1}\mathsf{df})$ as in the proof of Lemma 5.24, If $(\mathsf{po}^{-1}\mathsf{d}) \subseteq r_{\mathsf{d}}$ then $(r_{\mathsf{d}} \circ (\mathsf{s}^{-1}))^* = (\mathsf{pp}^{-1}) \in \mathcal{S}$, and the result follows from Lemma 5.23. Therefore we have $(\mathsf{o}^{-1}\mathsf{d}) \subseteq r_{\mathsf{d}} \subseteq (\mathsf{p}^{-1}\mathsf{m}^{-1}\mathsf{o}^{-1}\mathsf{df})$, and $(\mathsf{p}^{-1}\mathsf{m}^{-1}\mathsf{o}^{-1}\mathsf{df}) = r_{\mathsf{d}} \circ (\mathsf{s}^{-1}) \in \mathcal{S}$. By Assumption 3, no non-empty $r_1 \subseteq (\mathsf{p}^{-1}\mathsf{m}^{-1}\mathsf{f})$ belongs to $\mathcal{S}$, so, for every $r \in \mathcal{S}$, we have $r \cap (\mathsf{p}^{-1}\mathsf{m}^{-1}\mathsf{o}^{-1}\mathsf{f}) \neq \emptyset \Rightarrow (\mathsf{d}) \subseteq r$, which means that $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{d}}$. □

Lemma 5.27. *With the assumptions above, if* $r_{\mathsf{d}} \cap (\mathsf{oo}^{-1}\mathsf{dd}^{-1}\mathsf{ss}^{-1}) = (\mathsf{od})$ *and* $r_{\mathsf{d}} = r_{\mathsf{o}}$*, then* $\mathcal{S} \subseteq \mathcal{H}$.

PROOF. Again, it can be shown that $(\mathsf{od}) \subseteq r_{\mathsf{d}} \subseteq (\mathsf{pp^{-1}mm^{-1}od})$, as in the previous lemmas. Then $r^*_{\mathsf{d}} = \emptyset$ and $(\mathsf{pmod}) = (\mathsf{s}) \circ r_{\mathsf{d}} \in \mathcal{S}$. Further, we have $(\mathsf{oo^{-1}dd^{-1}ff^{-1}}) = ((\mathsf{s}^{-1}) \circ (\mathsf{pmod}))^* \in \mathcal{S}$ and $(\mathsf{od}) = (\mathsf{pmod}) \cap (\mathsf{oo^{-1}dd^{-1}ff^{-1}}) \in \mathcal{S}$. By Lemma 5.22, we know that every $r \in \mathcal{S}$ satisfies condition 3) of $\mathcal{H}$. Suppose some $r_1 \in \mathcal{S}$ does not satisfy condition 1) of $\mathcal{H}$. Then r_1 can be chosen so that

$$(\mathsf{s^{-1}f^{-1}}) \subseteq r_1 \subseteq (\equiv \mathsf{pmodss^{-1}ff^{-1}}) \text{ or } (\mathsf{os^{-1}}) \subseteq r_1 \subseteq (\equiv \mathsf{pmodss^{-1}ff^{-1}}).$$

If $(\mathsf{f}) \subseteq r_1$ then $(\mathsf{ff^{-1}}) = (r_1 \cap (\mathsf{oo^{-1}dd^{-1}ff^{-1}}))^* \in \mathcal{S}$, which contradicts Assumption 3. Further, if $(\mathsf{od}) \not\subseteq r_1$ then $r_1 \cap (\mathsf{oo^{-1}dd^{-1}ff^{-1}}) = (\mathsf{f}^{-1}) \in \mathcal{S}$, which contradicts Assumption 1. Therefore we may assume that $(\mathsf{ods^{-1}f^{-1}}) \subseteq r_1 \subseteq (\equiv \mathsf{pmodss^{-1}f^{-1}})$. Let $r_2 = r_1 \cap (r_1^{-1} \circ (\mathsf{o^{-1}d^{-1}}))$. Now it can be checked that relation r_2 satisfies $(\mathsf{ods^{-1}f^{-1}}) \subseteq r_2 \subseteq (\mathsf{pmods^{-1}f^{-1}})$, so $\{r_2\}$ is NP-complete by Lemma 5.21, which contradicts Assumption 2.

One can proceed similarly if condition 2) of $\mathcal{H}$ fails in $\mathcal{S}$. □

LEMMA 5.28. *With the assumptions above, if $r \cap (\mathsf{ss^{-1}}) \neq \emptyset$ for each non-trivial $r \in \mathcal{S}$ then $\mathcal{S} \subseteq \mathcal{A}_i$ for some $1 \leq i \leq 4$.*

PROOF. Case 1. $r_{\mathsf{p}} \cap (\mathsf{ss^{-1}}) = (\mathsf{s}^{-1})$.
We have $(\mathsf{ps^{-1}}) \subseteq r_{\mathsf{p}}$ and $(\mathsf{s}) \not\subseteq r_{\mathsf{p}}$. Let $r_1 = (\mathsf{s}^{-1}) \circ r_{\mathsf{p}} \in \mathcal{S}$. Then it is easy to check that $(\mathsf{pmod^{-1}s^{-1}f^{-1}}) \subseteq r_1$, and that $r_1 \cap (\equiv \mathsf{s}) = \emptyset$. It follows that $r_1 = (\mathsf{pmod^{-1}s^{-1}f^{-1}})$, since otherwise r_1^* is non-empty and $r_1^* \cap (\mathsf{ss^{-1}}) = \emptyset$. No non-empty subrelation of $(\mathsf{pmod^{-1}f^{-1}})$ can belong to $\mathcal{S}$. Therefore, for any $r \in \mathcal{S}$, we have $r \cap (\mathsf{pmod^{-1}f^{-1}}) \neq \emptyset \Rightarrow (\mathsf{s}^{-1}) \subseteq r$. Hence $\mathcal{S} \subseteq \mathcal{A}_1$.
Case 2. $r_{\mathsf{d}} \cap (\mathsf{ss^{-1}}) = (\mathsf{s}^{-1})$.
The proof is similar to Case 1; the only change is that $r_1 = r_{\mathsf{d}} \circ (\mathsf{s}^{-1})$, and we deduce that $r_1 = (\mathsf{p^{-1}m^{-1}o^{-1}ds^{-1}f})$, and, hence, $\mathcal{S} \subseteq \mathcal{A}_2$.
Case 3. $r_{\mathsf{o}} \cap (\mathsf{ss^{-1}}) = (\mathsf{s}^{-1})$.
In view of Cases 1 and 2 we may assume that

$$(\mathsf{os^{-1}}) \subseteq r_{\mathsf{o}} \subseteq (\equiv \mathsf{p^{-1}mm^{-1}od^{-1}s^{-1}ff^{-1}}).$$

Let $r_1 = (\mathsf{s}^{-1}) \circ r_{\mathsf{o}} \in \mathcal{S}$. It is easy to check that r_1 satisfies

$$(\mathsf{od^{-1}s^{-1}f^{-1}}) \subseteq r_1 \subseteq (\mathsf{p^{-1}m^{-1}oo^{-1}d^{-1}s^{-1}f^{-1}}).$$

Since $r_1^* \neq (\mathsf{oo^{-1}})$, we obtain $r_1 \subseteq (\mathsf{p^{-1}m^{-1}od^{-1}s^{-1}f^{-1}})$. It can straightforwardly be verified that if $r_1 \neq (\mathsf{od^{-1}s^{-1}f^{-1}})$ then relation $r_2 = (r_1 \circ r_1)^*$ contains $(\mathsf{pp^{-1}})$ and $r_2 \cap (\mathsf{ss^{-1}}) = \emptyset$, which contradicts the assumptions made. Therefore we have $r_1 = (\mathsf{od^{-1}s^{-1}f^{-1}})$, and $(\mathsf{pmod^{-1}s^{-1}f^{-1}}) = r_1 \circ r_1 \in \mathcal{S}$. Therefore, for any $r \in \mathcal{S}$, $r \cap (\mathsf{pmod^{-1}f^{-1}}) \neq \emptyset$ implies $(\mathsf{s}^{-1}) \subseteq r$, that is, $\mathcal{S} \subseteq \mathcal{A}_1$.
Case 4. $r_{\mathsf{m}} \cap (\mathsf{ss^{-1}}) = (\mathsf{s}^{-1})$.
Similarly to Case 3, we infer that $\mathcal{S} \subseteq \mathcal{A}_1$.
Case 5. (s) is contained in each of r_{p}, r_{d}, r_{o}, and r_{m}.
We have $r_{\mathsf{f}} \cap (\mathsf{ss^{-1}}) \neq \emptyset$. Then it follows that if $(\mathsf{s}) \subseteq r_{\mathsf{f}}$ then $\mathcal{S} \subseteq \mathcal{A}_3$. Otherwise, $(\mathsf{s}^{-1}) \subseteq r_{\mathsf{f}}$ and we have $\mathcal{S} \subseteq \mathcal{A}_4$. □

LEMMA 5.29. *With the assumptions above, $\mathcal{S}$ is contained in one of the subalgebras $\mathcal{H}$, $\mathcal{S}_{\mathsf{d}}$, $\mathcal{S}_{\mathsf{o}}$, $\mathcal{S}_{\mathsf{p}}$, $\mathcal{E}^*$, or in one of $\mathcal{A}_i$, $1 \leq i \leq 4$.*

PROOF. By Lemma 5.28 it suffices to consider the case when $\mathcal{S}$ contains a non-trivial relation r with $r \cap (\mathsf{ss}^{-1}) = \emptyset$.

By Lemma 5.23 we can assume that, for every non-trivial relation $r \in \mathcal{S}$, we have $r \cap (\mathsf{oo}^{-1}\mathsf{dd}^{-1}\mathsf{ss}^{-1}) \neq \emptyset$, so it suffices to consider cases when $r_{\mathsf{d}} \cap (\mathsf{ss}^{-1}) = \emptyset$ or $r_{\mathsf{o}} \cap (\mathsf{ss}^{-1}) = \emptyset$.

We claim that the result now follows from Lemmas 5.24-5.27. To establish this claim suppose that $r_{\mathsf{d}} \cap (\mathsf{ss}^{-1}) = \emptyset$, but r_{d} does not satisfy the conditions of Lemma 5.24. Then we have $(\mathsf{d}^{-1}) \not\subseteq r_{\mathsf{d}}$ and $r_{\mathsf{d}} \cap (\mathsf{oo}^{-1}) \neq \emptyset$. If $(\mathsf{oo}^{-1}) \subseteq r_{\mathsf{d}}$ then $r^*_{\mathsf{d}} \cap (\mathsf{oo}^{-1}\mathsf{dd}^{-1}) = (\mathsf{oo}^{-1})$, which implies that r_{o} satisfies the conditions of Lemma 5.25 and $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{o}}$. Otherwise we have $r_{\mathsf{o}} \subseteq r_{\mathsf{d}}$ or $r_{\mathsf{o}^{-1}} \subseteq r_{\mathsf{d}}$. If both of these inclusions are proper then it is easy to see that $r_{\mathsf{o}} \cap (\mathsf{oo}^{-1}\mathsf{dd}^{-1}) = (\mathsf{o})$ and, hence, $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{o}}$ by Lemma 5.25. Thus we only need to consider two cases: $r_{\mathsf{d}} = r_{\mathsf{o}^{-1}}$ and $r_{\mathsf{d}} = r_{\mathsf{o}}$ which are dealt with in Lemmas 5.26 and 5.27. The case when $r_{\mathsf{o}} \cap (\mathsf{ss}^{-1}) \neq \emptyset$ but r_{o} does not satisfy the conditions of Lemma 5.25 is similar and it is sufficient to consider the same two cases. □

6. SUBALGEBRAS WITH ONLY TRIVIAL BASIC RELATIONS

In this section, we consider subalgebras $\mathcal{S}$ of $\mathcal{A}$ such that $bas(\mathcal{S}) = \{(\equiv)\}$. We can assume that $\mathcal{S}$ contains a relation r' such that $(\equiv) \not\subseteq r'$; otherwise $\mathcal{S} \subseteq \mathcal{A}_{\equiv}$.

A relation r is *symmetric* if $r^* = r$ and it is *asymmetric* if $r^* = \emptyset$. If we choose the relation r' to be minimal, then this implies that r' is either asymmetric or symmetric. We consider these two cases in Subsections 6.1 and 6.2, respectively.

6.1 Asymmetric relations

In this subsection we prove the following proposition.

PROPOSITION 6.1. *Let $\mathcal{S}$ be a subalgebra of $\mathcal{A}$ such that $bas(\mathcal{S}) = \{\equiv\}$, which contains an asymmetric relation. Then $\mathcal{S}$ is tractable if it is contained in one of the 18 algebras listed in Table III. Otherwise $\mathcal{S}$ is NP-complete.*

To obtain this result we shall assume throughout this subsection that $\mathcal{S}$ is a subalgebra of $\mathcal{A}$ satisfying the following assumptions:

Assumption 1. $bas(\mathcal{S}) = \{\equiv\}$.

Assumption 2. $\mathcal{S}$ is not NP-complete.

Assumption 3. $r' \in \mathcal{S}$ is an asymmetric relation.

We first show that r' must have a very restricted form, and then show that the result holds for all possible cases in Lemmas 6.5-6.10.

A relation $r \in \mathcal{A}$ is said to be *acyclic* if, for every $k > 1$, the instance

$$x_1 r x_2, x_2 r x_3, \ldots, x_{k-1} r x_k, x_k r x_1$$

has no model. The acyclic relations are characterised in [Drakengren and Jonsson 1997b].

LEMMA 6.2 [DRAKENGREN AND JONSSON 1997B]. *A relation $r \in \mathcal{A}$ is acyclic if and only if r or r^{-1} is a subset of one of the following relations:*

$$(\mathsf{pmod}^{-1}\mathsf{sf}^{-1}),\ (\mathsf{pmod}^{-1}\mathsf{s}^{-1}\mathsf{f}^{-1}),\ (\mathsf{pmodsf})\ \textit{or}\ (\mathsf{pmodsf}^{-1}).$$

PROPOSITION 6.3. *If r is asymmetric, but not acyclic, then $\{r\}$ is NP-complete.*

The proof of this proposition can be found in Subsection A.2. By using this result, we can now restrict our attention to cases where r' is an acyclic relation. To complete the proof of Proposition 6.1, we will now consider all acyclic relations. The proofs rely on the following NP-completeness results.

LEMMA 6.4. *The following sets of relations are NP-complete:*
1) $\{(\mathsf{oo}^{-1}), r\}$ *where* $(\mathsf{d}) \subseteq r \subseteq (\mathsf{dsf})$*;*
2) $\{(\mathsf{dd}^{-1}), r\}$*, where* $(\mathsf{o}) \subseteq r \subseteq (\mathsf{pmosf}^{-1})$.

