Alcom-FT Technical Report Series
ALCOMFT-TR-03-188

Rectilinear Graphs and Angular Resolution *

Hans L. Bodlaender † Gerard Tel ‡

November 2003

Abstract

In this note we show that a planar graph with angular resolution at least $\pi/2$ can be drawn with all angles an integer multiple of $\pi/2$, that is, in a rectilinear manner. Moreover, we show that for $d \neq 4$, $d > 2$, having an angular resolution of $2\pi/d$ does *not* imply that the graph can be drawn with all angles an integer multiple of $2\pi/d$. We argue that the exceptional situation for $d = 4$ is due to the absence of triangles in the rectangular grid.

Keywords : Rectilinear drawing, plane graph, angular resolution, integer flow.

1 Introduction

In this note, we consider straight line drawings of plane graphs. In such a drawing, each vertex is represented by a point, and an edge is represented by the straight line segment between its endpoints. We want this *straight line drawing* to be *planar*, i.e., line segments may not intersect.

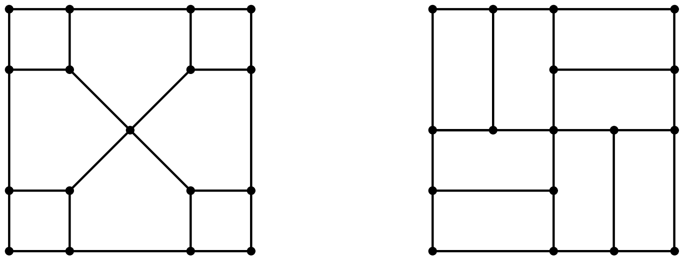
Part (a) of Figure 1 shows a drawing in which all angles formed by edges joining at a node are at least $\pi/2$; consequently, the drawing (and the graph) are said to have *angular resolution* $\pi/2$. Part (b) shows that the same plane graph can be drawn with all angles *an integer multiple* of $\pi/2$, and thus, with all edges parallel to one of the axes; such a drawing is said to be *rectilinear*. The main result of this note is that *all* plane graphs that admit a drawing of type (a) also admit a drawing of type (b).

Also, the more general question is answered: if a plane graph G has angular resolution at least $2\pi/d$, does it also have a planar straight line drawing with all angles an integer multiple of $2\pi/d$? We show that, apart from the trivial cases $d = 1$ and $d = 2$, 4 is the only value of d for which the answer is positive. Indeed, part (a) of Figure 2 shows a triangle with angular resolution $\pi/5$. However, because the three angles of the outer triangle sum

*Partially supported by the Future and Emerging Technologies programme of the EU under contract number IST-1999-14186 (ALCOM-FT).

†Institute of Information and Computing Sciences, Utrecht University, P.O. Box 80.089, 3508 TB Utrecht, The Netherlands, hansb@cs.uu.nl.

‡gerard@cs.uu.nl



(a) Angular resolution $\pi/2$ (b) Rectilinear drawing

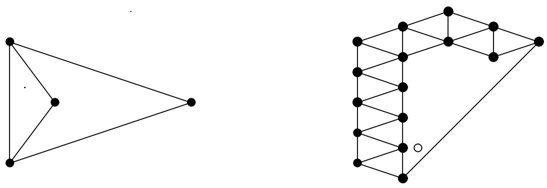
Figure 1: Graph with a.r. $\pi/2$ and a rectilinear drawing

up to π, and are subdivided into 5 subangles in total, the angles are fixed in any drawing of angular resolution $\pi/5$ of the triangle. *For the triangle this implies that its shape is fixed* and by connecting several of these "rigid triangles" together, a drawing can be forced to contain an angle that is not a multiple of $\pi/5$.

This is the organization of the paper. Section 2 contains definitions and presents the *flow model* for the angles in a drawing. Section 3 proves our main result. Section 4 contains the negative result for all angles other than $\pi/2$. Section 5 lists conclusions.

