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# Bounds on the max and min bisection of random cubic and random 4-regular graphs*

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**Abstract**

In this paper we present a randomized algorithm to compute the bisection width of cubic and 4-regular graphs. The analysis of the proposed algorithms on random graphs provides asymptotic upper bounds for the bisection width of random cubic and random 4-regular graphs with $n$ vertices, giving upper bounds of $0.174039n$ for random cubic, and of $0.333333n$ for random 4-regular. We also obtain asymptotic lower bounds for the size of the maximum bisection, for random cubic and random 4-regular graphs with $n$ vertices, of $1.32697n$ and $1.66667n$, respectively. The randomized algorithms are derived from initial greedy algorithm and their analysis is based on the differential equation method.

## 1 Introduction

Given a graph $G = (V, E)$ with $|V|$ even, a *bisection* of $G$ is a partition of $V$ into two parts with the same cardinality, and its *size* is the number of edges crossing between the parts. A *minimum bisection* is a bisection of $V$ with minimal size (to avoid parity problems, all through this paper we assume that $n = |V|$ is even). The size of a minimum bisection is called the *bisection width* and the *min bisection problem* consists of finding a minimum bisection in a given $G$. In the same manner, we can also consider a *maximum bisection*, i.e. a bisection that maximizes the number of crossing edges. A related problem is that of finding the largest bipartite subgraph of a graph, i.e. a bipartite subgraph with as many edges as possible. This problem is known as the *max cut Problem* (see for example [GJ79]). Given a graph, the size of a maximum bisection is clearly a lower bound on the size of a max cut in the graph.

The min bisection problem has received a lot of attention, as the bisection width plays an important role in finding lower bounds to the routing performance of a network. The decisional version of the problem is known to be NP-complete [GJ79], even for cubic graphs [BCLS87]. In [AKK95]it is shown that min bisection has a PTAS for everywhere dense graphs (graphs with minimum degree $\Omega(n)$). Moreover, there exists a $O(\log^2 n)$ approximation for the min bisection on general graphs and a $O(\log n)$ approximation for planar graphs [FK00]. The min bisection problem can be solved in polynomial time for bounded treewidth graphs

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[JKLS01]. Several other exact and heuristic positive results are known for particular cases of the problem (see for example [DPS02]).

With respect to lower bounds, it is well known the spectral lower bound of $\frac{\lambda_2 n}{4}$ to the bisection width of any graph, where $\lambda_2$ is the second eigenvalue of the Laplacian of the graph [Fie74]. Bollobas provided a lower bound of $(\frac{d}{2} - \sqrt{d \ln 2})\frac{n}{2}$, for almost all $d$-regular graphs [Bol84]. In the same paper, he gave a lower bound of $0.22n$ for the particular case of 4-regular graphs. In [KM92] it is shown that almost all cubic graphs have bisection width greater than $0.101n$. Recently, using spectral techniques, Bezrukov et al. have given lower bounds for of $0.082n$ for the bisection width of cubic Ramanujan graphs, and of $0.176n$ for the case of 4-regular Ramanujan graphs [BEM+00].

Regarding upper bounds, in [KM92] it is shown that asymptotically, all $d$ regular graphs have bisection width at most $\frac{d-2}{4}n + O(d\sqrt{n}\log n)$. Alon proved that for $n > 40d^9$, all $d$-regular graph have bisection width of at most $(\frac{d}{4} - \frac{2\sqrt{d}}{32\sqrt{2}})n$ [Alo97]. Recently, Monien and Preis [MP01] gave upperbounds to the bisection width of $(0.16666 + \epsilon)n$ for 3-regular graphs and of $(0.4 + \epsilon)n$, for any $\epsilon$, for $n$ bigger than a function of the choosen $\epsilon$.

The problem of finding the maximum bisection also received a lot of attention. The problem is again NP-hard even for planar graphs [JKLS01]. The problem is known to be solvable in polynomial time for bounded treewidth graphs [JKLS01]. There exists several approximations algorithms for the problem; the max bisection problem has a PTAS for planar graphs [KKL00]. In the case of regular graphs, there is a 0.795 approximation algorithm for the max bisection [FKL00]. In the same paper, the authors gave an 0.834 approximation algorithm for the special case of max bisection on cubic graphs. The best approximation algorithm for max bisection on general graphs has an approximation ratio of 0.7027 [FL01]. We are not aware of any non-trivial lower bounds on the size of the maximum bisection,

We wil use standard notation and we refer the reader to [LJR00], for the definitions of u.a.r. (uniformly at random) and a.a.s. (asymptotically almost surely).

In this paper we present asymptotic results results for the expected bisection width and the expected maximum bisection of random cubic and random 4-regular graphs. In particular we prove the following theorems,

**Theorem 1** *The bisection width of a random 4-regular graph on $n$ vertices is a.a.s. smaller than $0.3333333n$.*

This upper bound improves the best known previous upper bound of $0.4n$ by Monien and Preis. Recall that Bollobas' lower bound for 4-regular graphs is $0.22n$.

**Theorem 2** *The bisection width of a random cubic graph on $n$ vertices is a.a.s. smaller than $0.174039n$.*

This asymptotic upper bound basically coincides with the upper bound by Monien and Preis [MP01].

Modifying slightly the proposed algorithms and, using the same analysis as for the previous results, we can get the following lower bounds to the maximum bisection of random cubic and random 4-regular graphs,

**Theorem 3** *The maximum bisection of a random 4-regular graph with $n$ vertices is a.a.s. greater than $1.32697n$.*

**Theorem 4** *The maximum bisection of a random cubic graph with $n$ vertices is a.a.s. greater than* $1.6666667n$.

As mentioned above, the size of the maximum bisection is a trivial lower bound on the size of the maximum cut, but removing the balance constraint does not permit our method to obtain any better result. We conjecture that the largest balanced bipartite subgraph of a random d-regular is a.a.s. almost the same size as the largest bipartite subgraph. For the particular example of random cubic graphs, we can state this even more strongly, as follows.

**Conjecture 1** *For every $\epsilon > 0$, a.a.s. the largest bipartite subgraph of a random cubic graph has a 2-coloring with the difference in the numbers of vertices of the two colors less than $\epsilon n$.*

## 2 Greedy algorithms for the minimum bisection of random cubic and 4-regular graphs

In this section we give the basic randomized greedy procedures to find a bisection for random cubic and random 4-regular graphs. We also introduce some generic notation to be used in the analysis.

Given a graph, and given a partial assignment of colors red (R) and blue (B) to its vertices, we classify the non-colored vertices according with the number of their colored neighbors:

> A vertex is of **Type** $(r, b)$ if it has $r$ neighbors colored R
> and $b$ neighbors colored B.

We say that a pair of uncolored vertices is $(r, b)$-symmetric if their types are $(r, b)$ and $(b, r)$.