PROOF. Set $R = (\mathsf{o})$, $R_1 = r$, $R_2 = (\mathsf{oo}^{-1})$ in the first case, $R = (\mathsf{d})$, $R_1 = r$ and $R_2 = (\mathsf{dd}^{-1})$ in the second case, and apply Lemma 4.1. □

LEMMA 6.5. *With the assumptions above, if* $(\mathsf{pmods}) \subseteq r'$ *or* $(\mathsf{pmod}^{-1}\mathsf{f}^{-1}) \subseteq r'$ *then $\mathcal{S}$ is contained in one of the 18 maximal tractable subalgebras.*

PROOF. We will consider only the first case, the second one is dual.

By Lemma 6.2, there are only three possible choices for r': (pmods), (pmodsf) and (pmodsf^{-1}). The relations $r_{\mathsf{p}}, r_{\mathsf{m}}, r_{\mathsf{o}}, r_{\mathsf{d}}$, and r_{s} must all be contained in r'. We now consider how they are related to each other.

Assume first that one of these five sets is contained in the other four. If r_{m} is contained in all of $r_{\mathsf{p}}, r_{\mathsf{o}}, r_{\mathsf{d}}, r_{\mathsf{s}}$, then either r_{m} is one of (mf^{-1}) and (mf), and in this case $\mathcal{S} \subseteq \mathcal{B}_1$ or $\mathcal{S} \subseteq \mathcal{B}_2$, respectively, or else r_{m} coincides with one of $r_{\mathsf{p}}, r_{\mathsf{o}}, r_{\mathsf{d}}, r_{\mathsf{s}}$ because $(\mathsf{m}) \notin \mathcal{S}$, by Assumption 1. Further, if r_{p}, r_{o}, or r_{d} is contained in the other four relations, then $\mathcal{S}$ is contained in $\mathcal{E}_{\mathsf{p}}$, $\mathcal{E}_{\mathsf{o}}$, or $\mathcal{E}_{\mathsf{d}}$, respectively. If r_{s} is contained in all of $r_{\mathsf{p}}, r_{\mathsf{m}}, r_{\mathsf{o}}, r_{\mathsf{d}}$, then $\mathcal{S} \subseteq \mathcal{B}_1$ if $(\mathsf{f}^{-1}) \subseteq r_{\mathsf{s}}$, $\mathcal{S} \subseteq \mathcal{B}_2$ if $(\mathsf{f}) \subseteq r_{\mathsf{s}}$, and $\mathcal{S} \subseteq \mathcal{E}^*$ if $(\equiv \mathsf{f}) \subseteq r_{\mathsf{f}}$. In the remaining case, we have $r_{\mathsf{s}} = r_{\mathsf{m}} = (\mathsf{ms})$ is contained in $r_{\mathsf{p}}, r_{\mathsf{o}}$ and r_{d}, so if r_{f} contains (s) then $\mathcal{S} \subseteq \mathcal{A}_3$ and if r_{f} contains (s^{-1}) then $\mathcal{S} \subseteq \mathcal{A}_4$, otherwise $r_{\mathsf{f}} \subseteq (\mathsf{ff}^{-1})$, and we have $(\mathsf{ms}) \cap (r_{\mathsf{f}} \circ (\mathsf{ms})) = (\mathsf{m}) \in \mathcal{S}$, which contradicts Assumption 1.

Now assume to the contrary that there are two relations r_1 and r_2 amongst $r_{\mathsf{p}}, r_{\mathsf{m}}, r_{\mathsf{o}}, r_{\mathsf{d}}$ and r_{s} which are both minimal in the inclusion ordering. Note that both r_1 and r_2 are contained in one of (pmodsf) and (pmodsf^{-1}), since they are both subsets of r'. We consider the first case, the second one is dual.

By the choice of r_1, r_2, $r_1 \cap r_2$ must be r_{f} or empty. If r_{f} is contained in every possible choice of r_1 and r_2 then $\mathcal{S} \subseteq \mathcal{B}_2$, so we consider the case when $r_1 \cap r_2$ is empty.

Assume first that $r_1 \subseteq (\mathsf{odsf})$. If $r_1 \neq (\mathsf{sf})$ then it can be checked that we have $r_1^{-1} \circ r_1 = \nu \in \mathcal{S}$ where ν is defined in Table II. If $r_1 = (\mathsf{sf})$ then $r_3 = r_1 \circ r_1 = (\mathsf{dsf})$ and $\nu = r_3^{-1} \circ r_3$ belongs to $\mathcal{S}$ anyway. By the minimality of r_2, we have either $r_2 \cap \nu = \emptyset$ or $r_2 \subseteq \nu$. If $r_2 \cap \nu = \emptyset$ then $r_2 \subseteq (\mathsf{pm})$ and $r_2 \circ r_2 = (\mathsf{p}) \in \mathcal{S}$, which contradicts Assumption 1. If $r_2 \subseteq \nu$ then both r_1 and r_2 are contained in (odsf). We may assume without loss of generality that $(o) \subseteq r_1$. It can then be checked that for all possible choices of r_1, r_2, either $(r_1 \circ r_2) \cap r_2$ or $(r_1^{-1} \circ r_2) \cap r_2$ is a non-empty proper subset of r_2, which contradicts the minimality of r_2.

This completes the analysis of the case when $r_1 \subseteq (\mathsf{odsf})$.

Now we only need to consider the case when the two distinct minimal relations r_1 and r_2 are r_{p} and r_{m}. Then we have $(\mathsf{m}) \subseteq r_{\mathsf{m}} \subseteq (\mathsf{modsf})$, and it can be checked that, unless $r_{\mathsf{m}} = (\mathsf{mf})$, $r_4 = r_{\mathsf{m}}^{-1} \circ r_{\mathsf{m}}$ is either ν or $\nu \cup (\mathsf{mm}^{-1})$. In the former case $r_4 \cap r_{\mathsf{m}}$ is a non-empty proper subrelation of r_{m}, while in the latter one $r_4 \cap r_{\mathsf{p}}$ is a non-empty proper subrelation of r_{p}, which contradicts the choice of r_1 and r_2. If $r_{\mathsf{m}} = (\mathsf{mf})$ then $((\mathsf{mf}) \circ (\mathsf{m}^{-1}\mathsf{f}^{-1})) \cap r_{\mathsf{p}} = (\mathsf{p})$, which contradicts Assumption 1. □

Lemma 6.6. *With the assumptions above, if* $(\mathsf{pmosf}^{-1}) \in \mathcal{S}$ *then* $\mathcal{S}$ *is contained in one of the 18 maximal tractable subalgebras.*

Proof. Consider the relation r_{d}. If $r_{\mathsf{d}} = (\equiv \mathsf{d})$ then $r_{\mathsf{d}} \circ (\mathsf{pmosf}^{-1})$ satisfies the conditions of Lemma 6.5. If $(\mathsf{dd}^{-1}) \subseteq r_{\mathsf{d}} \subseteq (\equiv \mathsf{dd}^{-1})$ then, using

$$(\equiv \mathsf{dd}^{-1}) \cap ((\equiv \mathsf{dd}^{-1}) \circ (\mathsf{pmosf}^{-1})) = (\mathsf{dd}^{-1}),$$

we get $\mathcal{S}$ is NP-complete by Lemma 6.4(2), which contradicts Assumption 2. Hence, $r_{\mathsf{d}} \cap (\mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{oo}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1}) \neq \emptyset$.

The relations $r_{\mathsf{p}}, r_{\mathsf{m}}, r_{\mathsf{o}}, r_{\mathsf{s}}$, and $r_{\mathsf{f}^{-1}}$ must all be contained in (pmosf^{-1}). Assume first that one of these five relations is contained in the other four. If relation r_{m} is contained in all of $r_{\mathsf{p}}, r_{\mathsf{o}}, r_{\mathsf{s}}, r_{\mathsf{f}^{-1}}$, then r_{m} coincides with one of $r_{\mathsf{p}}, r_{\mathsf{o}}, r_{\mathsf{s}}, r_{\mathsf{f}^{-1}}$ because $(\mathsf{m}) \notin \mathcal{S}$, by Assumption 1. Further, if r_{p}, r_{o}, r_{s} or $r_{\mathsf{f}^{-1}}$ is contained in the other four relations, then it is also contained in r_{d} or $r_{\mathsf{d}^{-1}}$. Hence, $\mathcal{S}$ is contained in $\mathcal{S}_{\mathsf{p}}$, $\mathcal{S}_{\mathsf{o}}$, $\mathcal{A}_2$, $\mathcal{A}_4$, $\mathcal{E}_{\mathsf{p}}$, $\mathcal{E}_{\mathsf{o}}$, $\mathcal{B}_1$, or $\mathcal{B}_4$.

Now assume to the contrary that there are two relations r_1 and r_2 amongst $r_{\mathsf{p}}, r_{\mathsf{m}}, r_{\mathsf{o}}, r_{\mathsf{s}}$ and $r_{\mathsf{f}^{-1}}$ which are both minimal in the inclusion ordering. By the choice of r_1, r_2, $r_1 \cap r_2$ must be empty.

Assume first that $r_1 \subseteq (\mathsf{osf}^{-1})$. If $(\mathsf{o}) \subseteq r_1$ then it can be checked that we have $r_1^{-1} \circ r_1 = \nu \in \mathcal{S}$. Then, by the minimality of r_2, we have either $r_2 \cap \nu = \emptyset$ or $r_2 \subseteq \nu$. If $r_2 \cap \nu = \emptyset$ then $r_2 \subseteq (\mathsf{pm})$ and $r_2 \circ r_2 = (\mathsf{p}) \in \mathcal{S}$, which contradicts Assumption 1. If $r_2 \subseteq \nu$ then both r_1 and r_2 are contained in (osf^{-1}), which contradicts Assumption 1. Hence we may assume that $r_1 = (\mathsf{sf}^{-1})$, which implies that $(\mathsf{o}) \subseteq r_2 \subseteq (\mathsf{pmo})$ and hence $(r_1^{-1} \circ r_2)^* = (\mathsf{dd}^{-1})$, so $\mathcal{S}$ is NP-complete by Lemma 6.4(2), which contradicts Assumption 2. This completes the analysis of the case when $r_1 \subseteq (\mathsf{osf}^{-1})$.

Now we only need to consider the case when the two distinct minimal relations r_1 and r_2 are r_{p} and r_{m}. Then we have $(\mathsf{m}) \subseteq r_{\mathsf{m}} \subseteq (\mathsf{mosf}^{-1})$, and it can be checked that in all cases either $r_{\mathsf{m}}^{-1} \circ r_{\mathsf{m}}$ or $r_{\mathsf{m}} \circ r_{\mathsf{m}}^{-1}$ is a non-empty proper subrelation of r_{p}, which contradicts the choice of r_2. □

Lemma 6.7. *With the assumptions above, if* $r' \not\subseteq (\mathsf{dsf})^{\pm 1}$, $r' \not\subseteq (\mathsf{pmos})^{\pm 1}$, *and* $r' \not\subseteq (\mathsf{pmof}^{-1})^{\pm 1}$, *then* $\mathcal{S}$ *is contained in one of the 18 maximal tractable subalgebras.*

Proof. It follows from Lemma 6.2 that if $r' \cap (\mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{oo}^{-1}) = \emptyset$ then r' (or its converse) is one of $(\mathsf{ds}^{-1}), (\mathsf{df}^{-1}), (\mathsf{sf}^{-1}), (\mathsf{dsf}^{-1})$, or $(\mathsf{ds}^{-1}\mathsf{f})$. In all of these cases $r' \circ r'$ (or its converse) satisfies the conditions of Lemma 6.5 or Lemma 6.6.

Hence we may assume that $r' \cap (\mathsf{pmo}) \neq \emptyset$. Now, using Lemma 6.2, it can straightforwardly be checked that, except for $r' = (\mathsf{md})$ and (md^{-1}), the relation $r' \circ r'$ satisfies the conditions of Lemma 6.5 or Lemma 6.6. For $r' = (\mathsf{md})$ or (md^{-1}), the relation $(r' \circ r') \circ r'$ is (pmods) or $(\mathsf{pmod}^{-1}\mathsf{f}^{-1})$, respectively. Once again, Lemma 6.5 can be applied. □

LEMMA 6.8. *With the assumptions above, if $\mathcal{S}$ contain relations r_1 and r_2 such that $r_1 \subseteq (\mathsf{pmos})$ or $r_1 \subseteq (\mathsf{pmof}^{-1})$, and $r_2 \subseteq (\mathsf{dsf})$, then $\mathcal{S}$ is contained in one of the 18 maximal tractable subalgebras.*

PROOF. Let $r_3 = r_1 \circ r_1$ and $r_4 = r_2 \circ r_2$. Then we have $(\mathsf{p}) \subseteq r_3 \subseteq (\mathsf{pmos})$ or $(\mathsf{p}) \subseteq r_3 \subseteq (\mathsf{pmof}^{-1})$, and $(\mathsf{d}) \subseteq r_4 \subseteq (\mathsf{dsf})$. Now it is easy to check that either $r_3 \circ r_4$ or $r_4^{-1} \circ r_3$ satisfies the conditions of Lemma 6.5. $\square$

LEMMA 6.9. *With the assumptions above, if $r' \subseteq (\mathsf{dsf})$ then $\mathcal{S}$ is contained in one of the 18 maximal tractable subalgebras.*

PROOF. As noted in the proof of Lemma 6.5, we have $\nu \in \mathcal{S}$.

We consider the case when $r' = (\mathsf{ds})$; the other cases are similar. Then, for every $r \in \mathcal{S}$, we have $r \cap (\mathsf{ds})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{s})^{\pm 1} \subseteq r$.

Consider the relation r_{o}. If it is asymmetric then we get the required result by Lemmas 6.7 and 6.8, and by Proposition 6.3, so assume it is not. It is clear that $r_{\mathsf{o}} \subseteq \nu$. Suppose that $(\mathsf{ss}^{-1}) \not\subseteq r_{\mathsf{o}}$, say $(\mathsf{s}^{-1}) \not\subseteq r_{\mathsf{o}}$. Then $(\mathsf{o}) \subseteq r_{\mathsf{o}} \subseteq (\equiv \mathsf{oo}^{-1}\mathsf{dsff}^{-1})$. It is easy to check that if $r_1 = (\mathsf{ds}) \circ r_{\mathsf{o}}$ then $(\mathsf{o}) \subseteq r_1$ and $(\equiv) \not\subseteq r_1$. This implies that $(\equiv) \not\subseteq r_{\mathsf{o}}$. Since r_{o} is not asymmetric, we conclude that $(\mathsf{oo}^{-1}) \subseteq r_{\mathsf{o}}$ or $(\mathsf{ff}^{-1}) \subseteq r_{\mathsf{o}}$. If $(\mathsf{oo}^{-1}) \not\subseteq r_{\mathsf{o}}$ then $(\mathsf{ff}^{-1}) = r_{\mathsf{o}}^* \in \mathcal{S}$ and $(\mathsf{pmods}) = (\mathsf{ds}) \circ (\mathsf{ff}^{-1}) \in \mathcal{S}$, that is, we can make use of Lemma 6.5. Suppose now that $(\mathsf{oo}^{-1}) \subseteq r_{\mathsf{o}}$; then r_{o} is either (oo^{-1}) or $(\mathsf{oo}^{-1}\mathsf{ff}^{-1})$. The case $r_{\mathsf{o}} = (\mathsf{oo}^{-1}\mathsf{ff}^{-1})$ is impossible in view of $(\mathsf{oo}^{-1}\mathsf{ff}^{-1}) \cap ((\mathsf{ds}) \circ (\mathsf{oo}^{-1}\mathsf{ff}^{-1})) = (\mathsf{oo}^{-1}\mathsf{f})$. If $r_{\mathsf{o}} = (\mathsf{oo}^{-1})$ then $\mathcal{S}$ is NP-complete by Lemma 6.4(1), which contradicts Assumption 2. Therefore we may consider further in this proof that, for every $r \in \mathcal{S}$, $r \cap (\mathsf{oo}^{-1}) \neq \emptyset$ implies $(\mathsf{ss}^{-1}) \subseteq r$.