2 Preliminaries

Let $G = (V, E)$ be a graph. A graph is planar if it can be drawn (in the plane) without crossing edges. An *embedding* of a planar graph $G = (V, E)$ is a clockwise sequence of the incident edges for each vertex $v \in V$, such that G has a drawing where for each v, the clockwise order of its edges is as given. A *plane graph* is a planar graph given together with an embedding. In a drawing of a plane graph, we require that the embedding is respected.



(a) Rigid triangle (b) Enforcing a bad angle

Figure 2: Rigid triangle and impossible graph for $d = 10$

The faces of a plane graph G are defined as usual.

2.1 Classification of Drawings and Plane Graphs

Given a drawing, its *angular resolution* is the minimum angle made by line segments at a vertex. The angular resolution of a plane graph is the maximum angular resolution of any drawings.

A drawing is called *rectilinear* if all angles are a multiple of $\pi/2$ radians; a rectilinear drawing can be rotated to have all line segments parallel to either the x- or the y-axis. A plane graph is called *rectilinear* if it has a rectilinear drawing.

2.2 Angles and Angle Sums

A vertex of degree δ has δ angles: one between each two successive incident edges. Each angle belongs to one face. If v is a vertex, and f a face, then we let $d(v, f)$ denote the number of angles incident to v that belong to face f. Note that only when G is not biconnected, and v is a cutvertex, then there will be a face f for which $d(v, f) \geq 2$. For every face f, we let $a(f)$ denote the number of arcs that belong to f, i.e., $a(f)$ equals the number of times an edge is traversed when we go around the border of f once. In a biconnected graph, $a(f)$ also equals the number of edges at the border of f and it equals the number of vertices on the border of f; when G is not biconnected, then we sometimes count some edges or vertices twice.

2.3 The Flow Model for Angles

The embedding contained in a plane graph defines the position of the nodes in a qualitative manner, but to convert the embedding to a drawing, in addition two more things need be specified: the angles between the edges at each node, and the length of all edges.

Tamassia [Tam87] has shown that the angle values in a drawing satisfy the constraints of a suitably chosen *multi-source multi-sink flow network*. In any drawing, the angles around a node sum up to 2π and if an internal face is drawn as an a-gon its angles sum up to $\pi(a-2)$. (The angles of the outer face with a edges sum up to $\pi(a+2)$.)

Definition of flow networks and flows. A *(multi-source multi-sink) flow network* is a directed graph $H = (N, A)$, given together with functions $d : N \to \mathbb{R}$, $\ell : A \to \mathbb{R}_{0+}$, and $c : A \to \mathbb{R}_{0+}$. Here $d(v)$ is the *demand* at vertex v, $\ell(a)$ (*lower bound* for a) denotes the minimum flow across arc a, and $c(a)$ (*capacity* of a) is the maximum flow across a.

A *flow* in a multi-source multi-sink flow network is a function $\psi : A \to \mathbb{R}_{0+}$, satisfying

1. the *arc constraints*: for all $a \in A$, $\ell(a) \leq \psi(a) \leq c(a)$.
2. the *demand constraints*: for all $v \in N$, $\sum_{(w,v)\in A} \psi((w,v)) - \sum_{(v,w)\in A} \psi((v,w)) = d(v)$, i.e., the net flow into node v equals its demand.

Flow ψ is an *integer flow* if for each $a \in A$, $\psi(a) \in \mathbb{N}$.