The greedy procedures work by coloring vertices chosen randomly in symmetric pairs, to maintain balancedness, and repeatedly use one of the two operations: the *majority operation* (Maj), that colors each vertex with the majority color among its neighbors, and the *random operation* (Rand) that randomly color one vertex R and the other B.

The greedy procedure 4-min greedy for random 4-regular graphs is given in Figure 1, while the greedy procedure 3-min greedy for random cubic graphs is given in Figure 2.

The main difference in both algorithms is that while the algorithm for random 4-regular graphs consists of only one phase, for random cubic graphs the algorithm consists of 3 phases. This fact simplifyies the analysis for random 4-regular graphs, which is presented in the first place.

In phase 1, the algorithm 4-min greedy considers only (1,0), (2,0) and (2,1) symmetric pairs and gives priority to the (2,0) and (2,1) symmetric pairs of uncolored vertices over the (1,0)-symmetric pairs. Note that the size of the bisection is the number of *bicolored* edges, with one vertex of each color, so only each dual Maj operation on a (2,1)-symmetric pair contributes, with 2, to the bisection.

On the other hand, each phase of the algorithm 3-min greedy considers two types of symmetric pairs and gives priority to one of them. Observe that in the first phase there is no contribution to the bisection, in the second phase every time a (1,1)-symmetric pair is colored the bisection is increased by 2, and in the last phase, the bisection is increased by 2, every time a (2,1)-symmetric pair is colored.

One method of analyzing the performance of a randomized algorithm is to use a system of differential equations to express the expected changes in the variables describing the state of

*Initial step:* select two non-adjacent vertices u.a.r., color one with R and the other with B

*Phase 1:* **repeat**
  **if** there are vertices of both types (2,0) and (0,2)
    **or** vertices of types (2,1) and (1,2)
  **then**
    select u.a.r. a (2,0) or (2,1)-symmetric pair and perform Maj;
  **else if** there are vertices of both types (1,0) and (0,1)
    **then**
      select u.a.r. a (1,0)-symmetric pair and perform Maj;
  **until** no new vertex is colored

color any remaining uncolored vertices, half of them R and half B, in any manner, and output the bisection R, B.

Figure 1: Algorithm 4-min greedy for obtaining a bisection of 4-regular graphs

the algorithm during its execution. An exposition of this method can be found in [Wor99a], which includes various examples of graph-theoretic optimization problems. For purposes of exposition, we continue for the present to discuss the proposed algorithms, without giving full justification. After this, in order to reduce the complexity of the justification, it is in fact a different but related algorithm which we will analyze to yield our claimed bounds. We call this variation of algorithm a *deprioritized* algorithm as in [Wor03], where this technique was first used.

We use the pairing model to generate $n$-vertex $d$-regular graphs u.a.r. Briefly, to generate such a random graph, it is enough to begin with $dn$ *points* in $n$ *cells*, and choose a random perfect matching of the points, which we call a *pairing*. The corresponding pseudograph (possibly with loops or multiple edges) has the cells as vertices and the pairs as edges. Any property a.a.s. true of the random pseudograph is also a.a.s. true of the restriction to random graphs, with no loops or multiple edges, and this restricted probability space is uniform (see for example [Bol85, Wor99b] for a full description).

Without loss of generality, when stating such asymptotic results, we restrict $n$ to being even to avoid parity problems. We consider the greedy algorithms applied directly to the random pairing. As discussed in [Wor99a], the random pairing can be generated pair by pair, and at each step a point $p$ can be chosen by any rule whatsoever, as long as the other point in the pair is chosen u.a.r. from the remaining unused points. We call this step *exposing* the pair containing $p$.

At each point in the algorithm, let $Z_{rb}$ represent the number of uncolored vertices of type $(r, b)$, and let $W$ denote the number of points not yet involved in exposed pairs, for $d$-regular graphs we get

$$W = \sum_{r+b \leq d} (d - r - b) Z_{rb}.$$

To analyze the algorithm, when a vertex is colored we immediately expose all pairs involved in that vertex. In this way, the numbers $Z_{rb}$ are always determined. Furthermore, $W$ points are available for the pairs that will be exposed during the next step.

*Initial step:* select two non-adjacent vertices u.a.r., color one with R
and the other with B
*Phase 1:* **repeat**
**if** there are vertices of both types (2,0) and (0,2)
**then**
select u.a.r. a (2,0)-symmetric pair and perform Maj;
**else if** there are vertices of both types (1,0) and (0,1)
**then**
select u.a.r. a (1,0)-symmetric pair and perform Maj;
**until** no new vertex is colored
*Phase 2:* **repeat**
**if** there are vertices of both types (1,0) and (0,1)
**then**
select u.a.r. a (1,0)-symmetric pair and perform Maj;
**else if** there are at least two vertices of type (1,1)
**then**
select u.a.r. a (1,1)-symmetric pair and perform Rand;
**until** no new vertex is colored
*Phase 3:* **repeat**
**if** there are vertices of both types (3,0) and (0,3)
**then**
select u.a.r. a (3,0)-symmetric pair and perform Maj;
**else if** there are vertices of both types (2,1) and (1,2)
**then**
select u.a.r. a (2,1)-symmetric pair and perform Maj;
**until** no new vertex is colored
color any remaining uncolored vertices, half of them R and half B,
in any manner, and output the bisection R, B.

Figure 2: Algorithm 3-min greedy for obtaining a bisection of cubic graphs

# 3 Analysis of the algorithm for random 4-regular graphs

When considering rndom 4-regular graphs, at any time step, the number of points not yet involved in exposed pairs is

$$W = 4Z_{00} + 3Z_{10} + 3Z_{01} + 2Z_{02} + 2Z_{20} + 2Z_{11} + Z_{12} + Z_{21} + Z_{03} + Z_{30}.$$

Consider what happens when a vertex $u$ is newly colored R and one of the pairs containing a point $p$ in that cell is exposed. The other point will lie in some vertex $v$. Let $d_{rb}$ denote the expected contribution to the increment $\Delta(Z_{rb})$ in $Z_{rb}$ due to the change in the status of

$v$. Then, up to terms $O(1/W)$,

$$
\begin{array}{llll}
d_{00} = -\dfrac{4Z_{00}}{W} & & & \\
d_{10} = \dfrac{4Z_{00} - 3Z_{10}}{W} & d_{01} = -\dfrac{3Z_{01}}{W} & & \\
d_{20} = \dfrac{3Z_{10} - 2Z_{20}}{W} & d_{11} = \dfrac{3Z_{01} - 2Z_{11}}{W} & d_{02} = -\dfrac{2Z_{02}}{W} & \\
d_{30} = \dfrac{2Z_{20} - Z_{30}}{W} & d_{21} = \dfrac{2Z_{11} - Z_{21}}{W} & d_{12} = \dfrac{2Z_{02} - Z_{12}}{W} & d_{03} = -\dfrac{Z_{03}}{W} \\
d_{40} = \dfrac{Z_{30}}{W} & d_{31} = \dfrac{Z_{21}}{W} & d_{22} = \dfrac{Z_{12}}{W} & d_{13} = \dfrac{Z_{03}}{W}
\end{array}
\qquad d_{04} = 0.
$$

The error term $O(1/W)$ is due to adjustments occurring when $v$ happens to be the same as $u$.