Consider the relation r_{p}. If it is asymmetric then we get the required result by Lemmas 6.7 and 6.8, so assume it is not. Suppose that $(\mathsf{ss}^{-1}) \not\subseteq r_{\mathsf{p}}$. Then we have $r_{\mathsf{p}}^* \subseteq (\equiv \mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{ff}^{-1})$. If $(\mathsf{ff}^{-1}) \subseteq r_{\mathsf{p}}^*$ then $r_1 = r_{\mathsf{p}}^* \cap \nu$ $(\in \mathcal{S})$ is either $(\equiv \mathsf{ff}^{-1})$ or (ff^{-1}). In both cases $(\mathsf{ds}) \circ r_1 = (\mathsf{pmods}) \in \mathcal{S}$, and the result follows from Lemma 6.5. We have $((\mathsf{ds}) \circ (\equiv \mathsf{pp}^{-1}\mathsf{mm}^{-1})) \cap (\equiv \mathsf{pp}^{-1}\mathsf{mm}^{-1}) = (\mathsf{pp}^{-1}\mathsf{m}^{-1})$. Then, if $r_{\mathsf{p}}^* \neq (\equiv)$, we get $r_{\mathsf{p}} = (\mathsf{pp}^{-1})$, and $\mathcal{S}$ is NP-complete by Proposition 5.1(3), since $\{(\mathsf{pp}^{-1}), (\mathsf{ds})\}$ is contained in neither $\mathcal{S}_{\mathsf{p}}$ nor $\mathcal{E}_{\mathsf{p}}$. The only remaining choice for r_{p}^* is $(\equiv)$. It is impossible, since $(\mathsf{p}) \subseteq (\mathsf{ds}) \circ r_{\mathsf{p}}$, but $(\equiv) \not\subseteq (\mathsf{ds}) \circ r_{\mathsf{p}}$. Therefore we may consider further in this proof that, for every $r \in \mathcal{S}$, $r \cap (\mathsf{pp}^{-1}) \neq \emptyset$ implies $(\mathsf{ss}^{-1}) \subseteq r$.

A similar argument shows that if $(\mathsf{ss}^{-1}) \not\subseteq r_{\mathsf{m}}$ then the result follows. Assume that $(\mathsf{ss}^{-1}) \subseteq r_{\mathsf{m}}$. Now it can be easily verified that $\mathcal{S} \subseteq \mathcal{E}^*$ if $r_{\mathsf{f}} \subseteq (\equiv \mathsf{ff}^{-1})$; otherwise we have $r_{\mathsf{f}} \cap (\mathsf{ss}^{-1}) \neq \emptyset$, and $\mathcal{S} \subseteq \mathcal{B}_1$ if $(\mathsf{s}^{-1}) \subseteq r_{\mathsf{f}}$, while $\mathcal{S} \subseteq \mathcal{B}_2$ if $(\mathsf{s}) \subseteq r_{\mathsf{f}}$. $\square$

LEMMA 6.10. *With the assumptions above, if $r' \subseteq (\mathsf{pmos})$ or $r' \subseteq (\mathsf{pmof}^{-1})$, then $\mathcal{S}$ is contained in one of the 18 maximal tractable subalgebras.*

PROOF. We shall consider the case $r' \subseteq (\mathsf{pmos})$; the second case is dual. Note that $r' \neq (\mathsf{ms})$ and $r' \neq (\mathsf{mo})$; otherwise $((r')^{-1} \circ r') \cap r'$ is (s) or (o), respectively.
Case 1. $(\mathsf{p}) \not\subseteq r'$.
We have $(\mathsf{o}) \subseteq r'$; otherwise $r' = (\mathsf{ms})$ which contradicts our assumptions. If $r' = (\mathsf{mos})$ then $((r')^{-1} \circ r') \cap r' = (\mathsf{os}) \in \mathcal{S}$. We may therefore assume that $r' = (\mathsf{os})$. Then, the proof is very similar to the one of Lemma 6.9.

Case 2. $(\mathsf{p}) \subseteq r'$.
Since $(\mathsf{pm}) \circ (\mathsf{pm}) = (\mathsf{p})$, r' cannot be (pm) and we can assume that $(\mathsf{o}) \subseteq r'$ or $(\mathsf{s}) \subseteq r'$. Then $r' \circ r'$ is one of (pmo), (ps), (pms), and (pmos). We can also assume that no relation satisfying the condition of Case 1 is contained in $\mathcal{S}$. This implies that r_{p} is a minimal relation in $\mathcal{S}$.
Subcase 2.1. $r' \circ r' = (\mathsf{pmo})$.
We have $r_{\mathsf{p}} = (\mathsf{po})$ or $r_{\mathsf{p}} = (\mathsf{pmo})$. In both cases, every $r \in \mathcal{S}$ such that $r \cap (\mathsf{pmo}) \neq \emptyset$ satisfies $(\mathsf{p}) \subseteq r$ because, as shown above, $(\mathsf{mo}) \notin \mathcal{S}$.

Suppose that $\mathcal{S}$ contains a non-trivial relation r_1 such that $r_1 \subseteq (\equiv \mathsf{dd}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1})$. Define r_2 to be $(\mathsf{pmo}) \circ r_1$ if $r_1 \not\subseteq (\equiv \mathsf{sf}^{-1})$, and $(\mathsf{pmo}) \circ r_1^{-1}$ otherwise. Then it can be easily checked that either $r_2^* = (\mathsf{dd}^{-1})$ or else r_2 (or r_2^{-1}) satisfies the conditions of Lemma 6.5 or Lemma 6.6. If $(\mathsf{dd}^{-1}) \in \mathcal{S}$ then $\{(\mathsf{pmo}), (\mathsf{dd}^{-1})\} \subseteq \mathcal{S}$, so $\mathcal{S}$ is NP-complete by Lemma 6.4(2). We may assume now that every non-trivial $r \in \mathcal{S}$ satisfies $r \cap (\mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{oo}^{-1}) \neq \emptyset$ and, consequently, $r \cap (\mathsf{pp}^{-1}) \neq \emptyset$.

Arbitrarily choose $r \in \mathcal{S}$. If r is such that r^* is neither $\emptyset$ nor $(\equiv)$, then $(\mathsf{pp}^{-1}) \subseteq r$. Assume that $r^* = \emptyset$, that is, r is asymmetric. The required result follows from the previous lemmas if $r \not\subseteq (\mathsf{pmos})^{\pm 1}$ and $r \not\subseteq (\mathsf{pmof}^{-1})^{\pm 1}$. So we can assume that every asymmetric $r \in \mathcal{S}$ satisfies $(\mathsf{p}) \subseteq r \subseteq (\mathsf{pmosf}^{-1})$ or $(\mathsf{p}) \subseteq r^{-1} \subseteq (\mathsf{pmosf}^{-1})$.

Suppose now that we have $r^* = (\equiv)$; then $r \cap (\mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{oo}^{-1})$ is contained either in (pmo) or in $(\mathsf{p}^{-1}\mathsf{m}^{-1}\mathsf{o}^{-1})$. Without loss of generality, we may assume that $r \cap (\mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{oo}^{-1}) \subseteq (\mathsf{pmo})$. Then consider $r_3 = (\mathsf{pmo}) \circ r$. If $r \not\subseteq (\equiv \mathsf{pmosf}^{-1})$ then either $r_3^* = (\mathsf{dd}^{-1})$ (and then $\mathcal{S}$ is NP complete by Lemma 6.4(2)) or r_3 is one of the relations satisfying the conditions of Lemma 6.5 (and then $\mathcal{S}$ is contained in one of the 18 maximal tractable subalgebras). Now let $r \subseteq (\equiv \mathsf{pmosf}^{-1})$. By our assumption, $\mathcal{S}$ contains no non-trivial subrelation of $(\equiv \mathsf{sf}^{-1})$ which implies that $(\mathsf{p}) \subseteq r$ unless $r = (\equiv)$.

We conclude that $\mathcal{S}$ is contained in $\mathcal{S}_{\mathsf{p}}$ or in $\mathcal{E}_{\mathsf{p}}$.
Subcase 2.2. $r' \circ r' = (\mathsf{ps})$.
We have $r' = r_{\mathsf{p}} = (\mathsf{ps})$ so every $r \in \mathcal{S}$ such that $r \cap (\mathsf{ps}) \neq \emptyset$ satisfies $(\mathsf{p}) \subseteq r$. Suppose that $\mathcal{S}$ contains a non-trivial relation r_4 such that $r_4 \subseteq (\equiv \mathsf{mm}^{-1}\mathsf{oo}^{-1}\mathsf{dd}^{-1}\mathsf{ff}^{-1})$ and $r_4 \not\subseteq (\equiv \mathsf{ff}^{-1})$ Then it can verified that the relation $r_5 = (r' \circ r_4) \cap r_4$ is non-empty and $(\equiv) \not\subseteq r_5$. Therefore, we can assume that $(\equiv) \not\subseteq r_4$. This leads to a contradiction in view of $(r_4 \circ r') \cap r' = (\mathsf{p})$. The case $r_4 = (\mathsf{ff}^{-1})$ is impossible because $((\mathsf{ps}) \circ (\mathsf{ff}^{-1})) \cap (\mathsf{ps}) = (\mathsf{p})$. Finally, if r_4 is $(\equiv \mathsf{ff}^{-1})$ or $(\equiv \mathsf{f})$ then $(\mathsf{ps}) \circ r_4 = (\mathsf{pmods})$, and we can apply Lemma 6.5.

We may therefore assume that, for non-trivial every $r \in \mathcal{S}$, $r \cap (\mathsf{pp}^{-1}\mathsf{ss}^{-1}) \neq \emptyset$ and, consequently, $r \cap (\mathsf{pp}^{-1}) \neq \emptyset$.

Arbitrarily choose a non-trivial $r \in \mathcal{S}$. If r is such that r^* is neither $\emptyset$ nor $(\equiv)$, then $(\mathsf{pp}^{-1}) \subseteq r$. Suppose that $r^* = \emptyset$. If $r \not\subseteq (\mathsf{pmos})^{\pm 1}$ and $r \not\subseteq (\mathsf{pmof}^{-1})^{\pm 1}$, then we get the required result by previous lemmas so we can assume that every asymmetric $r \in \mathcal{S}$ satisfies $(\mathsf{p}) \subseteq r \subseteq (\mathsf{pmosf}^{-1})$ or $(\mathsf{p}) \subseteq r^{-1} \subseteq (\mathsf{pmosf}^{-1})$.

If $r^* = (\equiv)$ then $r \cap (\mathsf{pp}^{-1}\mathsf{ss}^{-1})$ is either (ps) or $(\mathsf{p}^{-1}\mathsf{s}^{-1})$. Without loss of generality, assume that $r \cap (\mathsf{pp}^{-1}\mathsf{ss}^{-1}) = (\mathsf{ps})$. Consider $r_6 = (\mathsf{ps}) \circ r$. If $r \cap (\mathsf{m}^{-1}\mathsf{o}^{-1}) \neq \emptyset$ or $(\mathsf{d}^{-1}\mathsf{f}) \subseteq r$ then r_6^* is a non-trivial subrelation of $(\mathsf{mm}^{-1}\mathsf{oo}^{-1}\mathsf{dd}^{-1}\mathsf{ff}^{-1})$ which contradicts our assumptions. Therefore, either $r \subseteq (\equiv \mathsf{pmodsf})$, or $r \subseteq (\equiv \mathsf{pmodsf}^{-1})$, or $r \subseteq (\equiv \mathsf{pmod}^{-1}\mathsf{sf}^{-1})$. Moreover, all $r \in \mathcal{S}$ such that $r \cap (\mathsf{pp}^{-1}\mathsf{ss}^{-1}) = (\mathsf{ps})$ have

to satisfy one (and the same) of these three conditions, since otherwise it is easy to generate a non-empty subrelation of (df) which would lead to a contradiction with our assumptions. We conclude that $\mathcal{S}$ is contained in $\mathcal{S}_{\mathsf{p}}$ or in $\mathcal{E}_{\mathsf{p}}$.
Subcase 2.3. $r' \circ r' = (\mathsf{pms})$ or $r' \circ r' = (\mathsf{pmos})$.
Similar to previous subcases. □

6.2 Symmetric relations

To conclude the proof of Theorem 2.3, in this subsection we prove the following proposition.

PROPOSITION 6.11. *Let $\mathcal{S}$ be a subalgebra of $\mathcal{A}$ such that $bas(\mathcal{S}) = \{\equiv\}$, which contains a symmetric relation r' such that $(\equiv) \not\subseteq r'$. Then $\mathcal{S}$ is tractable if it is contained in one of the 18 algebras listed in Table III. Otherwise $\mathcal{S}$ is NP-complete*

To obtain this result we shall assume throughout this subsection that $\mathcal{S}$ is a subalgebra of $\mathcal{A}$ satisfying the following assumptions:

Assumption 1. $bas(\mathcal{S}) = \{\equiv\}$.

Assumption 2. $\mathcal{S}$ is not NP-complete.

Assumption 3. $r' \in \mathcal{S}$ is a minimal symmetric relation such that $(\equiv) \not\subseteq r'$.

We show that the result holds for all possible choices of r' in Lemmas 6.15–6.16. These lemmas rely on the following NP-completeness results.

PROPOSITION 6.12. *If r, s are symmetric relations such that $r \cap s = \emptyset$, both r and s are contained in neither $(\equiv \mathsf{ss}^{-1})$ nor $(\equiv \mathsf{ff}^{-1})$, then $\{r, s\}$ is NP-complete.*

The proof of this proposition can be found in Subsection A.3.

LEMMA 6.13. *The following sets of relations are NP-complete:*
1) $\{(\mathsf{mm}^{-1})\}$*; and*
2) $\{r, (\mathsf{ss}^{-1}), (\mathsf{ff}^{-1})\}$ *when* $r \in \{(\mathsf{oo}^{-1}), (\mathsf{dd}^{-1})\}$.

PROOF. *1)* We note that the set $\{(\mathsf{mm}^{-1}), (\mathsf{pp}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1})\}$ has been shown NP-complete in [Drakengren and Jonsson 1998]. Let $r_1 = (\mathsf{mm}^{-1}) \circ (\mathsf{mm}^{-1})$. Then one can check that $(\mathsf{pp}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1}) = r_1 \cap ((\mathsf{mm}^{-1}) \circ r_1)$.
2) Let $r = (\mathsf{dd}^{-1})$ and consider the following set of constraints:

$$a(\mathsf{dd}^{-1})b, x(\mathsf{ss}^{-1})a, x(\mathsf{ff}^{-1})b, y(\mathsf{ff}^{-1})a, y(\mathsf{ss}^{-1})b.$$

It can be checked that the derived relation between x and y is (oo^{-1}) and NP-completeness of $\{(\mathsf{oo}^{-1}), (\mathsf{dd}^{-1})\}$ follows from Proposition 6.12.