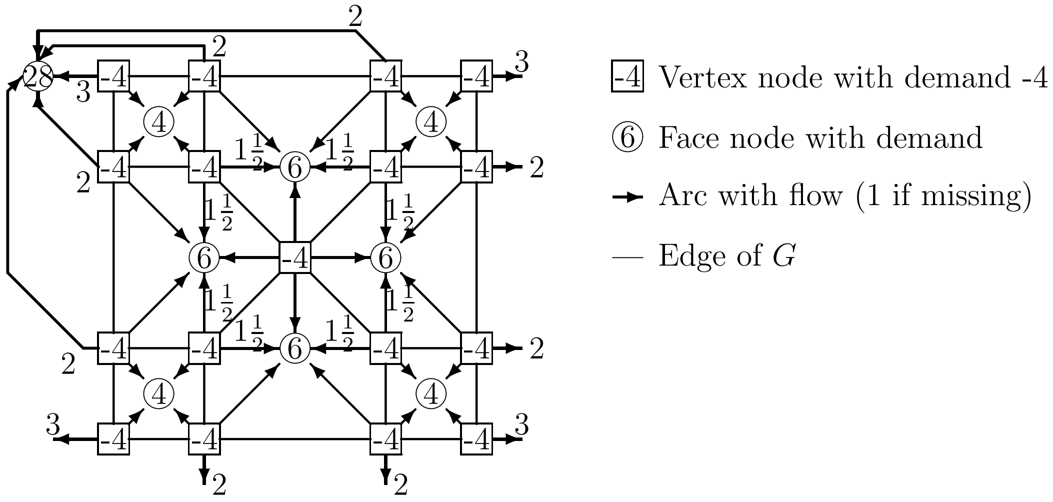


Figure 3: THE FLOW MODEL FOR THE GRAPH IN FIGURE 1(A)

Angle constraints modeled as flow. In the sequel we shall express angles in terms of the *angle unit* $\pi/2$ radians, thus an orthogonal angle has size 1 and a straight angle has size 2. The reason is that we want angles that are a multiple of $\pi/2$ radians to correspond to integer values in flows.

Tamassia's [Tam87] *flow model* for angle computations considers angles in a drawing to be flow transported from nodes to faces of the graph. That is, the nodes of H are the vertices and faces of G, and there is an arc in H from any vertex to each of the faces it is adjacent to. Let F be the set of faces of G; then $H = (N, A)$ where $N = V \cup F$ and

$$A = \{(v, f) : v \in V,\ f \in F,\ v \text{ is on the boundary of } f\}.$$

We now define the demands, minimum flows, and capacities. Because the angles around a node of G sum to 4 units, a vertex node has demand $d(v) = -4$, i.e., it produces 4 flow. Because the angles around an internal face of G with a points sum to $2a - 4$ units, a face node f has demand $d(f) = 2a(f) - 4$. If f is the exterior face, then $d(f) = 2a(f) + 4$. Because in a drawing with angular resolution $\pi/2$ radians, each angle measures at least 1 unit (and vertex v contributes $d(v, f)$ times to face f), arc (v, f) has flow lower bound $d(v, f)$. Because no angle can ever be larger than 4 units, arc (v, f) has capacity 4.

Relations between flows and drawings. The following two results are known.

Theorem 2.1 *If plane graph G has a drawing with angular resolution 1 unit ($\pi/2$ radians), then the associated flow network H admits a flow.*

Theorem 2.2 (Tamassia [Tam87]) *If the flow network H associated to plane graph G admits an* integer flow*, then G has a rectilinear drawing.*

3 Angular Resolution $\pi/2$ Implies Rectilinear

Our main result is obtained by combining Theorems 2.1 and 2.2 with a result from standard flow theory.

Theorem 3.1 *If a graph has angular resolution $\pi/2$, then it is rectilinear.*

Proof. Assume G has angular resolution $\pi/2$ radians. By definition it has a drawing with angular resolution 1 unit, hence by Theorem 2.1 the associated flow network admits a flow. It is known from flow theory (see, e.g., [Sch03, Chapter 10, 11]) that if a flow network with integer constraints admits a flow, then it admits an integer flow. By Theorem 2.2, G has a rectilinear drawing. △

We invite the reader to verify that the flow in Figure 3 can be modified into an integer flow by replacing the $1\frac{1}{2}$ values to 1 and 2 alternatingly and that the drawing in Figure 1(b) corresponds to the resulting integer flow.

4 Generalisation to Other Angles

This section answers the question for what values of d it is the case that any plane graph with angular resolution $2\pi/d$ radians has a drawing in which all angles are integer multiples of $2\pi/d$.

The cases $d = 1$ and $d = 2$ are somewhat trivial. Any drawing in which all angles are at least 2π (or π, respectively) radians, only contain angles that are a multiple of 2π (or π, respectively) radians. These drawings allow only one (or two, respectively) edges to join at a node. The classes of drawable graphs are collections of isolated edges (or paths, respectively).