The corresponding equations when a vertex is colored B form a symmetric set with these: they are the same but with the index pair on all variables swapped. Therefore, denoting by $\bar{d}_{r,b}$ the expected increments due to a dual step, consisting of coloring two vertices one R and the other B, we get

$$
\begin{array}{llll}
\bar{d}_{00} = -\dfrac{8Z_{00}}{W} & & & \\
\bar{d}_{10} = \dfrac{4Z_{00} - 6Z_{10}}{W} & \bar{d}_{01} = \dfrac{-6Z_{01} + 4Z_{00}}{W} & & \\
\bar{d}_{20} = \dfrac{3Z_{10} - 2Z_{20}}{W} & \bar{d}_{11} = \dfrac{3Z_{01} + 3Z_{10} - 4Z_{11}}{W} & \bar{d}_{02} = \dfrac{-4Z_{02} + 3Z_{01}}{W} & \bar{d}_{03} = \dfrac{-2Z_{03} + 2Z_{02}}{W} \\
\bar{d}_{30} = \dfrac{2Z20 - 2Z_{30}}{W} & \bar{d}_{21} = \dfrac{2Z_{11} - 2Z_{21} + 2Z_{20}}{W} & \bar{d}_{12} = \dfrac{2Z_{02} - 2Z_{12} + 2Z_{11}}{W} & \bar{d}_{13} = \dfrac{Z_{03} + Z_{12}}{W} \\
\bar{d}_{40} = \dfrac{Z_{30}}{W} & \bar{d}_{31} = \dfrac{Z_{21} + Z_{30}}{W} & \bar{d}_{22} = \dfrac{Z_{12} + Z_{21}}{W} & \bar{d}_{04} = \dfrac{Z_{03}}{W}.
\end{array}
\tag{1}
$$

We make the assumption of having *rb-symmetry*: for all $i$ and $j$, $Z_{ij} = Z_{ji}$. Later we will see how to remove it. With this last assumption, the equations are:

$$
\begin{array}{lll}
\bar{d}_{00} = -\dfrac{8Z_{00}}{W}, & \bar{d}_{01} = \dfrac{4Z_{00} - 6Z_{01}}{W}, & \bar{d}_{02} = \dfrac{3Z_{01} - 4Z_{02}}{W}, \\
\bar{d}_{03} = \dfrac{2Z_{02} - 2Z_{03}}{W}, & \bar{d}_{04} = \dfrac{Z_{03}}{W}, & \bar{d}_{11} = \dfrac{6Z_{01} - 4Z_{11}}{W}, \\
\bar{d}_{12} = \dfrac{2Z_{02} + 2Z_{11} - 2Z_{12}}{W}, & \bar{d}_{13} = \dfrac{Z_{03} + Z_{12}}{W}, & \bar{d}_{22} = \dfrac{2Z_{12}}{W}.
\end{array}
\tag{2}
$$

The rest of our discussion, until considering the deprioritized algorithm, is mainly motivation but also includes some derivations of formulae used later. The difficulty of analysis is caused by the prioritization. To proceed with the discussion, let $\phi_1$ be the probability of processing a $(1,0)$-symmetric pair, let $\phi_2$ be the probability of processing a $(2,0)$-symmetric pair, and let $\phi_3$ be the probability of processing a $(2,1)$-symmetric pair. Therefore,

$$\phi_1 + \phi_2 + \phi_3 = 1. \tag{3}$$

Moreover, every dual coloring of a $(1,0)$-symmetric pair produces in expectation $3\bar{d}_{20}$ $(2,0)$-symmetric pairs and $3\bar{d}_{21}$ $(2,1)$-symmetric pairs. Every dual coloring of a $(2,0)$-symmetric pair produces $2\bar{d}_{20}$ $(2,0)$-symmetric pairs and $2\bar{d}_{21}$ $(2,1)$-symmetric pairs; and every dual coloring of $(2,1)$-symmetric pair produces $\bar{d}_{20}$ $(2,0)$-symmetric pairs and $\bar{d}_{21}$ $(2,1)$-symmetric pairs. Therefore, the probability of processing a $(2,0)$-symmetric pair is given by,

$$\phi_2 = (3\phi_1 + 2\phi_2 + \phi_3)\bar{d}_{20}, \tag{4}$$

and the probability of processing a $(2,1)$-symmetric pair is given by,

$$\phi_3 = (3\phi_1 + 2\phi_2 + \phi_3)\bar{d}_{21}. \tag{5}$$

Solving the system given by equations (3), (4) and (5), we get

$$\phi_1 = \frac{1 - 2\bar{d}_{02} - \bar{d}_{12}}{1 + \bar{d}_{02} + 2\bar{d}_{12}};\ \phi_2 = \frac{3\bar{d}_{02}}{1 + \bar{d}_{02} + 2\bar{d}_{12}};\ \phi_3 = \frac{3\bar{d}_{12}}{1 + \bar{d}_{02} + 2\bar{d}_{12}}.$$

Phase 1 will finish when $\phi_1 = 0$, i.e. when $2\bar{d}_{20} + \bar{d}_{21} = 1$. Now we can compute the expected increments of the random variables $Z_{ij}$ in each iteration in phase 1 (assuming rb-symmetry). Using the linearity of the expectation,

$$\mathbf{E}\left[\Delta(Z_{ij})\right] = \bar{d}_{ij}(\phi_3 + 2\phi_2 + 3\phi_1) - \delta_{01}\phi - \delta_{02}\phi_2 - \delta_{03}\phi_3 \tag{6}$$

for any $i, j$ with $i \leq j$ and $i + j \leq 3$, where $-\delta_{pq} = 1$ if $(p,q) = (i,j)$, and 0 otherwise.