The case when $r = (\mathsf{oo}^{-1})$ is similar: $a(\mathsf{dd}^{-1})b$ is replaced by $a(\mathsf{oo}^{-1})b$, the derived relation is then (dd^{-1}), and again we have NP-completeness by Proposition 6.12. □

LEMMA 6.14. *With the assumptions above, if $\mathcal{S}$ contains $(\equiv \mathsf{ss}^{-1})$, $(\equiv \mathsf{ff}^{-1})$, and a non-trivial relation r_1 such that $r_1 \cap (\equiv \mathsf{ss}^{-1}\mathsf{ff}^{-1}) = \emptyset$ then $\mathcal{S}$ is contained in one of the 18 maximal tractable subalgebras.*

PROOF. Choose r_1 to be minimal. If $r_1 = (\mathsf{mm}^{-1})$ then $\mathcal{S}$ is NP-complete by Lemma 6.13(1). Hence we shall assume that $r_1 \neq (\mathsf{mm}^{-1})$.

We have $(\equiv \mathsf{ss}^{-1}) \circ (\equiv \mathsf{ff}^{-1}) = (\equiv \mathsf{pmoo}^{-1}\mathsf{dd}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1}) \in \mathcal{S}$. Then either r_1 is asymmetric or $r_1 = r_1^* \subseteq (\mathsf{oo}^{-1}\mathsf{dd}^{-1})$. In the former case, we get the required result by Proposition 6.1, so we shall assume that r_1 is symmetric.

If $r_1 = (\mathsf{oo}^{-1})$ or $r_1 = (\mathsf{dd}^{-1})$, thn $(\mathsf{ss}^{-1}) = (\equiv \mathsf{ss}^{-1}) \cap (r_1 \circ (\equiv \mathsf{ff}^{-1})) \in \mathcal{S}$. Similarly, $(\mathsf{ff}^{-1}) \in \mathcal{S}$. Hence, $\mathcal{S}$ is NP-complete by Lemma 6.13(2), which contradicts Assumption 2.

It follows that $r_1 = (\mathsf{oo}^{-1}\mathsf{dd}^{-1})$. Since r_1 is minimal, we have $(\mathsf{oo}^{-1}\mathsf{dd}^{-1}) \subseteq r$ for every $r \in \mathcal{S}$ such that $(\mathsf{oo}^{-1}\mathsf{dd}^{-1}) \cap r \neq \emptyset$. Further, we have

$$(\mathsf{pmoo}^{-1}\mathsf{dd}^{-1}) = ((r_1 \circ (\equiv \mathsf{ff}^{-1})) \cap ((\equiv \mathsf{ss}^{-1}) \circ r_1)) \in \mathcal{S}.$$

In view of $(\mathsf{pm}) \circ (\mathsf{pm}) = (\mathsf{p})$, no non-empty subrelation of (pm) can belong to $\mathcal{S}$. This implies that $(\mathsf{oo}^{-1}\mathsf{dd}^{-1}) \subseteq r$ for every $r \in \mathcal{S}$ with $(\mathsf{pmoo}^{-1}\mathsf{dd}^{-1}) \cap r \neq \emptyset$. Further, neither (sf) nor (sf^{-1}) can belong to $\mathcal{S}$ because they give (s) being intersected with $(\equiv \mathsf{ss}^{-1}) \in \mathcal{S}$. If $(\equiv \mathsf{sf}) \in \mathcal{S}$ or $(\equiv \mathsf{sf}^{-1}) \in \mathcal{S}$ then we can obtain $(\equiv \mathsf{s})$ and $(\equiv \mathsf{f})$, and, further, $(\equiv \mathsf{dsf})$. However, this contradicts the fact that every relation containing (d) must also contain $(\mathsf{oo}^{-1}\mathsf{dd}^{-1})$. Now it is straightforward to check that $\mathcal{S} \subseteq \mathcal{H}$. $\square$

By examining the composition table given in Table II, it is easy to verify that $(\equiv) \subseteq r_1 \circ r_2$ if and only if there exists a non-trivial basic relation b such that $(b) \subseteq r_1$ and $(b^{-1}) \subseteq r_2$. In the next two lemmas, we shall make use of this fact.

Lemma 6.15. *With the assumptions above, if $r' \not\subseteq (\mathsf{ss}^{-1}\mathsf{ff}^{-1})$, then $\mathcal{S}$ is contained in one of the 18 maximal tractable subalgebras.*

Proof. We may assume that $r' \not\subseteq (\mathsf{pp}^{-1}\mathsf{mm}^{-1})$; otherwise the result follows from Proposition 5.1(3).

By the minimality of r', for every non-trivial relation r_1 in $\mathcal{S}$ we have $r' \cap r_1 = r'$ or empty. Obviously, if every non-trivial relation in $\mathcal{S}$ contains r' then $\mathcal{S}$ is contained in one of the 18 maximal tractable subalgebras. Hence we shall assume that $\mathcal{S}$ contains a non-trivial relation r_1 such that $r' \cap r_1 = \emptyset$.

Case 1. r_1 is asymmetric.
Apply Proposition 6.1.

Case 2. r_1 is symmetric.
It follows from Proposition 6.12 that if $r_1 \not\subseteq (\equiv \mathsf{ss}^{-1})$ and $r_1 \not\subseteq (\equiv \mathsf{ff}^{-1})$ then $\mathcal{S}$ is NP-complete, which contradicts Assumption 2. We shall consider the case $r_1 \subseteq (\equiv \mathsf{ss}^{-1})$; the second case is dual.

Note that we have $r' \cap (\equiv \mathsf{ss}^{-1}) = \emptyset$. Also, $r' \neq (\mathsf{mm}^{-1}\mathsf{ff}^{-1})$, since otherwise $r' \cap (r' \circ (\equiv \mathsf{ss}^{-1})) = (\mathsf{mm}^{-1}\mathsf{f}) \in \mathcal{S}$, which contradicts the minimality of r'. Hence, r' contains (bb^{-1}) where b is one of p, o, and d.

We have $r_1 \circ r_1^{-1} = (\equiv \mathsf{ss}^{-1}) \in \mathcal{S}$. Let $r_2 = r' \circ (\equiv \mathsf{ss}^{-1})$. Obviously, $r' \subseteq r_2$. Moreover, it can be easily checked by examining Table II that $r_2 \cap (\equiv \mathsf{ss}^{-1}) = \emptyset$ and that, for every basic relation b_1 such that $b_1 \notin \{\equiv, \mathsf{s}, \mathsf{s}^{-1}\}$, at least one of (b_1) and (b_1^{-1}) is contained in r_2. If $(bb^{-1}) \subseteq r$ for every $r \in \mathcal{S}$ such that $r \not\subseteq (\equiv \mathsf{ss}^{-1})$ then $\mathcal{S}$ is contained in one of $\mathcal{S}_{\mathsf{p}}$, $\mathcal{S}_{\mathsf{o}}$, and $\mathcal{S}_{\mathsf{d}}$. Otherwise $\mathcal{S}$ contains a non-empty relation r_3 such that $r_3 \not\subseteq (\equiv \mathsf{ss}^{-1})$ and $r_3 \cap r' = \emptyset$ (because r' is minimal). Then $r_3 \cap r_2$ or $r_3 \cap r_2^{-1}$ is non-empty. Denote this non-empty relation by r_4. Then we have $r_4 \cap (\equiv \mathsf{ss}^{-1}) = \emptyset$ and $r_4 \cap r' = \emptyset$. Consider a minimal relation r_5 contained in r_4.

This minimal relation must be either symmetric or asymmetric. Therefore, unless $r_5 = (\mathsf{ff}^{-1})$, we get the required result by Proposition 6.12 or 6.1, respectively. If $r_5 = (\mathsf{ff}^{-1})$ then $r_5 \circ r_5 = (\equiv \mathsf{ff}^{-1}) \in \mathcal{S}$ and $r' \cap (\equiv \mathsf{ss}^{-1}\mathsf{ff}^{-1}) = \emptyset$, and we can apply Lemma 6.14.

Case 3. r_1 is neither symmetric nor asymmetric.
We may assume that r_1 is minimal. Then, by minimality, we have $r_1^* = (\equiv)$. Since r' is symmetric and $r' \cap r_1 = \emptyset$, we obtain that $(\equiv) \not\subseteq r' \circ r_1$ and that $(\equiv) \not\subseteq r' \circ r_1^{-1}$. It follows that if one of $r' \circ r_1$ and $r' \circ r_1^{-1}$ has a non-empty intersection with r_1 or with r_1^{-1} then we get a contradiction with minimality of r_1.

Now it can be checked that we indeed get this contradiction except when r_1 (or r_1^{-1}) is one of the relations $(\equiv \mathsf{m})$, $(\equiv \mathsf{s})$, and $(\equiv \mathsf{f})$. If $r_1 = (\equiv \mathsf{m})$ then $r_6 = (\equiv \mathsf{pm}) = r_1 \circ r_1 \in \mathcal{S}$, and arguing as in the previous paragraph we can obtain a non-empty subrelation r_7 of pm which leads to a contradiction because $r_7 \circ r_7 = (\mathsf{p})$. If $r_1 = (\equiv \mathsf{s})$ then $r' \cap (\equiv \mathsf{ss}^{-1}) = \emptyset$. Further $(\equiv \mathsf{ss}^{-1}) = r_1 \circ r_1^{-1} \in \mathcal{S}$, and Case 2 applies. If $r_1 = (\equiv \mathsf{f})$ then the argument is dual. □

Lemma 6.16. *With the assumptions above, if $r' \subseteq (\mathsf{ss}^{-1}\mathsf{ff}^{-1})$, then $\mathcal{S}$ is contained in one of the 18 maximal tractable subalgebras.*

Proof. Case 1. $r' = (\mathsf{ss}^{-1}\mathsf{ff}^{-1})$.
We have $(\mathsf{ss}^{-1}\mathsf{ff}^{-1}) \subseteq r$ for every $r \in \mathcal{S}$ such that $r \cap (\mathsf{ss}^{-1}\mathsf{ff}^{-1}) \neq \emptyset$. If every non-trivial $r \in \mathcal{S}$ satisfies $(\mathsf{ss}^{-1}\mathsf{ff}^{-1}) \subseteq r$ then $\mathcal{S} \subseteq \mathcal{A}_1$. Suppose that $\mathcal{S}$ has a non-trivial relation r_1 such that $r_1 \cap (\mathsf{ss}^{-1}\mathsf{ff}^{-1}) = \emptyset$. Then consider the relation $r_2 = r_1 \cap (r_1 \circ (\mathsf{ss}^{-1}\mathsf{ff}^{-1}))$. It is easy to check that r_2 is a non-empty subrelation of r_1 and that $(\equiv) \not\subseteq r_2$ (in fact, $r_2 = r_1 \setminus (\equiv)$). The relation r_2 contains some minimal relation that must be either symmetric or asymmetric. Now we obtain the required result by Lemma 6.15 or Proposition 6.1.
Case 2. Both (ss^{-1}) and (ff^{-1}) belong to $\mathcal{S}$.
The relations $(\mathsf{ss}^{-1}) \circ (\mathsf{ss}^{-1}) = (\equiv \mathsf{ss}^{-1})$ and $(\mathsf{ff}^{-1}) \circ (\mathsf{ff}^{-1}) = (\equiv \mathsf{ff}^{-1})$ both belong to $\mathcal{S}$. Furthermore, $(\mathsf{oo}^{-1}\mathsf{dd}^{-1}) = ((\mathsf{ss}^{-1}) \circ (\mathsf{ff}^{-1}))^* \in \mathcal{S}$, and the result follows from Lemma 6.14.
Case 3. Exactly one of (ss^{-1}) and (ff^{-1}) belongs to $\mathcal{S}$.
Assume that $(\mathsf{ss}^{-1}) \in \mathcal{S}$ and $(\mathsf{ff}^{-1}) \notin \mathcal{S}$, the second case is dual.

If (ss^{-1}) is the only minimal relation in $\mathcal{S}$ then every non-trivial relation in $\mathcal{S}$ contains (ss^{-1}), and we have $\mathcal{S} \subseteq \mathcal{A}_1$. Suppose that there exists a minimal relation $r_3 \in \mathcal{S}$ such that $r_3 \cap (\mathsf{ss}^{-1}) = \emptyset$. Then we may assume that $(\equiv) \subseteq r_3$; otherwise we get the required result by Proposition 6.1 or by Lemma 6.15. Let $r_4 = r_3 \circ (\mathsf{ss}^{-1})$. We have $(\equiv) \not\subseteq r_4$.

It is easy to verify that if $r_3 \not\subseteq (\equiv \mathsf{ff}^{-1})$ then, for every basic relation b_1 such that $b_1 \notin \{\equiv, \mathsf{s}, \mathsf{s}^{-1}, \mathsf{f}, \mathsf{f}^{-1}\}$ and $(b_1) \subseteq r_3$, at least one of (b_1) and (b_1^{-1}) is contained in r_4. This leads to a contradiction with minimality of r_3.

If $r_3 \subseteq (\equiv \mathsf{ff}^{-1})$ then every $r \in \mathcal{S}$ such that $r \cap (\mathsf{ff}^{-1}) \neq \emptyset$ also satisfies $(\equiv) \subseteq r$. Then we have $r_3 \circ r_3^{-1} = (\equiv \mathsf{ff}^{-1})$ and $(\mathsf{pmoo}^{-1}\mathsf{dd}^{-1}\mathsf{ss}^{-1}) = (\mathsf{ss}^{-1}) \circ (\equiv \mathsf{ff}^{-1}) \in \mathcal{S}$. If some non-empty subrelation of $(\mathsf{pmoo}^{-1}\mathsf{dd}^{-1})$ belongs to $\mathcal{S}$ then we get the required result by Lemma 6.15 or Proposition 6.1. Else, every $r \in \mathcal{S}$ such that $r \cap (\mathsf{pmoo}^{-1}\mathsf{dd}^{-1}) \neq \emptyset$ also satisfies $(\mathsf{ss}^{-1}) \subseteq r$, and we have $\mathcal{S} \subseteq \mathcal{E}^*$. □

7. CONCLUSION

We have now completed the classification of complexity in Allen's algebra and shown that there exist exactly eighteen forms of tractability in this algebra. We did this by applying a technique from general algebra which has not been previously used in this context.

Both the result and the method can be used to classify the complexity in other temporal and spatial formalisms; a first application is given in Section 3. There are also strong connections with the analysis of complexity in temporal logics ([Bellini et al. 2000]) which deserve further investigation.

It has already been established that the maximal tractable subalgebra, $\mathcal{H}$, can be used to speed up backtracking algorithms [Nebel 1997b]. We believe that the complete description of tractability in Allen's algebra which is presented here may lead to new methods in approximate temporal reasoning, as one can uniquely loosen, in a minimal way, any set of interval constraints to obtain an instance of a given tractable case.