The case of *odd d* is somewhat degenerate as, while drawings are supposed to be built up of straight lines, a straight angle at a vertex is not allowed. Let C_k denote the cycle with k vertices and k edges.

Proposition 4.1 *Let $d \geq 3$ odd; then C_{2d+1} has angular resolution $\frac{2d-1}{2d+1}\pi > 2\pi/d$, but has no drawing with all angles an integer multiple of $2\pi/d$.*

Proof. The total of all inside angles of any drawing of C_{2d+1} (necessarily as a face with $2d+1$ points) equals $(2d-1)\cdot\pi$.

So, if we draw C_{2d+1} as a regular $(2d+1)$-gon, then all inside angles have the same size $(2d-1)\pi/(2d+1) > 2\pi/d$, and all outside angles are larger than π, hence the angular resolution of C_{2d+1} is at least $2\pi/d$.

But C_{2d+1} does not have a drawing with all angles an integer multiple of $2\pi/d$, as $(2d-1)\cdot\pi$ is not an integer multiple of $2\pi/d$. △

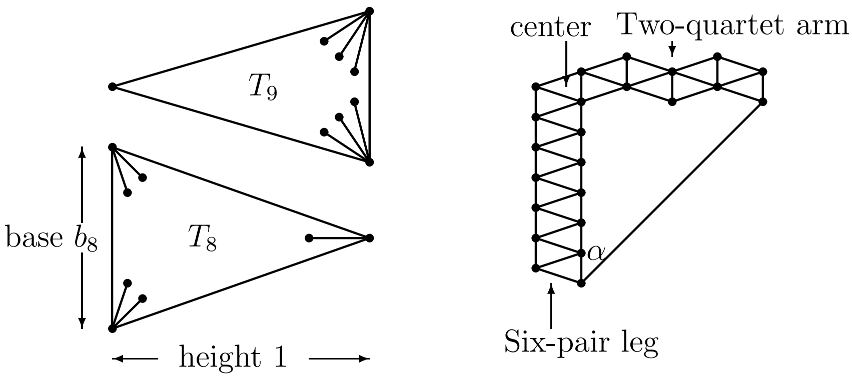


Figure 4: RIGID TRIANGLES AND THE CRANE $C_{d',6,2}$

In the remainder of this section $d = 2d'$ is even and larger than 4. Measuring angles in units of π/d' radians, we observe that the angles of any triangle add to exactly d' units. By dissecting the angles into d' parts, each of which must be at least one unit in a drawing with angular resolution $2\pi/d$ radians, we in fact *fix* the shape of the triangle in any such drawing, and we call these gadgets *rigid triangles.*

To simplify the construction, we take these basic elements to be isosceles. The graph $T_{d'}$ has a top node t and base nodes b_1 and b_2, forming a cycle. For even d', t has a third neighbor i, inserted between b_2 and b_1 in the planar embedding. Each base node has $\lfloor \frac{d'-3}{2} \rfloor$ neighbors, also located inside the triangle; see Figure 4. The graph $T_{d'}$ has an angular resolution $2\pi/d$ radians, and in every drawing with that resolution, each segment of each angle measures exactly π/d' radians, that is, the proportions of $T_{d'}$ are fixed in every drawing. The ratio between base length and height of the rigid triangle in such a drawing, $b_{d'}$ can be computed as $b_{d'} = \frac{2}{\tan(\lfloor \frac{d'-1}{2} \rfloor \cdot (\frac{\pi}{d'}))}$.

The crane graph $C_{d',k,l}$ contains $1 + 2k + 4l$ copies of $T_{d'}$ joined together. A central triangle is extended with a leg consisting of k pairs of triangles on one side, and an arm consisting of l quartets of triangles on the other side. In any drawing where all internal angles of the triangles satisfy the constraint for angular resolution π/d', the angle α at the bottom of the drawing satisfies $\tan \alpha = \frac{k \cdot b_{d'}}{2l}$. By choosing k and l, any angle between π/d' and $\pi/2$ radians can be approximated arbitrarily closely, contradicting the possibility to draw any crane with all angles a multiple of π/d' radians.