Without justification at this point, we may express the above expected increments as a set of differential equations, where each $\mathbf{E}\left[\Delta(Z_{ij})\right]$ is expressed as the differential $Z'_{ij}$ (all as functions of the number $t$ of iterations). If we scale both time and the variables by dividing by $n$, and denote $Z_{ij}/n$ by $z_{ij}$ $x = t/n$ and $W/n$ by $w$. Notice the bisection size at the end of phase 1 will be given by the sum of the expected $\bar{d}_{21}$ at each iteration, therefore let $Y$ be a random variable keeping track of the number of times of this sum and let $y = Y/n$. Then the equations are

$$\begin{aligned}
z'_{00} &= -8Z_{00}\frac{D(x)}{w(x)}, & z'_{01} &= (4Z_{00} - 6Z_{01})\frac{D(x)}{w(x)},\\
z'_{02} &= (3Z_{01} - 4Z_{02})\frac{D(x)}{w(x)}, & z'_{03} &= (2Z_{02} - 2Z_{03})\frac{D(x)}{w(x)},\\
z'_{04} &= Z_{03}\frac{D(x)}{w(x)}, & z'_{11} &= (6Z_{01} - 4Z_{11})\frac{D(x)}{w(x)},\\
z'_{12} &= (2Z_{02} + 2Z_{11} - 2Z_{12})\frac{D(x)}{w(x)}, & z'_{13} &= (Z_{03} + Z_{12})\frac{D(x)}{w(x)},\\
z'_{22} &= 2Z_{12}\frac{D(x)}{w(x)}.
\end{aligned}$$

where $D(x) = \phi_3(x) + 2\phi_2(x) + 3\phi_1(x)$.

As said, phase 1 will end when $\phi_1 \leq 0$, so it will run up to

$$x_1 = \sup\{x \,|\, \phi_1 = 0\ \}.$$

Solving numerically using a Runge-Kutta algorithm with the initial conditions,

$$z_{00}(0) = 1,\ z_{r,b}(0) = 0,\ \text{for } 0 < r \leq b, r + b \leq 4 \text{ and } y(0) = 0, \tag{7}$$

we get the solution

$$x_1 = 0.5, \quad z_{rb} \approx 0.00000, \quad y = 0.33363. \tag{8}$$

So in the solution of the differential equation system at the end of phase 1, all nodes are colored, and the expected bisection is $3.33363n$.

Now we are in position to carry out the formal analysis via a deprioritized algorithm. For a given sufficiently small $\epsilon > 0$, consider the deprioritized algorithm given in Figure 3. Notice that pre-phase 1, where a large number of, vertices with no colored neighbors are colored in pairs, just to ensures a good supply of vertices of types $(0,1),(1,0),(2,0),(0,2),(2,1),(1,2)$, before entering phase 1.

*Pre-phase 1:* **do** the following $\lfloor \epsilon n \rfloor$ times:
select u.a.r. a non-adjacent (0,0)-symmetric pair,
and perform Rand;

*Phase 1:* **while** all of $Z_{01}$ and $Z_{10}$; $Z_{02}$ and $Z_{20}$; $Z_{21}$ and $Z_{12}$ are non-zero
let $\phi_1 = \dfrac{1-2\bar{d}_{20}-\bar{d}_{21}}{1+\bar{d}_{20}+2\bar{d}_{21}}$; $\phi_2 = \dfrac{3\bar{d}_{20}}{1+\bar{d}_{20}+2\bar{d}_{21}}$; $\phi_3 = \dfrac{3\bar{d}_{21}}{1+\bar{d}_{20}+2\bar{d}_{21}}$;
**with probability** $\phi_1$
select u.a.r. a $(0,1)$-symmetric pair and perform Maj;
**with probability** $\phi_2$
select u.a.r. a $(0,2)$-symmetric pair and perform Maj;
**with probability** $\phi_3$
select u.a.r. a $(0,1)$-symmetric pair and perform Maj;

color any remaining uncolored vertices, half of them R and half B,
in any manner, and output the bisection R, B.

Figure 3: Algorithm deprioritized 4-min greedy for bisection on random 4-regular

In pre-phase 1, we have a unique operation, working as before the expected increment of $Z_{ij}$, is given by

$$\mathbf{E}\left[\Delta(Z_{ij})\right] = 4\bar{d}_{ij} - \delta_{00}. \tag{9}$$

At this stage, we entirely avoid using the rb-symmetry assumption. Referring back to (1), this requires only the adjustment of the formulae. The result is

$$\begin{aligned}
z'_{00} &= -\frac{32z_{00}}{w(x)} - 1 \\
z'_{10} &= \frac{16z_{00} - 24z_{10}}{w(x)} & z'_{01} &= \frac{-24z_{01} + 16z_{00}}{w(x)} \\
z'_{20} &= \frac{12z_{10} - 8z_{20}}{w(x)} & z'_{11} &= \frac{12z_{01} + 12z_{10} - 16z_{11}}{w(x)} & z'_{02} &= \frac{-16z_{02} + 12z_{01}}{w(x)} & z'_{03} &= \frac{-2Z_{03} + 2Z_{02}}{w(x)} \\
z'_{30} &= \frac{8z20 - 8z_{30}}{w(x)} & z'_{21} &= \frac{8z_{11} - 8z_{21} + 8z_{20}}{w(x)} & z'_{12} &= \frac{8z_{02} - 8z_{12} + 8z_{11}}{w(x)} & z'_{13} &= \frac{Z_{03} + Z_{12}}{w(x)} \\
z'_{40} &= \frac{4z_{30}}{w(x)} & z'_{31} &= \frac{4z_{21} + 4z_{30}}{w(x)} & z'_{22} &= \frac{4z_{12} + 4z_{21}}{w(x)} & z'_{04} &= \frac{Z_{03}}{w(x)},
\end{aligned} \tag{10}$$

where $w(x) = 4z_{00}(x) + 3z_{01}(x) + 3z_{10}(x) + 2z_{02}(x) + 2z_{20}(x) + 2z_{11}(x) + z_{30}(x) + z_{03}(x)$.

It follows that the unique solution (which is unique by standard theory of differential equations) must be the symmetric one, which satisfies $z_{ij}(t) = z_{ji}(t)$ for all $i$, $j$ and $t$, as well as the stated equations.

Let $z_{ij}(t/n)$ denote $Z_{ij}/n$ after $t$ steps. Thus the previous equations give the expected one-step change in the variables $Z_{ij}$ with error $O(1/n)$. This error is due to the changing value of the variables between when one vertex of type $(0,0)$ is chosen and the next. This applies with initial conditions given in (7). We write $\tilde{z}_{ij}(x)$ for the (unique) solutions of this initial value problem, $0 \le x \le \epsilon$. We may now apply the differential equation method (using, for example, [Wor95, Theorem 1] or [Wor99a, Theorem 5.1]) to deduce that during pre-phase 1, we have a.a.s.

$$Z_{ij}(t) = n\tilde{z}_{ij}(t/n) + o(n) \tag{11}$$

for each $i$ and $j$, where $Z_{ij}(t)$ is the value of $Z_{ij}$ after $t$ steps. This applies until either $t = \lfloor \epsilon n \rfloor$ or one of the derivatives approaches a singularity, which we can prevent by restricting to a domain in which $\theta > -1 + \epsilon$ and $w > \epsilon$, or the differential equations no longer apply for some other reason, which in this case only occurs if $Z_{00}$ reaches 0. Note that the derivatives are all $O(1)$, so $\tilde{z}_{00}(x)$ stays close to 1 for $x < \epsilon$ (recall $\epsilon > 0$ is sufficiently small). We conclude that a.a.s.