In this paper, we considered the problem of satisfiability of temporal constraints. However, there are other important tasks in temporal (and spatial) reasoning, for example, the task of answering queries in different types of constraint networks (see, e.g., [Koubarakis and Skiadopoulos 2000]). The method and the results presented in this paper can contribute to further progress in tackling such tasks.

Finally, we note that many other constraint formalisms (not just temporal ones) are based on manipulating objects with intrinsic structural properties which can be captured by an appropriate algebra. This prompts us to conjecture that algebraic approaches to constraint manipulation, such as the one taken in this paper, or those presented in [Bulatov 2002; Bulatov et al. 2000; 2001], provide the appropriate reasoning tools across many different areas of constraint reasoning and artificial intelligence.

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A. APPENDIX

This appendix contains the proofs of Propositions 6.3 and 6.12. In the sequel, we will make frequent (but implicit) use of Proposition 5.2 in the following way. With the help of Lemma 6.2, it is easy to check that the relations r mentioned in Proposition 6.3 and pairs of relations $\{r, s\}$ mentioned in Proposition 6.12 are not contained in one of the 18 tractable subalgebras in Table III. Therefore, we conclude that $\{r\}$ or $\{r, s\}$ is NP-complete whenever we can derive, from this set, either a non-trivial basic relation or some set of relations whose NP-completeness was shown before. Recall that derivation is introduced in Section 4. B. Nebel's CSP solver [Nebel 1997a] can considerably simplify calculating the derivations. Recall also that if a relation can be obtained from a given set by several derivations then it can be obtained by using a single derivation.

A.1 Model Transformations

This subsection contains the basics of *model transformations* which is a method for proving NP-completeness results. It is based on transforming a solution of one problem to a solution of a related problem. This method will be used many times in the proofs of Propositions 6.3 and 6.12.

Suppose T is a mapping on models of $\mathcal{A}$-SAT-instances with the same set of variables and let f_T be a function from the set of all basic relations to $\mathcal{A}$ such that the following holds: for any model f of an $\mathcal{A}$-SAT-instance over a set V of variables, for any $x, y \in V$, and for any basic relation b, if $f(x)$ is related to $f(y)$ by (b) then $T(f(x))$ is related to $T(f(y))$ by $f_T(b)$. Then we say that T is a *model transformation* with *description* f_T. A description f_T can be extended to handle all relations $r \in \mathcal{A}$ in the obvious way: $f_T(r) = \bigcup_{b \in r} f_T(b)$.

The following lemma gives us a way of proving NP-completeness by using model transformations.

LEMMA A.1. *Let $\mathcal{R} = \{r_1, \ldots, r_n\} \subseteq \mathcal{A}$ and $\mathcal{R}' = \{r'_1, \ldots, r'_n\} \subseteq \mathcal{A}$ be such that $r'_k \subseteq r_k$ for all $1 \leq k \leq n$. If there exists a model transformation T with description f_T such that $f_T(r_k) \subseteq r'_k$ for every $1 \leq k \leq n$, then $\mathcal{R}$ is NP-complete if and only if $\mathcal{R}'$ is NP-complete.*

PROOF. The proof of the only-if direction can be found in [Drakengren and Jonsson 1998] and the proof of the other direction is analogous. □

Let S be a finite set of real numbers. The *minimal distance in* S, $\mathrm{MD}(S)$, is defined as

$$\mathrm{MD}(S) = \min\{x - y \mid x, y \in S \wedge x > y\}.$$

For a model f of an $\mathcal{A}$-SAT-instance over a set V of variables, we define

$$\mathrm{MD}(f) = \mathrm{MD}(\{f(x^-), f(x^+) \mid x \in V\}),$$

where $f(x^-)$ and $f(x^+)$ denote the starting and the ending point of the interval $f(x)$, as in Section 3.

We continue by defining a number of model transformations. We shall use them with fixed descriptions which can be found in Table IV. We define the model transformation shrink as follows. Let f be a model of an $\mathcal{A}$-SAT-instance over $\{x_1, \ldots, x_n\}$ and let $\epsilon = \mathrm{MD}(f)/3$. Then $\mathsf{shrink}(f) = f'$ where, for $1 \leq i \leq n$,

$$f'(x_i) = [f(x_i^-) + \epsilon, f(x_i^+) - \epsilon].$$

We can analogously define a model transformation expand by subtracting ϵ from $f(x^-)$ and adding ϵ to $f(x^+)$.

By ordering the intervals with respect to their length, we can obtain a number of useful model transformations. We define the model transformation $\mathsf{ordshrink}$ as follows. Let f be a model of an $\mathcal{A}$-SAT-instance over $\{x_1, \ldots, x_n\}$, set ϵ to be $\mathrm{MD}(f)/(2n)$ and rename the variables so that $|f(x_1)| \geq \ldots \geq |f(x_n)|$. Then $\mathsf{ordshrink}(f) = f'$ where, for $1 \leq i \leq n$,

$$f'(x_i) = [f(x_i^-) + i\epsilon, f(x_i^+) - i\epsilon].$$

	$\equiv$	p	p^{-1}	m	m^{-1}	o	o^{-1}	d	d^{-1}	s	s^{-1}	f	f^{-1}
shrink	$\equiv$	p	p^{-1}	p	p^{-1}	o	o^{-1}	d	d^{-1}	s	s^{-1}	f	f^{-1}
expand	$\equiv$	p	p^{-1}	o	o^{-1}	o	o^{-1}	d	d^{-1}	s	s^{-1}	f	f^{-1}
ordshrink	dd^{-1}	p	p^{-1}	p	p^{-1}	o	o^{-1}	d	d^{-1}	d	d^{-1}	d	d^{-1}
ordexpand	dd^{-1}	p	p^{-1}	o	o^{-1}	o	o^{-1}	d	d^{-1}	o	o^{-1}	o^{-1}	o
leftordshrink	ff^{-1}	p	p^{-1}	p	p^{-1}	o	o^{-1}	d	d^{-1}	o	o^{-1}	f	f^{-1}
leftordexpand	ff^{-1}	p	p^{-1}	o	o^{-1}	o	o^{-1}	d	d^{-1}	d	d^{-1}	f	f^{-1}

Table IV. Model transformations.

We analogously define a model transformation ordexpand.

We will also use model transformations that only change one of the endpoints of an interval. We define the model transformation leftordshrink as follows. Let f be a model of an $\mathcal{A}$-SAT-instance over $\{x_1, \ldots, x_n\}$, let $\epsilon = \mathrm{MD}(f)/(n+1)$ and rename the variables so that $|f(x_1)| \leq \ldots \leq |f(x_n)|$. Then leftordshrink$(f) = f'$, where, for $1 \leq i \leq n$,

$$f'(x_i) = [f(x_i^-) + i\epsilon, f(x_i^+)].$$

A.2 Proof of Proposition 6.3

We will now show that if r is asymmetric, but not acyclic, then $\{r\}$ is NP-complete. The proof is largely based on the use of different derivations. Throughout this subsection, each time we use variables a_1, a_2, a_3 we will assume that they are newly introduced. Let $C_9(r)$ denote the relation (between x and y) derived from the following set of constraints:

$$\left\{\begin{array}{lll} xra_1 & a_1ra_2 & a_1ry \\ xra_2 & a_2ra_3 & a_2ry \\ xra_3 & a_3ra_1 & a_3ry \end{array}\right\}.$$

$C_{9b}(r)$ is the relation derived from the same set of constraints but with xra_2 replaced by $xr^{-1}a_2$ and a_2ry replaced by $a_2r^{-1}y$. Finally, $C_{14}(r)$ denotes the relation derived from

$$\left\{\begin{array}{lll} xr^{-1}a_1 & a_1r^{-1}a_2 & a_1ry \\ xr^{-1}a_2 & a_1ra_3 & a_2r^{-1}y \\ xr^{-1}a_3 & a_1r^{-1}a_4 & a_3r^{-1}y \\ xra_4 & a_2r^{-1}a_3 & a_4r^{-1}y \\ & a_2ra_4 & \\ & a_3r^{-1}a_4 & \end{array}\right\}$$

Before the proof, we need one auxiliary lemma.

LEMMA A.2. $\{(\mathsf{pp}^{-1}\mathsf{dd}^{-1}), (\mathsf{oo}^{-1})\}$ *is NP-complete.*

PROOF. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs such that $V_1 = V_2$ and $E_1 \subseteq E_2$; in this case, we say that G_2 is a *supergraph* of G_1. The GRAPH

Sandwich Problem for Property Π is defined as follows:

Instance: Two graphs, $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$, such that G_2 is a supergraph of G_2.
Question: Is there a graph $G = (V, E)$ such that $E_1 \subseteq E \subseteq E_2$ and (V, E) has property Π?

Two intervals *overlap*[6] if their intersection is non-empty but neither one of them properly contains the other. An *overlap* graph (also known as a *circle* graph [Gavril 1973]) is an undirected graph $G = (V, E)$ for which there is an assignment of an interval to each vertex such that two vertices are adjacent iff the corresponding intervals overlap. The graph sandwich problem for overlap graphs (SP-Overlap) is NP-complete [Golumbic et al. 1995]. Clearly, this problem can be transformed to the satisfiability problem for $\{(\equiv \mathsf{mm}^{-1}\mathsf{oo}^{-1}), (\mathsf{pp}^{-1}\mathsf{dd}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1})\}$: given an instance $G_1 = (V, E_1)$, $G_2 = (V, E_2)$ of SP-Overlap, construct an instance as follows:

(1) for each node $v \in V$, introduce a variable v;
(2) for each $(v, w) \in E_1$, add the constraint $v(\equiv \mathsf{mm}^{-1}\mathsf{oo}^{-1})w$; and
(3) for each $(v, w) \notin E_2$, add the constraint $v(\mathsf{pp}^{-1}\mathsf{dd}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1})w$.

It is easy to see that the resulting instance has a model if and only if the given instance of SP-Overlap has a solution. Thus, it follows that the set of relations $S = \{(\equiv \mathsf{mm}^{-1}\mathsf{oo}^{-1}), (\mathsf{pp}^{-1}\mathsf{dd}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1})\}$ is NP-complete. By applying Lemma A.1 to the set S and the model transformation expand, we obtain that $S' = \{(\equiv \mathsf{oo}^{-1}), (\mathsf{pp}^{-1}\mathsf{dd}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1})\}$ is NP-complete.

Define the model transformation T_1 as follows: $T_1(f) = f'$, where f' is obtained from f by first setting $\epsilon = \mathrm{MD}(f)/((n+1) \cdot c)$ with $c = \max\{|f(x_i)|\ ;\ 1 \le i \le n\}$ and defining

$$f'(x_i) = [f(x_i^-) - |f(x_i)| \cdot \epsilon, f(x_i^+) + |f(x_i)| \cdot \epsilon].$$

Then, T_1 has the description f_{T_1} with the following properties: $f_{T_1}(\equiv) = (\equiv)$, $f_{T_1}(\mathsf{mm}^{-1}) = (\mathsf{pp}^{-1}\mathsf{oo}^{-1})$, $f_{T_1}(\mathsf{ss}^{-1}) = f_{T_1}(\mathsf{ff}^{-1}) = (\mathsf{dd}^{-1})$ and $f_{T_1}(b) = b$ for $b \in \{\mathsf{p}, \mathsf{p}^{-1}, \mathsf{o}, \mathsf{o}^{-1}, \mathsf{d}, \mathsf{d}^{-1}\}$. By applying Lemma A.1 to this transformation and S', we see that $S'' = \{(\equiv \mathsf{oo}^{-1}), (\mathsf{pp}^{-1}\mathsf{dd}^{-1})\}$ is NP-complete. Note that we cannot replace T_1 with the very similar $\mathsf{ordexpand}$ transformation since $\mathsf{ordexpand}$ changes the $(\equiv)$ relation.

Finally, define the model transformation T_2 as follows: $T_2(f) = f'$, where f' is obtained from f by first setting $\epsilon = \mathrm{MD}(f)/(n+1)$ and defining

$$f'(x_i) = [f(x_i^-) + i\epsilon, f(x_i^+) + i\epsilon].$$

It is easy to see that T_2 has the description f_{T_2} with the following properties: $f_{T_2}(\equiv) = (\mathsf{oo}^{-1})$, $f_{T_2}(\mathsf{mm}^{-1}) = (\mathsf{pp}^{-1}\mathsf{oo}^{-1})$, $f_{T_2}(\mathsf{ss}^{-1}) = f_{T_2}(\mathsf{ff}^{-1}) = (\mathsf{oo}^{-1}\mathsf{dd}^{-1})$ and $f_{T_2}(b) = b$ for $b \in \{\mathsf{p}, \mathsf{p}^{-1}, \mathsf{o}, \mathsf{o}^{-1}, \mathsf{d}, \mathsf{d}^{-1}\}$. By applying Lemma A.1 to this transformation and S'', we have shown that $\{(\mathsf{oo}^{-1}), (\mathsf{pp}^{-1}\mathsf{dd}^{-1})\}$ is NP-complete. □

[6]Note that this notion is different from the more restricted notion of 'overlap' used in definition of the basic relation o.

PROOF. (of Proposition 6.3) It is clear that sets $\{r\}$ and $\{r^{-1}\}$ are NP-complete simultaneously, so it suffices to consider only one of them.

We assume first that $r \cap (\mathsf{p}^{-1}\mathsf{m}^{-1}\mathsf{o}^{-1}) = \emptyset$. If $r \cap (\mathsf{pmo}) = \emptyset$, then it follows from Lemma 6.2 that r or r^{-1} equals $(\mathsf{ds}^{-1}\mathsf{f}^{-1})$; however, $C_9((\mathsf{ds}^{-1}\mathsf{f}^{-1})) = (\mathsf{d})$. Assume now that $r \cap (\mathsf{pmo}) \neq \emptyset$. By using Lemma 6.2 once again, one of the following holds:

(1) $(\mathsf{ds}^{-1}) \subseteq r$;
(2) $(\mathsf{d}^{-1}\mathsf{f}) \subseteq r$; or
(3) $(\mathsf{s}^{-1}\mathsf{f}) \subseteq r$.