Theorem 4.2 *For each $d' > 2$, $\beta \geq \pi/d'$, $\epsilon > 0$, there exists a graph $G = C_{d',k,l}$ such that*

1. *G has angular resolution π/d' radians;*

2. *each drawing of G with angular resolution π/d' contains an angle α such that $|\alpha - \beta| < \epsilon$.*

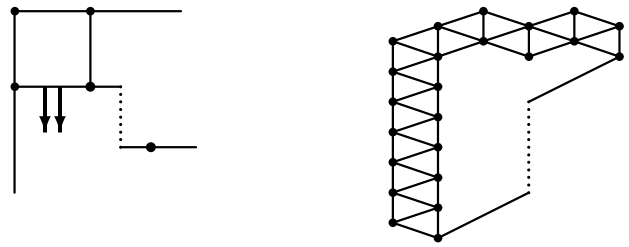


Figure 5: Orthogonal and non-orthogonal movements

5 Conclusions

To convert a plane graph into a drawing, two things need be done, namely, computing all angles between edges, and computing the edge lengths. The angle values must satisfy vertex constraints (the angles around a vertex have sum 2π) and face constraints (the angles around f have sum $\pi(f(a) - 2)$), which can be expressed in a flow model.

Our note compares two types of drawings, namely (1) those where all angles are at least $2\pi/d$ (angular resolution) and (2) those where all angles are an integer multiple of $2\pi/d$. The flow model introduced by Tamassia [Tam87] implies that if angles can be assigned satisfying (1), then it is also possible to assign all angles satisfying (2). However, only in the special case $d = 4$ it is also possible to assign edge lengths in such a way that a drawing results.

When drawing graphs in a rectilinear way, the drawing is built up from rectangular elements and in such drawings it is possible to shift parts of the drawing without disrupting angles in other parts; see Figure 5. Indeed, a rectangle can be stretched in one direction while preserving the orthogonality of its angles. In a drawing containing triangles, this is not possible, because stretching a triangle in one direction changes its angles. We therefore conclude that the special position of $d = 4$ in the studied problem is due to the absence of triangles in a rectilinear grid.

A plane graph G with for each angle its size specified is called an *angle graph*. It is not only so that it is sometimes not possible to draw an angle graph in a planar way, as follows from Section 4, but determining for an angle graph whether one can assign edge lengths such that it gets a planar drawing, is NP-hard. This was shown by Garg [Gar95, Gar96], and holds even if all angles are a multiple of $\pi/12$, or when G is triconnected and angles are a multiple of $\pi/36$. Garg also gives a polynomial time algorithm to test planarity of series parallel angle graphs. The testing for planarity for other special cases of angle graphs (e.g., k-outerplanar graphs) remains an interesting open problem.

The observations in this note can be extended to the situation where the embedding is free; that is, only a (planar) graph is given and the question is, what type of drawings does it admit. For the positive result (for $d = 4$), if the graph has some drawing with angular resolution $\pi/2$, it can be drawn rectilinearily with the same embedding. Our triangles $T_{d'}$ loose their rigidity if the embedding is free, because one can draw the internal nodes on the outside and then modify the triangle shape. By connecting the internal nodes in a cycle this can be prevented; in fact the triangles are modified to enforce the same embedding.

Acknowledgement: We thank the participants of the Graph Drawing seminar at Utrecht University for discussions.

References

[Gar95] GARG, A. On drawing angle graphs. In proc. *Proceedings 3rd International Symposium on Graph Drawing GD'94* (1995), Springer Verlag, Lecture Notes in Computer Science, vol. 894, pp. 84–95.

[Gar96] GARG, A. Where to draw the line. Ph.D. thesis, Department of Computer Science, Brown University, 1996.

[Sch03] SCHRIJVER, A. *Combinatorial Optimization. Polyhedra and Efficiency.* Springer, 2003.

[Tam87] TAMASSIA, R. On embedding a graph in the grid with the minimum number of bends. *SIAM J. Comput.* **16** (1987), 421–444.