$$Z_{ij}(t_0) = n\tilde{z}_{ij}(t_0/n) + o(n), \qquad t_0 := \lfloor \epsilon n \rfloor. \tag{12}$$

We also note that $z'_{01}$ must be strictly positive, and so $\tilde{z}_{01}$, $\tilde{z}_{02}$ and $\tilde{z}_{12}$ are strictly positive on $(0, \epsilon)$. Thus, in particular, for sufficiently small $\epsilon_1 = \epsilon_1(\epsilon) > 0$,

$$\tilde{z}_{01}(\epsilon) \ge \epsilon_1, \quad \tilde{z}_{02}(\epsilon) \ge \epsilon_1, \quad \tilde{z}_{12}(\epsilon) \ge \epsilon_1, \tag{13}$$

Now consider phase 1. Arguing as above, the expected changes in the $Z_{ij}$ are given, with error $O(1/W)$, by the right hand sides of the equations in (3), with $w = 4z_{00} + 3z_{01} + 3z_{10} + 2z_{02} + 2z_{20} + 2z_{11} + z_{30} + z_{03}$ and the replacement equation

$$z'_{11} = \frac{3z_{01} + 3z_{10} - 4z_{11}}{w} D$$

to avoid the rb-symmetry assumption. Again the symmetrically reversed functions have symmetrically reversed equations (the $\phi_i$ stays the same). Continue the definition of the functions $\tilde{z}_{ij}(x)$ for $x > \epsilon$ by the solution of these equations with initial conditions given by the values of these functions at $x = \epsilon$ as determined above.

Note that setting $z_{ij} = z_{ji}$ for all $i$ and $j$ in the equations, again makes the formulae for $z'_{ij}$ and $z'_{ji}$ identical. It follows that again the unique solution is symmetric, with $\tilde{z}_{ij} = \tilde{z}_{ji}$. We deduce that the equations (3) are satisfied, with the symmetric definitions of $w$ and $z_{11}$, and we may restrict attention to the variables appearing there.

Again applying the differential equation method, we deduce that (11) holds a.a.s. as long as the solution set $\tilde{z}_{ij}$ stays within a predefined closed domain which does not contain singularities of the derivatives, and also the variables $Z_{01}$ and $Z_{02}$ stay positive. We may select the domain $L$ satisfying $\tilde{z}_{01} \ge \epsilon_1$, $\tilde{z}_{02} \ge \epsilon_1$, $w > \epsilon$. By (13), the first two of these inequalities hold at $x = \epsilon$, and the other two also hold for $\epsilon$ sufficiently small by boundedness of derivatives. Arguing by continuity, $z'_{01}$ as given in (3) is strictly positive for $x < \delta$, where $\delta > 0$ is an absolute constant independent of $\epsilon$. By definition of $\theta$, $z'_{02}$ is identical to 0, and

so $\tilde{z}_{02} \equiv \tilde{z}_{02}(\epsilon)$. Thus, for $\epsilon$ sufficiently small, the solution set $\tilde{z}_{ij}$ stays within $L$ for $x < \delta$, and can only leave $L$ when, for $x > \delta$,

$$\tilde{z}_{01} = \epsilon_1, \ w = \epsilon \text{ or } \theta = 1 - \epsilon. \tag{14}$$

Note that for $\epsilon$ and $\epsilon'$ sufficiently small, the initial conditions for $\tilde{z}_{ij}$ are arbitrarily close to (7). Let us denote the solutions with initial conditions (7) by $\bar{z}_{ij}$. By standard theory of first order systems of differential equations, it follows that the functions $\tilde{z}_{ij}$ can be made arbitrarily close to $\bar{z}_{ij}$ in the domain $L$, by taking $\epsilon$ and $\epsilon_1$ sufficiently small. By numerical computation, the conditions corresponding to (14) are not reached by the solution $\bar{z}_{ij}$, until $x$ approaches $x_1$ given in (8), at which point all $\bar{z}_{ij}$ reaches 0. It follows that, as $\epsilon$ and $\epsilon_1 \to 0$, the values are given in the limit by the values of $\bar{z}_{ij}$ in (8).

We also introduce the variable $y$, to keep track of the number $Y$ of times the pair $(2,1) - (1,2)$ is processed (as we have seing, it will give us the bisection). The conclusion is that there is a deprioritized algorithm in which the values of the variables $Z_{ij}$ are a.a.s. $\tilde{z}_{ij}n + o(n)$, where the functions $\tilde{z}_{ij}$ and $\tilde{y}$ solve (3). Notice that if $Y = yn + o(n)$ a.a.s. and $|\bar{z}_{ij}(x_2 - \delta') - \bar{z}_{ij}(x_2)| = O(\delta')$ for arbitrary small $\delta'$ (since the derivatives are all bounded). It follows that the size of the bisection produced is a.a.s.

$$y(x_1)n + O(\delta' n).$$

By (8), this completes the proof of Theorem 1.

# 4 Analysis of the algorithm for random cubic graphs

In the same way as we have done for 4-regular graphs, when a vertex $u$ is newly colored R and one of the pairs containing a point $p$ in that cell is exposed, the other point will lie in some vertex $v$. Let $d_{rb}$ denote the expected contribution to the increment $\Delta(Z_{rb})$ in $Z_{rb}$ due to the change in the status of $v$. Then, up to terms $O(1/W)$,

$$\begin{aligned} d_{00} &= -\frac{3Z_{00}}{W,}, \ d_{01} = -\frac{2Z_{01}}{W}, \ d_{02} = -\frac{Z_{02}}{W}, \ d_{03} = 0, \\ d_{10} &= \frac{3Z_{00} - 2Z_{10}}{W}, \ d_{11} = \frac{2Z_{01} - Z_{11}}{W}, d_{12} = \frac{Z_{02}}{W}, \\ d_{20} &= \frac{2Z_{10} - Z_{20}}{W}, \ d_{21} = \frac{Z_{11}}{W}, \ d_{30} = \frac{Z_{02}}{W}, \end{aligned} \tag{15}$$

where $W = 3Z_{00} + 2Z_{01} + 2Z_{10} + Z_{02} + Z_{20} + Z_{11}$.