Case 1: $(\mathsf{ds}^{-1}) \subseteq r$.
If $r \notin \{(\mathsf{mds}^{-1}), (\mathsf{mds}^{-1}\mathsf{f}), (\mathsf{mds}^{-1}\mathsf{f}^{-1})\}$, then $C_9(r) = (\mathsf{pmods})$ and it follows that $C_9(r) \cap r^{-1} = (\mathsf{s})$. If $r = (\mathsf{mds}^{-1}\mathsf{f})$ or $r = (\mathsf{mds}^{-1}\mathsf{f}^{-1})$, then $C_9(r) = (\mathsf{d})$ or (pd), respectively, and $C_9(r) \cap r = (\mathsf{d})$. Finally, if $r = (\mathsf{mds}^{-1})$, then $(r^{-1} \circ r) \cap r = (\mathsf{ds}^{-1})$ and $((\mathsf{ds}^{-1}) \circ (\mathsf{ds}^{-1})) \cap (\mathsf{m}^{-1}\mathsf{d}^{-1}\mathsf{s}) = (\mathsf{m}^{-1})$.
Case 2: $(\mathsf{d}^{-1}\mathsf{f}) \subseteq r$.
Dual to case 1.
Case 3: $(\mathsf{s}^{-1}\mathsf{f}) \subseteq r$.
We can assume that $(\mathsf{dd}^{-1}) \cap r = \emptyset$ since one of the previous two cases applies otherwise. If $r \neq (\mathsf{ms}^{-1}\mathsf{f})$, then one of the following holds:

(1) $(\mathsf{o}) \not\subseteq r$. This implies that $C_9(r) = (\mathsf{p})$ and NP-completeness follows immediately.
(2) $(\mathsf{o}) \subseteq r$. Then, $C_9(r) = (\mathsf{pmo})$, $(\mathsf{pmo}) \circ r = (\mathsf{pmodd}^{-1}\mathsf{sf}^{-1})$ and we can obtain the relation (dd^{-1}). Let $R = (\mathsf{d})$, $R_1 = (\mathsf{pmo})$, $R_2 = (\mathsf{dd}^{-1})$ and NP-completeness of $\{r\}$ follows from Lemma 4.1.

Finally, if $r = (\mathsf{ms}^{-1}\mathsf{f})$, then consider the following set of constraints:

$$\left\{\begin{array}{l} xra_1 \\ a_1ra_2 \\ a_2rx \\ yra_1 \\ yra_2 \end{array}\right\}$$

The relation between x and y derived from these constraints is $(\mathsf{pp}^{-1}\mathsf{dd}^{-1})$. Next, the relation (oo^{-1}) is derived from the following set of constraints:

$$\left\{\begin{array}{lll} xra_1 & a_1r_1a_2 & a_1ry \\ xr_1a_2 & a_2r_1a_3 & a_2r_1y \\ xra_3 & a_3r_1a_1 & a_3ry \end{array}\right\}$$

Consequently, NP-completeness of $\{(\mathsf{ms}^{-1}\mathsf{f})\}$ follows from Lemma A.2.

Finally, we consider the case when $r \cap (\mathsf{pmo}) \neq \emptyset$ and $r \cap (\mathsf{p}^{-1}\mathsf{m}^{-1}\mathsf{o}^{-1}) \neq \emptyset$; then Lemma 6.2 implies that r is not acyclic. The proof considers four cases depending on the value of $r \cap (\mathsf{pp}^{-1}\mathsf{oo}^{-1}\mathsf{mm}^{-1})$. For every $r_1 \in \mathcal{A}$, let r_1^2 denote the relation $r_1 \circ r_1$. Recall, that $\top$ denotes the union of all basic relations.

In the following, Lemma 4.2 significantly reduces the number of cases to be considered. For instance, by showing that $\{(\mathsf{mo}^{-1}\mathsf{ds})\}$ is NP-complete, we also know that $\{(\mathsf{m}^{-1}\mathsf{odf})\}$ is NP-complete.

<u>Case 1:</u> $r \cap (\mathsf{pp}^{-1}\mathsf{oo}^{-1}\mathsf{mm}^{-1}) \in \{(\mathsf{po}^{-1}), (\mathsf{pmo}^{-1}), (\mathsf{pm}^{-1}\mathsf{o}^{-1})\}$.

Suppose first that $r = r' \cup (\mathsf{po}^{-1})$.

r'	$C_9(r)$	NP-completeness of $\{r\}$
$\emptyset$	(p)	$C_9(r) = (\mathsf{p})$
(d)	(pmods)	$C_9(r) \cap r^{-1} = (\mathsf{o})$
(s)	(pds)	$C_9(r)^2 \cap r^{-1} = (\mathsf{o})$
(s^{-1})	(p)	$C_9(r) = (\mathsf{p})$
(ds)	(pmods)	$C_9(r) \cap r^{-1} = (\mathsf{o})$
(ds^{-1})	(pmods)	$(C_9(r) \cap r^{-1})^2 \cap r = (\mathsf{p})$
(df)	(pmods)	$C_9(r) \cap r^{-1} = (\mathsf{o})$
(df^{-1})		$C_{14}(r)^* \cap r = (\mathsf{o}^{-1}\mathsf{df}^{-1})$
(sf)	(pds)	$C_9(r)^2 \cap r^{-1} = (\mathsf{o})$
(sf^{-1})	$\top \setminus (\equiv \mathsf{s}^{-1}\mathsf{f})$	$C_9(r)^* \cap r = (\mathsf{po}^{-1})$
$(\mathsf{s}^{-1}\mathsf{f})$	(p)	$C_9(r) = (\mathsf{p})$
(dsf)	(pmods)	$C_9(r) \cap r^{-1} = (\mathsf{o})$
(dsf^{-1})	$\top \setminus (\equiv \mathsf{s}^{-1})$	$C_9(r)^* \cap r = (\mathsf{po}^{-1}\mathsf{df}^{-1})$
$(\mathsf{ds}^{-1}\mathsf{f})$	(pmods)	$(C_9(r) \cap r^{-1})^2 \cap r = (\mathsf{p})$
$(\mathsf{ds}^{-1}\mathsf{f}^{-1})$	$\top \setminus (\equiv \mathsf{s}^{-1})$	$C_9(r) \cap r = (\mathsf{po}^{-1}\mathsf{df}^{-1})$

Note that, in the table above, if $r' = (\mathsf{df}^{-1})$, that is, if $r = (\mathsf{po}^{-1}\mathsf{df}^{-1})$ then $C_{14}(r) = \top \setminus (\equiv \mathsf{s}^{-1})$, and we can obtain $(\mathsf{o}^{-1}\mathsf{df}^{-1})$ as shown in the table; NP-completeness of $(\mathsf{o}^{-1}\mathsf{df}^{-1})$ follows from Lemma 5.21 by using Lemma 4.2. If $r \cap (\mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{oo}^{-1})$ is (pmo^{-1}) or $(\mathsf{pm}^{-1}\mathsf{o}^{-1})$, then NP-completeness of $\{r\}$ follows from the proof above by using Lemma A.1 with model transformations shrink and expand, respectively.

<u>Case 2:</u> $r \cap (\mathsf{pp}^{-1}\mathsf{oo}^{-1}\mathsf{mm}^{-1}) = (\mathsf{pm}^{-1})$. Let $r = r' \cup (\mathsf{pm}^{-1})$.

r'	$C_9(r)$	NP-completeness of $\{r\}$
$\emptyset$	(p)	$C_9(r) = (\mathsf{p})$
(d)	(pmods)	$C_9(r) \cap r^{-1} = (\mathsf{m})$
(s)	(p)	$C_9(r) = (\mathsf{p})$
(s^{-1})	(p)	$C_9(r) = (\mathsf{p})$
(ds)	(pmods)	$C_9(r) \cap r^{-1} = (\mathsf{m})$
(ds^{-1})	(pmods)	$C_9(r) \cap r^{-1} = (\mathsf{ms})$
(df)	(pmods)	$C_9(r) \cap r^{-1} = (\mathsf{m})$
(df^{-1})	(pmods)	$C_9(r) \cap r^{-1} = (\mathsf{m})$
(sf)	(pds)	$C_9(r)^2 \cap r^{-1} = (\mathsf{m})$
(sf^{-1})	(pmo)	$C_9(r) \cap r^{-1} = (\mathsf{m})$
($\mathsf{s}^{-1}\mathsf{f}$)	(p)	$C_9(r) = (\mathsf{p})$
(dsf)	(pmods)	$C_9(r) \cap r^{-1} = (\mathsf{m})$
(dsf^{-1})	(pmods)	$C_9(r) \cap r^{-1} = (\mathsf{m})$
($\mathsf{ds}^{-1}\mathsf{f}$)	(pmods)	$C_9(r) \cap r^{-1} = (\mathsf{ms})$
($\mathsf{ds}^{-1}\mathsf{f}^{-1}$)	($\mathsf{pp}^{-1}\mathsf{modd}^{-1}\mathsf{sf}^{-1}$)	$C_9(r) \cap r^{-1} = (\mathsf{p}^{-1}\mathsf{md}^{-1}\mathsf{s})$

The relation (ms) generates (s) since $((\mathsf{m}^{-1}\mathsf{s}^{-1}) \circ (\mathsf{ms})) \cap (\mathsf{ms}) = (\mathsf{s})$.

<u>Case 3:</u> $r \cap (\mathsf{pp}^{-1}\mathsf{oo}^{-1}\mathsf{mm}^{-1}) = (\mathsf{pm}^{-1}\mathsf{o})$. Let $r = r' \cup (\mathsf{pm}^{-1}\mathsf{o})$.

r'	$C_9(r)$	NP-completeness of $\{r\}$
$\emptyset$	(p)	$C_9(r) = (\mathsf{p})$
(d)	(pmods)	$C_9(r) \cap r^{-1} = (\mathsf{m})$
(s)	(p)	$C_9(r) = (\mathsf{p})$
(s^{-1})	(pmo)	$C_9(r) \cap r^{-1} = (\mathsf{m})$
(ds)	(pmods)	$C_9(r) \cap r^{-1} = (\mathsf{m})$
(ds^{-1})	($\mathsf{pp}^{-1}\mathsf{moo}^{-1}\mathsf{dd}^{-1}\mathsf{s}$)	$(C_9(r) \cap r)^2 \cap r^{-1} = (\mathsf{ms})$
(df)	(pmods)	$C_9(r) \cap r^{-1} = (\mathsf{m})$
(df^{-1})	(pmods)	$C_9(r) \cap r^{-1} = (\mathsf{m})$
(sf)	(pmods)	$C_9(r) \cap r^{-1} = (\mathsf{m})$
(sf^{-1})	(pmo)	$C_9(r) \cap r^{-1} = (\mathsf{m})$
($\mathsf{s}^{-1}\mathsf{f}$)	($\mathsf{pp}^{-1}\mathsf{moo}^{-1}\mathsf{dd}^{-1}$)	$C_9(r) \cap r^{-1} = (\mathsf{p}^{-1}\mathsf{mo}^{-1})$
(dsf)	(pmods)	$C_9(r) \cap r^{-1} = (\mathsf{m})$
(dsf^{-1})	(pmods)	$C_9(r) \cap r^{-1} = (\mathsf{m})$
($\mathsf{ds}^{-1}\mathsf{f}$)	($\mathsf{pp}^{-1}\mathsf{moo}^{-1}\mathsf{dd}^{-1}\mathsf{sf}$)	$C_9(r) \cap r^{-1} = (\mathsf{p}^{-1}\mathsf{mo}^{-1}\mathsf{d}^{-1}\mathsf{s})$
($\mathsf{ds}^{-1}\mathsf{f}^{-1}$)	($\mathsf{pp}^{-1}\mathsf{moo}^{-1}\mathsf{dd}^{-1}\mathsf{sf}^{-1}$)	$C_9(r) \cap r^{-1} = (\mathsf{p}^{-1}\mathsf{mo}^{-1}\mathsf{d}^{-1}\mathsf{s})$

<u>Case 4:</u> $r \cap (\mathsf{pp}^{-1}\mathsf{oo}^{-1}\mathsf{mm}^{-1}) = (\mathsf{mo}^{-1})$. Let $r = r' \cup (\mathsf{mo}^{-1})$.

r'	$C_9(r)$	NP-completeness of $\{r\}$
$\emptyset$	$\emptyset$	$r^2 \cap r = (\mathsf{o}^{-1})$
(d)	$\emptyset$	$(r^2 \cap r)^2 \cap r^{-1} = (\mathsf{m}^{-1})$
(s)	$\emptyset$	$r^2 \cap r^{-1} = (\mathsf{m}^{-1}\mathsf{o})$
(s^{-1})	$\emptyset$	$((r^{-1} \circ r) \cap r)^2 \cap r^{-1} = (\mathsf{m}^{-1})$
(ds)	$\emptyset$	$((r^{-1} \circ r) \cap r)^2 \cap r^{-1} = (\mathsf{m}^{-1})$
(ds^{-1})	$\emptyset$	$((r^{-1} \circ r) \cap r)^2 \cap r^{-1} = (\mathsf{m}^{-1})$
(df)	(d)	$C_9(r) = (\mathsf{d})$
(df^{-1})	$(\mathsf{pp}^{-1}\mathsf{m}^{-1}\mathsf{oo}^{-1}\mathsf{f})$	$(C_9(r) \cap r)^2 \cap r^{-1} = (\mathsf{m}^{-1}\mathsf{f})$
(sf)	(d)	$C_9(r) = (\mathsf{d})$
(sf^{-1})	$(\mathsf{pp}^{-1}\mathsf{m}^{-1}\mathsf{o}^{-1}\mathsf{dd}^{-1})$	$C_9(r) \cap r = (\mathsf{o}^{-1})$
$(\mathsf{s}^{-1}\mathsf{f})$	$\emptyset$	$C_{9b}(r) = (\mathsf{mo}^{-1})$
(dsf)	(d)	$C_9(r) = (\mathsf{d})$
(dsf^{-1})	$(\mathsf{pp}^{-1}\mathsf{m}^{-1}\mathsf{oo}^{-1}\mathsf{dd}^{-1}\mathsf{sf})$	$C_9(r) \cap r^{-1} = (\mathsf{m}^{-1}\mathsf{od}^{-1}\mathsf{f})$
$(\mathsf{ds}^{-1}\mathsf{f})$	(d)	
$(\mathsf{ds}^{-1}\mathsf{f}^{-1})$	$(\mathsf{pp}^{-1}\mathsf{m}^{-1}\mathsf{oo}^{-1}\mathsf{dd}^{-1}\mathsf{f})$	$C_9(r) \cap r^{-1} = (\mathsf{m}^{-1}\mathsf{od}^{-1}\mathsf{f})$

□

A.3 Proof of Proposition 6.12

This subsection contains the proof of Proposition 6.12. We will use a short-hand notation for relations of the type (bb^{-1}) by writing (O) to denote (oo^{-1}), (S) to denote (ss^{-1}) and so on. We will also use combinations of these macro relations—for instance, by writing (PO), we mean the relation $(\mathsf{pp}^{-1}\mathsf{oo}^{-1})$.

Let s be a symmetric relation such that $s \not\subseteq (\equiv \mathsf{S})$ and $s \not\subseteq (\equiv \mathsf{F})$. We write '$\widehat{s}$ is NP-complete' to denote that, for every symmetric relation r such that $r \cap s = \emptyset$, $r \not\subseteq (\equiv \mathsf{S})$ and $r \not\subseteq (\equiv \mathsf{F})$, $\{r, s\}$ is NP-complete. When we show results of the form '$\widehat{s}$ is NP-complete', we will tacitly assume that r' is an arbitrary symmetric relation satisfying the requirements stated above. Thus, we can formulate Proposition 6.12 as follows:

if s is a symmetric relation such that $s \not\subseteq (\equiv \mathsf{S})$ and $s \not\subseteq (\equiv \mathsf{F})$, then $\widehat{s}$ is NP-complete.