The corresponding equations when a vertex is colored B form a symmetric set with these, and the expected increments $(\bar{d}_{r,b})$ due to a dual step, consisting of one new pair from a vertex of each color, have the equations:

$$\begin{aligned} \bar{d}_{00} &= -\frac{6Z_{00}}{W}, \ \bar{d}_{01} = \frac{3Z_{00} - 4Z_{01}}{W}, \ \bar{d}_{02} = \frac{2Z_{01} - 2Z_{02}}{W}, \ \bar{d}_{03} = \frac{Z_{02}}{W}, \\ \bar{d}_{10} &= \frac{3Z_{00} - 4\mathbb{Z}_{10}}{W}, \ db_{11} = \frac{2Z_{01} + 2Z_{10} - 2Z_{11}}{W}, \ \bar{d}_{12} = \frac{Z_{02} + Z_{11}}{W}, \\ \bar{d}_{20} &= \frac{2Z_{10} - 2Z_{20}}{W}, \ \bar{d}_{21} = \frac{Z_{20} + Z_{11}}{W}, \ \bar{d}_{30} = \frac{Z_{20}}{W}, \\ &\text{where } W = 3Z_{00} + 2Z_{01} + 2Z_{10} + Z_{02} + Z_{20} + Z_{11}. \end{aligned} \tag{16}$$

The difference in the analysis for algorithm 3-min greedy, given in Figure 2, with respect to the analysis of the previous section is that we must analyze each phase separatly, feeding to it the solutions to the differential equation in the previous phase.

In phase 1 the algorithm performs the Maj operation on one of two types of pairs. If vertices of types $(0,2)$ and $(2,0)$ exist, these have priority. In practice, for a random graph there are never very many vertices of types $(0,2)$ and $(2,0)$; as soon as there are some of each, they are processed, and the number of new ones arising tends to be less than the number used up. Which leaves a few of one type or the other, waiting for the matching ones to be created. To proceed with the discussion, we assume that a given iteration in phase 1 colors a (2,0)-symmetric pair with probability $\phi$, and therefore processes (1,0)-symmetric pairs with probability $1-\phi$. We assume that at any point in the process, $\phi$ is such that the variables $Z_{02}$ and $Z_{20}$ are kept roughly constant (virtually at zero), but that there are no problems with vertices of the appropriate type existing. We also assume as before *rb-symmetry*. These are all unjustified simplifications, but gives an accurate representation of the results, which we justify later using as before a deprioritized algorithm.

Define

$$\theta = \frac{2Z_{01} - 2Z_{02}}{W}.$$

Due to the coloring of a (0,1)-symmetric pair the expected number of new vertices of type (0,2) arising from this is $2\bar{d}_{02} = \frac{4Z_{01}-4Z_{02}}{W} = 2\theta$. Using rb-symmetry there also are $2\theta$ new vertices of type (2,0). it is also $2\theta$. So each iteration of phase 1 processing a (1,0)-symmetric pair gives rise on average to $2\theta$ steps processing a (2,0)-symmetric pair.

However, for each iteration processing a (2,0)-symmetric we expect $\theta$ extra steps of the same operation as this is the expected number of new vertices of type (2,0)-(0,2). For $\theta < 1$, the expected number of iterations coloring a (2,0)-symmetric pair after coloring a (2,0)-symmetric pair is $\sum_{i=0}^{\infty} 2\theta(\theta)^i = \frac{2\theta}{1-\theta}$. Therefore, $\frac{2\theta}{1-\theta}\phi = 1-\phi$, and we can conclude that

$$\phi = \frac{1-\theta}{1+\theta}. \tag{17}$$

Using this, we can compute the expected increments of the random variables $Z_{ij}$ in each iteration in phase 1.

$$\mathbf{E}\left[\Delta(Z_{ij})\right] = \phi(2\bar{d}_{ij} - \delta_{01}) + (1-\phi)(\bar{d}_{ij} - \delta_{02}) = (1+\phi)\bar{d}_{ij} - \phi\delta_{01} - (1-\phi)\delta_{02} \tag{18}$$

for any $i, j$ with $i \le j$ and $i + j = 3$, where $-\delta_{pq} = 1$ if $(p,q) = (i,j)$, and 0 otherwise. The equations for $i > j$ are the symmetric ones.

In (18) we may use $\bar{d}_{ij}$ as given in (16) but with $\bar{d}_{11} = (4Z_{01} - 2Z_{11})/W$ by rb-symmetry. Expressing the above expected increments as a set of differential equations, where each $\mathbf{E}\left[\Delta(Z_{ij})\right]$ is expressed as the differential $Z'_{ij}$ (all as functions of the number $t$ of iterations), and again scaling both time and the variables by dividing by $n$, and denote $Z_{ij}/n$ by $z_{ij}$ and $W/n$ by $w$, then the differential equations are

$$z'_{00} = -(1+\phi)\frac{6z_{00}}{w},\ z'_{01} = (1+\phi)\frac{3z_{00} - 4z_{01}}{w} - \phi,\ z'_{02} = (1+\phi)\frac{2z_{01} - 2z_{02}}{w} + (\phi - 1),$$
$$z'_{11} = (1+\phi)\frac{4z_{01} - 2z_{11}}{w},\ z'_{12} = (1+\phi)\frac{z_{02} + z_{11}}{w},\ \text{where } w = 3z_{00} + 4z_{01} + 2z_{02} + z_{11},$$

by rb-symmetry, $\phi = \frac{1-\theta}{1+\theta}$ and $\theta = \frac{2z_{01}-2z_{02}}{w}$.

The initial conditions at $x = t/n = 0$ are

$$z_{00}(0) = 1,\ z_{01}(0) = 0,\ z_{02}(0) = 0,\ z_{03}(0) = 0,\ z_{11}(0) = 0,\ z_{12}(0) = 0. \tag{19}$$

Note that by the way we defined $\phi$ in (17), $z'_{02} \equiv 0$ and hence $z_{02} \equiv 0$. We are interested in the point that $z_{01}$ first goes negative, which by numerical solution occurs when

$$x = x_1 \approx 0.41178,\ z_{00} \approx 0.002405,\ z_{11} \approx 0.046633,\ z_{12} \approx 0.063700. \tag{20}$$

The whole algorithm "takes off" at the start because the derivative of $z_{01}$ is strictly positive, so a.a.s. phase 1 does not quickly use up all vertices to be processed.