To prove this result, we begin by showing that $\widehat{s}$ is NP-complete for all $s \subseteq (\equiv \mathsf{MSF})$ such that $s \not\subseteq (\equiv \mathsf{S})$ and $s \not\subseteq (\equiv \mathsf{F})$; this proof can be found in Subsection A.3.1.

Next, we show that we do not have to care about the $(\equiv)$ relation. More precisely, assume that $\{s, s'\}$ is NP-complete for all choices of s, s' such that

(1) s and s' are symmetric relations;

(2) $s \cap s' = \emptyset$;

(3) $(\equiv) \not\subseteq s$ and $(\equiv) \not\subseteq s'$;

(4) $s \not\subseteq (\mathsf{S})$ and $s' \not\subseteq (\mathsf{S})$; and
(5) $s \not\subseteq (\mathsf{F})$ and $s' \not\subseteq (\mathsf{F})$.

We show that $X = \{s \cup (\equiv), s'\}$ is NP-complete for all choices of s, s' satisfying the requirements above. If $s \subseteq (\mathsf{MSF})$ or $s' \subseteq (\mathsf{MSF})$, then X is NP-complete by Lemma A.11. Hence, we assume that each of s and s' contains at least one of the relations P, D, and O. Let $s_\equiv = s \cup (\equiv)$. It is easy to realize that $(\equiv) \not\subseteq s_\equiv \circ s'$ by inspecting the composition table. Furthermore, the following result can easily be shown: if $B, B' \in \{\mathsf{P}, \mathsf{D}, \mathsf{O}\}$, $B \neq \mathsf{D}$ and $B' \neq \mathsf{P}$, then $(B) \subseteq B \circ B'$. If $(\mathsf{D}) \not\subseteq s_\equiv$, then $s_1 = s_\equiv \cap (s_\equiv \circ s')^*$ is a non-trivial symmetric relation not containing $(\equiv)$ and it can be checked that the set $\{s_1, s'\}$ satisfies the conditions above and implying that it is NP-complete. Hence, $\{s_\equiv, s'\}$ is NP-complete. Now assume $(\mathsf{D}) \subseteq s_\equiv$. If $s' \cap (\mathsf{OSF}) \neq \emptyset$, then $(\mathsf{D}) \subseteq s_\equiv \circ s'$ and we can reason as above to show that $\{s_\equiv, s'\}$ is NP-complete. Otherwise, $s' \subseteq (\mathsf{PM})$ and NP-completeness follows from Proposition 5.1(3), since $\{s_\equiv, s'\}$ is contained in neither $\mathcal{S}_\mathsf{p}$ nor $\mathcal{E}_\mathsf{p}$, or from Lemma 6.13(1).

Hence, we can now restrict our attention to pairs of relations s, s' satisfying conditions 1-5 and such that $s \cap (\mathsf{POD}) \neq \emptyset$ and $s' \cap (\mathsf{POD}) \neq \emptyset$. These proofs are collected in Subsection A.3.2.

A.3.1 *Below* $(\equiv \mathsf{MSF})$

LEMMA A.3. *$\{(\mathsf{OD}), (\mathsf{SF})\}$ and $\{(\mathsf{OD}), (\equiv \mathsf{SF})\}$ are NP-complete.*

PROOF. We note that $((\mathsf{OD}) \circ (\equiv \mathsf{SF})) \cap (\equiv \mathsf{SF}) = (\mathsf{SF})$ so it is sufficient to give a proof for the set $\{(\mathsf{OD}), (\mathsf{SF})\}$. The proof is by a polynomial-time reduction from the NP-complete problem NOT-ALL-EQUAL SATISFIABILITY [Garey and Johnson 1979].

An instance of NOT-ALL-EQUAL SATISFIABILITY consists of a set U of Boolean variables, and a collection C of *clauses* over U, where each clause is a set of three *literals*, and a literal is a variable or a negated variable. The question is whether there is an assignment of truth variables to the variables such that each clause contains at least one true literal and at least one false literal.

To obtain the reduction from NOT-ALL-EQUAL SATISFIABILITY, we design three 'gadgets', that is, small sets of constraints with convenient properties. The first corresponds to a Boolean variable, the second corresponds to a clause, and the third ensures that the variables are connected to the clauses in the appropriate way. Hence there are three parts to the construction.

Let P be an instance of NOT-ALL-EQUAL SATISFIABILITY. We construct a corresponding instance I of $\mathcal{A}$-SAT$(\{(\mathsf{OD}), (\mathsf{SF})\})$ as follows:

(1) For each variable $u \in U$,
 —introduce variables v_{u_1}, v_{u_2} and v_{u_3} in V;
 —impose the constraint (SF) on the edges (v_{u_1}, v_{u_2}) and (v_{u_2}, v_{u_3});
 —impose the constraint (OD) on the edge (v_{u_1}, v_{u_3}).
(2) For each clause $c \in C$,
 —introduce variables $v_{c_1}, v_{c_2}, v_{c_3}$ and v_{c_4} in V;
 —impose the constraint (SF) on the edges (v_{c_1}, v_{c_2}), (v_{c_2}, v_{c_3}) and (v_{c_3}, v_{c_4});
 —impose the constraint (OD) on the edge (v_{c_1}, v_{c_4}).

(3) For each literal c_i, $i = 1,2,3$ in each clause c,
—introduce variables v'_{c_i} and v''_{c_i} in V and impose the constraint (SF) on the edge (v'_{c_i}, v''_{c_i});
—impose the constraint (SF) on the edge (v_{c_i}, v''_{c_i}) and impose the constraint (OD) on the edges (v_{c_i}, v'_{c_i}) and $(v_{c_{i+1}}, v''_{c_i})$;
—if c_i is the (unnegated) variable u, then impose the constraint (SF) on the edge (v'_{c_i}, v_{u_2}) and impose the constraint (OD) on the edges (v'_{c_i}, v_{u_1}) and (v''_{c_i}, v_{u_2});
—If c_i is the negated variable u, then impose the constraint (SF) on the edge (v'_{c_i}, v_{u_3}) and impose the constraint (OD) on the edges (v'_{c_i}, v_{u_2}) and (v''_{c_i}, v_{u_3}).

Clearly, this construction can be carried out in polynomial time and we will now show that I has a solution if and only if P has a solution. First, assume that I has a solution. We will use this to construct a corresponding solution to P. Consider a variable $u \in U$. Because of the constraints imposed in part (1) of the above construction, exactly one of the pairs (v_{u_1}, v_{u_2}) and (v_{u_2}, v_{u_3}) must be related by f or f^{-1}. If it is the pair (v_{u_1}, v_{u_2}), then we assign the value T (true) to u, otherwise we assign the value F (false) to u.

Now consider each clause $c \in C$. Because of the constraints imposed in part (3) of the construction above, the relation between v_{c_i} and $v_{c_{i+1}}$ is f or f^{-1} if and only if the corresponding literal is assigned the value T. Finally, because of the constraints imposed in part (2) of the above construction, v_{c_i} and $v_{c_{i+1}}$ must be related by f or f^{-1} for at least one and at most two of the 3 possibilities $i = 1,2,3$. Hence the chosen assignment gives at least one true literal and one false literal in each clause, and so is a solution to P.

Conversely, assume that P has a solution σ. We will use this to construct a corresponding solution to I.

Consider the variables v_{u_1}, v_{u_2} and v_{u_3} in V, which are associated with the variable $u \in U$. Assign v_{u_1} the interval $[5,8]$, and assign v_{u_3} the interval $[6,7]$. If u is assigned the value T in σ, then assign v_{u_2} the interval $[6,8]$, otherwise assign v_{u_2} the value $[5,7]$.

Now consider the variables $v_{c_1}, v_{c_2}, v_{c_3}$ and v_{c_4} in V, which are associated with the clause $c \in C$. Since σ is a solution, it must assign values to the literals c_1, c_2, c_3 which contain at least one true value and at least one false value. There are therefore 6 possibilities, and we assign intervals to the variables $v_{c_1}, v_{c_2}, v_{c_3}$ and v_{c_4} in each case according to the following table:

c_1	c_2	c_3	v_{c_1}	v_{c_2}	v_{c_3}	v_{c_4}
T	T	F	$[1,12]$	$[2,12]$	$[3,12]$	$[3,10]$
T	F	T	$[1,12]$	$[2,12]$	$[2,10]$	$[3,10]$
F	T	T	$[1,12]$	$[1,10]$	$[2,10]$	$[3,10]$
F	F	T	$[1,12]$	$[1,11]$	$[1,10]$	$[3,10]$
F	T	F	$[1,12]$	$[1,11]$	$[3,11]$	$[3,10]$
T	F	F	$[1,12]$	$[3,12]$	$[3,11]$	$[3,10]$

It is easy to check that each of these assignments satisfies all constraints on the variables $v_{c_1}, v_{c_2}, v_{c_3}$ and v_{c_4}.

It remains to show that each of these assignments can be extended to a complete solution to I by assigning the remaining variables $v'_{c_1}, v''_{c_1}, v'_{c_2}, v''_{c_2}, v'_{c_3}$ and v''_{c_3} appropriately. To show this we note that, for $i = 1, 2, 3$, if the literal c_i is assigned the value T in σ, then in order to satisfy the constraints on v'_{c_i} and v''_{c_i} the starting point of v'_{c_i} must equal the starting point of v_{u_1} or v_{u_2} and the starting point of v''_{c_i} must equal the starting point of v_{c_i}. In this case we simply assign the ending points of both of these intervals the value 13. Similarly, if the literal c_i is assigned the value F in σ, then the ending points of these intervals are forced to take certain (distinct) values, and we assign the starting points of these intervals the value 0. It is easy to check that this assignment satisfies all the constraints. For example, if $c = \{x, \neg y, z\}$ and $\sigma(x) = T, \sigma(y) = T$, and $\sigma(z) = F$ then these variables are assigned the values $[6, 13], [1, 13], [0, 7], [0, 12], [0, 7]$ and $[0, 11]$, respectively. □

Lemma A.4. $\widehat{(\mathsf{SF})}$ *is NP-complete.*

Proof. we have $(B) \subseteq (\mathsf{SF}) \circ (B)$ when $B \in \{\mathsf{P}, \mathsf{M}, \mathsf{O}, \mathsf{D}\}$. It can be easily verified that $(\equiv) \not\subseteq r_1 = (r' \cap ((\mathsf{SF}) \circ r'))^*$. Consequently, we may assume that $(\equiv) \not\subseteq r'$. If $r' \subseteq (\mathsf{PM})$ then NP-completeness follows from Proposition 5.1(3) or Lemma 6.13(1), so we assume that $r' \cap (\mathsf{OD}) \neq \emptyset$.

To show NP-completeness of the remaining cases, we introduce the OD-*switch*. The switch is an instance Γ on five variables a, b, c, x, y:

$$\{a(\mathsf{SF})b, a(\mathsf{SF})c, x(\mathsf{SF})b, x(\mathsf{SF})c, y(\mathsf{SF})b, y(\mathsf{SF})c\}.$$

It has the following properties (which can easily be checked):

(1) if $b(\mathsf{PM})c$ holds, then $x(\equiv)y$;
(2) if $b(\mathsf{D})c$ holds, then $x(\equiv \mathsf{O})y$; and
(3) if $b(\mathsf{O})c$ holds, then $x(\equiv \mathsf{D})y$.

It can be checked that one of the relations $(\equiv \mathsf{O})$, $(\equiv \mathsf{D})$, $(\equiv \mathsf{OD})$ is derived from $\Gamma \cup \{br'c\}$. Therefore we can obtain one of (O), (D), (OD) as described above. For (OD), apply Lemma A.3. If we obtain (O) or (D) then apply Lemma 4.1 with $R = (\mathsf{o})$, $R_1 = (\mathsf{SF})$, $R_2 = (\mathsf{O})$ or with $R = (\mathsf{d})$, $R_1 = (\mathsf{SF})$, $R_2 = (\mathsf{D})$ If $r' = (\mathsf{D})$, respectively. □

Lemma A.5. $\widehat{(\equiv \mathsf{SF})}$ *is NP-complete.*

Proof. This proof is analogous to the previous one because the properties of the OD-switch are the same if we replace the relation (SF) with $(\equiv \mathsf{SF})$. □

Lemma A.6. $\widehat{(\equiv \mathsf{M})}$ *is NP-complete.*

Proof. Consider the following:

$$\begin{array}{lcllcl}
(\equiv \mathsf{M}) \circ (\mathsf{P}) & = & (\mathsf{PMOd^{-1}s^{-1}f^{-1}}) & (\equiv \mathsf{M}) \circ (\mathsf{O}) & = & (\mathsf{POdsf}) \\
(\equiv \mathsf{M}) \circ (\mathsf{D}) & = & (\mathsf{PODsf}) & (\equiv \mathsf{M}) \circ (\mathsf{S}) & = & (\mathsf{p^{-1}mo^{-1}dSf}) \\
(\equiv \mathsf{M}) \circ (\mathsf{F}) & = & (\mathsf{pm^{-1}odsF}) & & &
\end{array}$$

If $(\mathsf{PSF}) \cap r' \neq \emptyset$, then $(\equiv) \not\subseteq r_1 = (\equiv \mathsf{M}) \circ r'$ but at least one of $(\mathsf{m}), (\mathsf{m}^{-1})$ is a member of r_1. Consequently, the relation $r_1 \cap (\equiv \mathsf{M})$ implies NP-completeness by either Proposition 5.2 or Lemma 6.13(1). Otherwise, $r' \subseteq (\mathsf{OD})$ and we have $(\mathsf{PO}) \subseteq ((\equiv \mathsf{M}) \circ r')^* \subseteq (\mathsf{POD})$. Since $(\equiv \mathsf{M}) \circ (\equiv \mathsf{M}) = (\equiv \mathsf{PSF})$, we can obtain the relation (P) and NP-completeness is a consequence of Proposition 5.1(3) because $\{(\equiv \mathsf{M}), (\mathsf{P})\}$ is contained in neither $\mathcal{S}_\mathsf{p}$ nor $\mathcal{E}_\mathsf{p}$. □

Lemma A.7. $\widehat{(\mathsf{MS})}$ *and* $\widehat{(\mathsf{MF})}$ *are NP-complete.*

Proof. We show the result for $\widehat{(\mathsf{MS})}$; the other case follows by using Lemma 4.2. Consider the symmetric relation $r_1 = (r' \cap ((\mathsf{MS}) \circ r'))^*$. It is easily verified that $(\equiv) \not\subseteq r_1$. Furthermore r_1 is non-empty since $(B) \subseteq (\mathsf{MS}) \circ (B)$ when $B \in \{\mathsf{P}, \mathsf{O}, \mathsf{D}\}$ and we know that $r' \not\subseteq (\equiv \mathsf{F})$. Consequently, we can assume that $(\equiv) \not\subseteq r'$.