At this point, since $z_{01}$ and $z_{02}$ are both 0, phase 2 is entered. The situation is similar to phase 1, but with different pairs. Following the discussion above, let $\theta_2 = \frac{3Z_{00}-4Z_{01}}{W}$. Then, due to coloring a (1,1)-symmetric pair, there is one pair exposed from each of two vertices of type $(1,1)$, and the expected number of new vertices of type (0,1) arising from this is $2\bar{d}_{01} = \frac{4Z_{01}-4Z_{02}}{W} = 2\theta_2$, where two vertices of type $(1,1)$ are used in this operation. When processing a (1,0)-symmetric pair, the expected number of new vertices of type (1,0) created is $2\theta_2$. Letting $\phi_2$ denote the probability that at a given time a (1,0)-symmetric pair is colored in phase 2, we have $\frac{\theta_2}{1-2\theta_2}\phi_2 = 1-\phi_2$, giving

$$\phi_2 = \frac{1-2\theta_2}{1-\theta_2}.$$

In place of (18) we now have

$$\mathbf{E}\left[\Delta(Z_{ij})\right] = \phi_2(\bar{d}_{ij} - \delta_{11}) + (1-\phi_2)(2\bar{d}_{ij} - \delta_{01}) = (2-\phi_2)\bar{d}_{ij} - \phi_2\delta_{11} - (1-\phi_2)\delta_{01}. \tag{21}$$

The coloring of a (1,1)-symmetric pair increases de bisection by two. To set the differential equations, we add a new random variable $Y$ that keeps track of the number of times a (1,1)-symmetric pair is colored. Scaling we let $y = Y/n$. Then

$$z'_{00} = -(2-\phi_2)\frac{6z_{00}}{w},\ z'_{01} = (2-\phi_2)\frac{3z_{00}-4z_{01}}{w} + (\phi_2 - 1),\ z'_{02} = (2-\phi_2)\frac{2z_{01}-2z_{02}}{w},$$
$$z'_{11} = (2-\phi_2)\frac{4z_{01}-2z_{11}}{w} - 2\phi_2,\ z'_{12} = (2-\phi_2)\frac{z_{02}+z_{11}}{w},\ y' = 2\phi_2,$$

with $w$ as before and with initial conditions given by (20) and $z_{01} = z_{02} = 0$.

By the choice of $\phi_2$, $z_{01} \equiv 0$, so $z_{02} \equiv 0$. The point of interest is

$$x_2 = \sup\{x\ :\ z_{11} > 0,\ w > 0,\ \theta_2 < 1\}. \tag{22}$$

Numerically, we find that $\theta_2$ does not reach 1 before $z_{11}$ reaches 0, which clearly must therefore hold at $x_2$. This corresponds to the beginning of phase 3. During phase 3 the number of bicolored edges created is 2 for every pair of vertices of types $(1,2)$ and $(2,1)$ (using rb-symmetry) and at most 6 for every other pair colored except types $(0,3)$ and $(3,0)$, which give none. Since $z_{01} = z_{02} = z_{11} = 0$ at $x_2$, our upper bound for the size of the bisection is thus $(6z_{00} + 2z_{12} + 2y)n$ where the variables are evaluated at $x_2$. Solving numerically, we find

$$6z_{00}(x_2) + 2z_{12}(x_2) + 2y(x_2) < 0.1740381057, \tag{23}$$

where the constant is correct to ten decimal places.

Now we are in position to carry out the formal analysis via a deprioritized algorithm. For a given sufficiently small $\epsilon > 0$, consider the deprioritized algorithm in Figure 4. Notice that pre-phase 1 ensures a good supply of vertices of types $(0,1)$, $(1,0)$, $(0,2)$ and $(2,0)$.

The expected changes in the variables $Z_{ij}$ for each edge exposed are given in (2). In pre-phase 1, the derivation of (4) applies, but with $\phi$ redefined 1 at all times, and with different

*Pre-phase 1:* **do** the following $\lfloor \epsilon n \rfloor$ times:
select u.a.r. a non-adjacent (0,0)-symmetric pair,
and perform Rand;
*Phase 1:* **while** all of $Z_{01}$, $Z_{10}$, $Z_{02}$ and $Z_{20}$ are non-zero
let $\theta = \frac{2Z_{01}-2Z_{02}}{W}$ and $\phi = \frac{1-\theta}{1+\theta}$;
**with probability** $\phi$
select a (2,0)-symmetric pair and perform Maj;
**otherwise**
select a (1,0)-symmetric pair and perform Maj;
*Pre-phase 2:* **do** $\lfloor \epsilon n \rfloor$ steps as in Pre-phase 1;
*Phase 2:* **while** $Z_{01} > 0$, $Z_{10} > 0$ and $Z_{11} > 1$
let $\theta_2 = \frac{3Z_{00}-4Z_{01}}{W}$ and $\phi_2 = \frac{1-2\theta_2}{1-\theta_2}$;
**with probability** $\phi_2$
select a (1,0)-symmetric pair an perform Maj;
**otherwise**
select a (1,1)-symmetric pair an perform Rand;
*Phase 3:* as for Algorithm 3-min greedy.

Figure 4: Algorithm deprioritized 3-min greedy for bisection

terms in the equations for $z'_{00}$ and $z'_{01}$ due to the fact that the vertex being processed is type $(0,0)$ rather than $(0,1)$ or $(0,2)$. At this stage, we entirely avoid using the rb-symmetry assumption. Referring back to (2), this requires only the adjustment of the formulae for $z'_{11}$ and $w$. The result is

$$z'_{00} = 1 - \frac{12z_{00}}{w},\ z'_{01} = 2\frac{3z_{00} - 4z_{01}}{w},\ z'_{02} = 2\frac{2z_{01} - 2z_{02}}{w},$$
$$z'_{11} = 2\frac{2z_{01} + 2z_{10} - 2z_{11}}{w},\ z'_{12} = 2\frac{z_{02} + z_{11}}{w},\ w = 3z_{00} + 2z_{01} + 2z_{10} + z_{02} + z_{20} + z_{11}.$$

Other derivatives $z'_{ji}$ are the symmetric versions of $z'_{ji}$ (with indices swapped); $z'_{03}$ and $z'_{30}$ are not needed. It follows that the unique solution must be the symmetric one, which satisfies $z_{ij}(t) = z_{ji}(t)$ for all $i$, $j$ and $t$, as well as the stated equations.

Let $z_{ij}(t/n)$ denote $Z_{ij}/n$ after $t$ steps. Thus the previous equations give the expected one-step change in the variables $Z_{ij}$ with error $O(1/n)$. This error is due to the changing value of the variables between when one vertex of type $(0,0)$ is chosen and the next. This applies with initial conditions given in (19). We write $\tilde{z}_{ij}(x)$ for the (unique) solutions of this initial value problem, $0 \le x \le \epsilon$. We may now apply the differential equation method (using, for example, [Wor95, Theorem 1] or [Wor99a, Theorem 5.1]) to deduce that during pre-phase 1, we have a.a.s.

$$Z_{ij}(t) = n\tilde{z}_{ij}(t/n) + o(n) \tag{24}$$

for each $i$ and $j$, where $Z_{ij}(t)$ is the value of $Z_{ij}$ after $t$ steps. This applies until either $t = \lfloor \epsilon n \rfloor$ or one of the derivatives approaches a singularity, which we can prevent by restricting to a domain in which $\theta > -1 + \epsilon$ and $w > \epsilon$, or the differential equations no longer apply for some other reason, which in this case only occurs if $Z_{00}$ reaches 0. Note that the derivatives are all $O(1)$, so $\tilde{z}_{00}(x)$ stays close to 1 for $x < \epsilon$ assuming that $\epsilon > 0$ is sufficiently small. We conclude that a.a.s.

$$Z_{ij}(t_0) = n\tilde{z}_{ij}(t_0/n) + o(n), \qquad t_0 := \lfloor \epsilon n \rfloor. \tag{25}$$

We also note that $z'_{01}$ must be strictly positive, and so $\tilde{z}_{01}$ and hence $\tilde{z}_{02}$ are strictly positive on $(0, \epsilon)$. Thus, in particular, for sufficiently small $\epsilon_1 = \epsilon_1(\epsilon) > 0$,

$$\tilde{z}_{01}(\epsilon) \geq \epsilon_1, \quad \tilde{z}_{02}(\epsilon) \geq \epsilon_1. \tag{26}$$

Now consider phase 1. Arguing as above, the expected changes in the $Z_{ij}$ are given, with error $O(1/W)$, by the right hand sides of the equations in (4), with

$$w = 3z_{00} + 2z_{01} + 2z_{10} + z_{02} + z_{20} + z_{11}, \ \phi = \frac{1-\theta}{1+\theta}, \ \theta = \frac{2z_{01} - 2z_{02}}{w},$$

and the replacement equation

$$z'_{11} = (1+\phi)\frac{2z_{01} + 2z_{10} - 2z_{11}}{w}$$

to avoid the rb-symmetry assumption. Again the symmetrically reversed functions have symmetrically reversed equations (except that $\theta$ stays the same). Continue the definition of the functions $\tilde{z}_{ij}(x)$ for $x > \epsilon$ by the solution of these equations with initial conditions given by the values of these functions at $x = \epsilon$ as determined above.

Note that setting $z_{ij} = z_{ji}$ for all $i$ and $j$ in the equations, except in the definition of $\theta$, again makes the formulae for $z'_{ij}$ and $z'_{ji}$ identical, despite the asymmetrical definition of $\theta$. It follows that again the unique solution is symmetric, with $\tilde{z}_{ij} = \tilde{z}_{ji}$. We deduce that the equations (4) are satisfied, with the symmetric definitions of $w$ and $z_{11}$, and we may restrict attention to the variables appearing there.

Again applying the differential equation method, we deduce that (24) holds a.a.s. as long as the solution set $\tilde{z}_{ij}$ stays within a predefined closed domain which does not contain singularities of the derivatives, and also the variables $Z_{01}$ and $Z_{02}$ stay positive. We may select the domain $D$ satisfying $\tilde{z}_{01} \geq \epsilon_1$, $\tilde{z}_{02} \geq \epsilon_1$, $w > \epsilon$ and $\theta > 1 - \epsilon$. By (26), the first two of these inequalities hold at $x = \epsilon$, and the other two also hold for $\epsilon$ sufficiently small by boundedness of derivatives. Arguing by continuity, $z'_{01}$ as given in (4) is strictly positive for $x < \delta$, where $\delta > 0$ is an absolute constant independent of $\epsilon$. By definition of $\theta$, $z'_{02}$ is identical to 0, and so $\tilde{z}_{02} \equiv \tilde{z}_{02}(\epsilon)$. Thus, for $\epsilon$ sufficiently small, the solution set $\tilde{z}_{ij}$ stays within $D$ for $x < \delta$, and can only leave $D$ when, for $x > \delta$,

$$\tilde{z}_{01} = \epsilon_1, \ w = \epsilon \text{ or } \theta = 1 - \epsilon. \tag{27}$$

Note that for $\epsilon$ and $\epsilon'$ sufficiently small, the initial conditions for $\tilde{z}_{ij}$ are arbitrarily close to (19). Let us denote the solutions with initial conditions (19) by $\bar{z}_{ij}$. By standard theory of first order systems of differential equations, it follows that the functions $\tilde{z}_{ij}$ can be made arbitrarily close to $\bar{z}_{ij}$ in the domain $D$, by taking $\epsilon$ and $\epsilon_1$ sufficiently small. By numerical computation, the conditions corresponding to (27) are not reached by the solution $\bar{z}_{ij}$, until $x$ approaches $x_1$ given in (20), at which point $\bar{z}_{01}$ reaches 0. It follows that, as $\epsilon$ and $\epsilon_1 \to 0$, the exit point of the $\tilde{z}_{ij}$ from $D$ also tends towards $x_1$, and the values are given in the limit by the values of $\bar{z}_{ij}$ in (20).

A similar argument applies for phase 2, and at this point we also introduce the variable $y$, to keep track of the number $Y$ of times Op3 is performed. The conclusion is that there is a deprioritized algorithm in which the values of the variables $Z_{ij}$ are a.a.s. $\tilde{z}_{ij}n + o(n)$, where the functions $\tilde{z}_{ij}$ and $\tilde{y}$ solve (4). They can be made arbitrarily close to $\bar{z}_{ij}$ for all $x < x_2 - \delta'$, where $x_2$ is given in (22) and $\delta'$ is an arbitrary positive quantity. Additionally,

$Y = yn + o(n)$ a.a.s. Note that $|\bar{z}_{ij}(x_2 - \delta') - \bar{z}_{ij}(x_2)| = O(\delta')$ since the derivatives are all bounded. Examining phase 3 as for Algorithm simple min greedy, it follows that the size of the bisection produced is a.a.s.

$$(z_{00}(x_2) + 2z_{12}(x_2) + 2y(x_2))n + O(\delta' n),$$

where $\delta'$ can be chosen arbitrarily small. By (23), this completes the proof of Theorem 2.

## 5 Maximum Bisection

Let us consider the variation of the alorithms 4-min greedy and 3-min greedy (given in Figure 1 and 2,respectively) obtained by changing the meaning of Maj, now we will color a vertex with the minority color among its colored neighbors.

Let us say that an edge is *fully colored* when both its ends are finally colored. A fully colored edge is *monocolored* if both ends have the same color and *bicolored* if both ends have different color. So the monocolored edges by min greedy get bicolored by max greedy and vice versa, whenever the vertices of the graph are treated in the same order (which happens with the same probability, in both cases). That is, every edge that counts in the bisection for one algorithm does not count in the other and vice versa. Therefore, taking into account that the total number of edges in a 4-regular graph is $2n$, and in a cubic graph is $1.5n$, we have proved Therorems 3 and 4.

## 6 Further remarks

We have shown how an application of the differential equation method to analyze the expected bisection of random cubic and random 4-regular graphs, provided reasonable bounds both for max and min bisection.

One natural problem remains open, find the exact solution of the systems of differential equations obtained in the paper. By doing so, more accurate constants will be obtained, up to know we have been only able to solve them numerically.

It is also pof interest to study whether the presented approach can be used to derive bounds for min and max bisection for $d$-regular graphs for some (or even better any) $d > 4$.

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