Case 1: $(\mathsf{P}) \cap r' = \emptyset$ or $(\mathsf{F}) \cap r' = \emptyset$. We note the following:

$$\begin{array}{llll} (\mathsf{MS}) \circ (\mathsf{P}) = (\mathsf{PMOd^{-1}s^{-1}f^{-1}}) & (\mathsf{MS}) \circ (\mathsf{O}) = (\mathsf{PmODsF}) \\ (\mathsf{MS}) \circ (\mathsf{D}) = (\mathsf{PmODsF}) & (\mathsf{MS}) \circ (\mathsf{F}) = (\mathsf{pMODs}) \end{array}$$

It follows that $r_1 = ((\mathsf{MS}) \circ r') \cap (\mathsf{MS})$ equals either (ms), $(\mathsf{ms^{-1}})$, (mS), (Ms), or $(\mathsf{Ms^{-1}})$. If $r_1 = (\mathsf{Ms})$ or $r_1 = (\mathsf{Ms^{-1}})$, then $r_1^* = (\mathsf{M})$ and NP-completeness follows from Lemma 6.13(1). If $r_1 = (\mathsf{ms})$ or $r_1 = (\mathsf{ms^{-1}})$, then

$$((\mathsf{m^{-1}s^{-1}}) \circ (\mathsf{ms})) \cap (\mathsf{ms}) = ((\mathsf{m^{-1}s}) \circ (\mathsf{ms^{-1}})) \cap (\mathsf{m^{-1}s}) = (\mathsf{s}).$$

Finally, if $r_1 = (\mathsf{mS})$, then $(\mathsf{S}) = r_1^*$ and

$$\begin{array}{llll} (\mathsf{S}) \circ (\mathsf{P}) = (\mathsf{Pmod^{-1}f^{-1}}) & (\mathsf{S}) \circ (\mathsf{O}) = (\mathsf{pmODF}) \\ (\mathsf{S}) \circ (\mathsf{D}) = (\mathsf{pmODF}) & (\mathsf{S}) \circ (\mathsf{F}) = (\mathsf{pmOD}) \end{array}$$

Hence, $((\mathsf{S}) \circ r') \cap (\mathsf{MS}) = (\mathsf{m})$ and NP-completeness follows immediately.
Case 2: $(\mathsf{PF}) \subseteq r'$. Consider the following instance Γ over the variables x, y, a, b:

$$x(\mathsf{MS})y, y(\mathsf{MS})a, a(\mathsf{MS})x, b(\mathsf{MS})x, b(\mathsf{MS})y, br'a.$$

One can show $x(\mathsf{S})y$ is derived from Γ so $((\mathsf{S}) \circ r') \cap (\mathsf{MS}) = (\mathsf{m})$ and NP-completeness follows. □

Lemma A.8. $(\equiv \widehat{\mathsf{MS}})$ *and* $(\equiv \widehat{\mathsf{MF}})$ *are NP-complete.*

Proof. We show the result for $(\equiv \widehat{\mathsf{MS}})$; the other case follows by applying Lemma 4.2. It can be shown that

$$r_1 = ((\equiv \mathsf{MS}) \circ r') \cap (\equiv \mathsf{MS}) \in \{(\mathsf{m}), (\mathsf{ms}), (\mathsf{Ms}), (\mathsf{Ms^{-1}})\}.$$

All four cases lead to NP-completeness as shown in Lemma A.7. □

Lemma A.9. $\widehat{(\mathsf{MSF})}$ *is NP-complete.*

Proof. It is easy to see that $B \subseteq (\mathsf{MSF}) \circ B$ when $B \in \{(\mathsf{P}), (\mathsf{O}), (\mathsf{D})\}$. Hence, $r' \subseteq (\mathsf{MSF}) \circ r'$. By inspecting the composition table, one can see that if $(\equiv)$ is a member of the composition of two symmetric relations, then these two relations cannot be disjoint. It follows that $(\equiv) \not\subseteq (\mathsf{MSF}) \circ r'$ so we can obtain the relation $r' \setminus (\equiv) = r' \cap ((\mathsf{MSF}) \circ r')$. We assume henceforth that $(\equiv) \not\subseteq r'$ and continue by noting that $(\mathsf{MSF}) \circ (\mathsf{P}) = (\mathsf{PMOd}^{-1}\mathsf{s}^{-1}\mathsf{f}^{-1})$ and $(\mathsf{MSF}) \circ (\mathsf{O}) = (\mathsf{MSF}) \circ (\mathsf{D}) = \mathsf{neq}$ where $\mathsf{neq} = \top \setminus (\equiv)$.

If $r' = (\mathsf{P})$, then NP-completeness follows from Proposition 5.1(3). Otherwise, $(\mathsf{MSF}) \circ r' = \mathsf{neq}$ and we can show NP-completeness by a polynomial-time reduction from Graph 4-colourability. Let $G = (V, E)$ be an arbitrary, undirected graph and construct a set of Allen constraints as follows:

(1) introduce two variables x, y and the constraint $xr'y$;
(2) for each $v \in V$, introduce a variable v and the constraints $v(\mathsf{MSF})x$ and $v(\mathsf{MSF})y$; and
(3) for each $(v, w) \in E$, add the constraint $v\mathsf{neq}w$.

It is a routine verification to see that the resulting instance is satisfiable iff G is 4-colourable. □

Lemma A.10. $(\widehat{\equiv \mathsf{MSF}})$ *is NP-complete.*

Proof. Note that $(\equiv \mathsf{MSF}) \circ (\mathsf{P}) = (\mathsf{PMOd}^{-1}\mathsf{s}^{-1}\mathsf{f}^{-1})$ and

$$(\equiv \mathsf{MSF}) \circ (\mathsf{O}) = (\equiv \mathsf{MSF}) \circ (\mathsf{D}) = \mathsf{neq}.$$

If $r' = (\mathsf{P})$, then NP-completeness follows from Proposition 5.1(3). Otherwise, $(\equiv \mathsf{MSF}) \circ r' = \mathsf{neq}$ so we can generate the relation (MSF) and NP-completeness follows from Lemma A.9. □

Lemma A.11. *If* $s \subseteq (\equiv \mathsf{MSF})$, *then* $\widehat{s}$ *is NP-complete.*

Proof. Combine Lemmas 6.13 and A.6–A.10. □

A.3.2 *Remaining cases.* This subsection considers the relations not covered in the previous section. We will henceforth assume that $r' \not\subseteq (\equiv \mathsf{MSF})$ and we will only consider $\widehat{s}$ where $s \not\subseteq (\equiv \mathsf{MSF})$. In the proofs, we will make use of a derivation $C_{39}(r_1, r_2)$ that denotes the relation (between x and y) derived from the set of constraints found in Table V.

• $\widehat{(\mathsf{PS})}$ and $\widehat{(\mathsf{PF})}$: We give the proof for $\widehat{(\mathsf{PS})}$; the other case follows by applying Lemma 4.2. Now, $(\mathsf{PS}) \circ (B) = (\mathsf{PMODsF})$ when $B \in \{\mathsf{M}, \mathsf{O}, \mathsf{D}\}$ and $(\mathsf{PS}) \circ (\mathsf{F}) = (\mathsf{PmODs})$. Thus, $(((\mathsf{PS}) \circ r')) \cap (\mathsf{PS}))^* = (\mathsf{P})$ and the result follows from Proposition 5.1.

• $\widehat{(\mathsf{PSF})}$: Recall that $r' \neq (\mathsf{M})$ and consider the following instance Γ on $\{x, y, a, b\}$:

$$\{x(\mathsf{PSF})a, ar'y, x(\mathsf{PSF})b, br'y, ar'b.\}$$

Let $r_1 = \top \setminus (\equiv \mathsf{s}^{-1}\mathsf{f}^{-1})$. One can show that xr_1y is derived from Γ which implies that $(\mathsf{P}) = ((\mathsf{PSF}) \cap r_1)^*$ can be obtained. NP-completeness follows from Proposition 5.1(3).

• $\widehat{s}$ when $(\mathsf{PM}) \subseteq s \subseteq (\mathsf{PMSF})$: Follows from previous results by applying Lemma A.1 with model transformation shrink.

We will from now assume that r' does not satisfy the inclusion $(\mathsf{P}) \subseteq r' \subseteq (\mathsf{PMSF})$; the previous results together with Proposition 5.1(3) allows us to do this without

$$\left\{\begin{array}{lll}
x r_1 a_1 & a_7 r_1 a_1 & y r_1 a_1 \\
a_1 r_1 a_2 & a_7 r_2 x & y r_1 a_2 \\
x r_1 a_2 & a_7 r_2 a_2 & y r_1 a_3 \\
 & & \\
a_3 r_2 x & a_8 r_1 x & y r_1 a_4 \\
a_3 r_2 a_1 & a_8 r_2 a_1 & y r_2 a_5 \\
a_3 r_1 a_2 & a_8 r_2 a_2 & y r_2 a_6 \\
 & & \\
a_4 r_1 x & a_9 r_1 a_1 & y r_1 a_7 \\
a_4 r_2 a_1 & a_9 r_2 x & y r_1 a_8 \\
a_4 r_2 a_2 & a_9 r_2 a_2 & y r_1 a_9 \\
 & & \\
a_5 r_1 x & a_7 r_1 a_8 & \\
a_5 r_1 a_1 & a_8 r_1 a_9 & \\
a_5 r_1 a_2 & a_7 r_2 a_9 & \\
 & & \\
a_6 r_1 x & & \\
a_6 r_2 a_1 & & \\
a_6 r_2 a_2 & & \\
 & & \\
a_4 r_1 a_5 & & \\
a_5 r_1 a_6 & & \\
a_4 r_2 a_6 & &
\end{array}\right\}$$

Table V. Definition of $C_{39}(r_1, r_2)$.

loss of generality.

• $\widehat{(\mathsf{D})}$: By setting $R = (\mathsf{d})$, $R_1 = r'$ and $R_2 = (\mathsf{D})$ and applying Lemma 4.1, we have that $\{(\mathsf{D}), r'\}$ is NP-complete.

• $\widehat{(\mathsf{O})}$: Case 1. $(\mathsf{PD}) \subseteq r' \subseteq (\mathsf{PMDSF})$. The case $r' = (\mathsf{PD})$ is considered in Lemma A.2. Otherwise apply Lemma A.2 and Lemma A.1 with model transformation ordshrink.
Case 2. $(\mathsf{D}) \subseteq r' \subseteq (\mathsf{DSF})$. Apply Lemma A.1 with model transformation ordshrink and then Lemma 4.1 with $R = (\mathsf{d})$, $R_1 = (\mathsf{O})$, $R_2 = (\mathsf{D})$.
Case 3. $(\mathsf{DM}) \subseteq r' \subseteq (\mathsf{DMSF})$. Now, $(\mathsf{D}) \subseteq C_{39}(r', (\mathsf{O})) \subseteq (\mathsf{PD})$ so we can obtain the relation (D).

• $\widehat{(\mathsf{PM})}$ and $\widehat{(\mathsf{MO})}$: Using Lemma A.1 with model transformations shrink and expand, the result follows from the earlier results.

• $\widehat{(\mathsf{PD})}$: Obviously, $(\mathsf{O}) \subseteq r' \subseteq \top \setminus (\mathsf{PD})$. Apply Lemma A.1 with model transformation ordexpand and then use Lemma A.2.
• $\widehat{(\mathsf{PO})}$: Case 1: $(\mathsf{D}) \subseteq r' \subseteq (\mathsf{DSF})$. Apply Lemma A.1 with model transformation ordshrink and then Lemma 4.1 with $R = (\mathsf{d})$, $R_1 = (\mathsf{PO})$, $R_2 = (\mathsf{D})$.

Case 2: $(\mathsf{DM}) \subseteq r' \subseteq (\mathsf{DMSF})$. In this case, $(\mathsf{DSF}) \subseteq C_{39}(r', (\mathsf{PO})) \subseteq (\mathsf{PODSF})$ and we can obtain a relation r_1 such that $(\mathsf{D}) \subseteq r_1 \subseteq (\mathsf{DSF})$. Hence, NP-completeness follows from the previous case.

• $(\widehat{\mathsf{OS}})$ and $(\widehat{\mathsf{OF}})$: We prove the result for $(\widehat{\mathsf{OS}})$; the other case follows by applying Lemma 4.2.
Case 1. $(\mathsf{PD}) \subseteq r' \subseteq (\mathsf{PMDF})$. Apply Lemma A.1 thrice with model transformations shrink, leftordshrink, and ordshrink, consecutively, and then use Lemma A.2.
Case 2. $(\mathsf{D}) \subseteq r' \subseteq (\mathsf{DMF})$. Now, $(\mathsf{ODS}) \subseteq C_{39}(r', (\mathsf{OS})) \subseteq (\mathsf{PODS})$ so we can obtain the relation (D).

• $(\widehat{\mathsf{DS}})$ and $(\widehat{\mathsf{DF}})$: We prove the result for $(\widehat{\mathsf{DS}})$; the other case follows by applying Lemma 4.2.
Case 1. $(\mathsf{PO}) \subseteq r' \subseteq (\mathsf{PMOF})$. Apply Lemma A.1 thrice with model transformations shrink, leftordexpand, and ordexpand, consecutively, and then use Lemma A.2.
Case 2. $(\mathsf{O}) \subseteq r' \subseteq (\mathsf{OMF})$. Use Lemma A.1 with model transformation expand and the earlier results.

• $(\widehat{\mathsf{MD}})$: Case 1. $(\mathsf{O}) \subseteq r' \subseteq (\mathsf{OSF})$. In this case, $(\mathsf{D}) \subseteq C_{39}((\mathsf{MD}), r') \subseteq (\mathsf{ODSF})$ and NP-completeness follows since we can obtain the relation (D).
Case 2. $(\mathsf{PO}) \subseteq r' \subseteq (\mathsf{POSF})$. Now, $(\mathsf{DSF}) \subseteq C_{39}((\mathsf{MD}), r') \subseteq (\mathsf{PODSF})$ so we can obtain the relation (D).

Now, it remains to consider pairs r, s of disjoint relations such that neither r nor s contains $(\equiv)$ and both of them contain exactly six basic relations. There are 10 such pairs. NP-completeness of $\{(\mathsf{MSF}), (\mathsf{POD})\}$ was proved in Lemma A.9. NP-completeness of $\{(\mathsf{PMS}), (\mathsf{ODF})\}$, $\{(\mathsf{PMF}), (\mathsf{ODS})\}$, and $\{(\mathsf{PSF}), (\mathsf{MOD})\}$ was shown earlier in this subsection. Here we consider the remaining six pairs.

For the pairs $\{(\mathsf{PMO}), (\mathsf{DSF})\}$, $\{(\mathsf{PMD}), (\mathsf{OSF})\}$, and $\{(\mathsf{PDF}), (\mathsf{MOS})\}$, apply Lemma A.1 with model transformations shrink, leftordshrink, and expand, respectively, and then use earlier results. NP-completeness of $\{(\mathsf{PDS}), (\mathsf{MOF})\}$ follows from that of $\{(\mathsf{PDF}), (\mathsf{MOS})\}$ by using Lemma 4.2.

Consider $\{(\mathsf{POF}), (\mathsf{MDS})\}$. It can be verified that $C_{39}((\mathsf{MDS}), (\mathsf{POF})) = (\mathsf{PODSF})$, and we can obtain the relation (DS) and use earlier results. Finally, NP-completeness of $\{(\mathsf{POS}), (\mathsf{MDF})\}$ follows from that of $\{(\mathsf{POF}), (\mathsf{MDS})\}$ by using Lemma 4.2 once again.

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