Efficient Colouring of Squares of Planar Graphs

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Abstract

In this work we provide a new colouring algorithm, which colours the square of any planar graph G using at most $1.5\Delta(G) + c$ colours, where $c \leq 26$, and the maximum degree of G (i.e. $\Delta(G)$) is greater than 8.

In 1977 Wegner conjectured the above bound for $c = 1$ in [15]. The best currently known upper bound for the chromatic number of the square of G is $1.66\Delta(G) + 24$ due to Molloy and Salavatipour [10].

1 Introduction

The communication in wireless systems, such as radio networks and ad-hoc mobile networks, is accomplished through exploitation of a rather limited range of the frequency spectrum. One of the mechanisms utilised in this direction is to reuse frequencies where this does not result in unacceptable levels of signal interference. In graph theoretic terms, the interference between transmitters is usually modelled by the *interference graph* $G = (V, E)$, where V corresponds to the set of transmitters and E represents distance constrains. E.g. if two nearby vertices in G get the same or close frequencies, then this causes unacceptable level of interference. In most real life cases, the network topology has some special properties, e.g. G is a lattice network or G is a planar graph. Here, we consider the case where the interference graph is planar.

The *Frequencies Assignment Problem* (FAP) is usually modelled by variations of the fundamental graph theoretic problem of *vertex colouring*, like the *distance two colouring* [8, 13]. In FAP the available frequencies are represented by a set of colours. In addition, the colour of each vertex of G, in any particular assignment, gets an integer value, which has to satisfy certain inequalities compared to the colours assigned to its nearby vertices in this graph.

In the (*vertex*) *colouring problem* one would like to find an assignment of colours (most of the times are represented by integers) to the vertices of a given graph using the minimum number of colours, so that adjacent vertices have different colours. A *colouring* of G is a function $C : V \rightarrow \{1, 2, ..., k\}$, such that $C(v) \neq C(u)$ if $(v, u) \in E$. The *chromatic number* of G is the smallest integer k for which there exists a colouring of G and is denoted by $\chi(G)$. In the *distance k colouring problem* of graph G one would like to find an assignment of colours, C', to the vertices of G, so that $|C'(u) - C'(v)| \geq k - d(u, v) + 1$, where $u, v \in V$ and $d(u, v)$ is the distance between u and v in G (i.e. the length of the shortest path joining them). This problem

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has two optimisation versions. In the first one, called *min order*, the objective is to be used the minimum number of colours. On the other hand, at its second version, called *span*, the objective is to be used the minimum range of colours, namely the difference between the biggest and the smallest colour used. It is easily seen, that the *min order distance two colouring problem* on any graph G is equivalent to the colouring problem on the *square* of G [6].

Definition 1.1 *The square of a graph G is a graph defined on the same vertex set as G and has an edge between any pair of vertices having distance at most two in G. This graph is denoted by G^2.*

Graph G is called a square root of graph G^2. Let $G' = G - e$, where e is any edge of G. G is a minimal square root of graph G^2, if there are at least two adjacent vertices in G^2 which are non-adjacent in G'^2.

Let G^2 be the square of any graph $G(V, E)$. Note that there are graphs, with more vertices than graph G, that have G^2 as subgraph of their squares. Let H be any such graph. Then, G^2 is a subgraph of H^2. For example, the graph H can be the graph that is obtained from graph G by replacing each of its edges with a new vertex and joining this vertex with the endpoints of the corresponding edge. We call the graph $H = (V', E')$ *pseudo square root* of graph G^2, when each vertex $v \in V' - V$ (we call it *auxiliary vertex*) joins adjacent vertices of G^2, which are not neighbours in H and they do not have a common neighbour in V.

It is shown in [14, 5] that the colouring problem on squares of planar graphs is an NP-complete problem. Let G be a planar graph of maximum degree $\Delta(G) \geq 8$. Wegner conjectured in 1977, that $\chi(G^2) \leq 1.5\Delta(G)+1$, see [15]. The best currently known upper bound for $\chi(G^2)$ is $1.66\Delta(G)+24$ due to Molloy and Salavatipour [10]. We are aware of another two approximation algorithms designed to colour these graphs [2, 9]. Their approximation ratios are 1.8 and 2, respectively. These algorithms are essentially based on showing that always there is a vertex which satisfies some restrictions on its degree in G^2 or in G. We note, that these algorithms cannot be easily generalized in order to colour the square of any graph or an arbitrary graph. On the other hand, the algorithm that we provide in this work colours any graph.

In this work we design a new colouring algorithm and we analyse its performance on the square of any planar graph G with $\Delta(G) \geq 8$. We call it *MDsatur*, because it is inspired by the algorithm *Dsatur* of Brélaz [3]. *Dsatur* is a *greedy* colouring algorithm with very good performance on classes of graphs whose chromatic number is strongly related to their clique number. *Dsatur* colours the vertices of a given graph G one at a time based on a dynamic order, O', of them. At each point, i.e. i, it finds which of the current uncoloured vertices of G will be the *ith* element of O' (denoted by O'_i). Then, it correctly colours O'_i with the smallest *allowable* (*available*) colour. This colour has not been assigned to any neighbour of vertex O'_i which has already been coloured. If $O'_i = v$, then v is the current uncoloured vertex of G which has the biggest value on its *degree of saturation.*

Definition 1.2 *The degree of saturation of vertex v is denoted by $ds(v)$ and is the number of distinct colours that have been assigned to its neighbours.*

Our algorithm, *MDsatur*, is a non-trivial modification of algorithm *Dsatur*. It also colours the vertices of a given graph, i.e. G^2, in the same way as algorithm *Dsatur*. Namely, greedily choosing the smallest available colour for each vertex. However, it follows a different dynamic order, O, on the vertices of this graph. Our algorithm differs from the original algorithm *Dsatur* in the way of growing order O. More precisely, if i is the current point of *MDsatur* during its application on graph G^2, then it orders the currently uncoloured vertices of this graph by taking into account not only their degree of saturation, but also by considering how '*close*' is every one of the potential vertices for the position i of order O to the currently last vertex which has been coloured, let us call it $CurrLast$. To do this, $MDsatur$ follows a procedure similar to Breadth First Search. The most '*closer*' neighbour of a vertex u is the one whose colouring is determined at most by the colours of the other neighbours of u. In most of the cases the location of this neighbour of vertex u, in an *embedding* of graph G in the plane, is very near (close) to the location of u. Observe that there are vertices which are not in the same face in an embedding of graph G, but they are neighbours in G^2.

Here, we prove that $MDsatur$ colours the square of any planar graph G using at most $1.5\Delta(G) + 26$ if the maximum degree of G is greater than 8. If $\Delta(G) < 8$, then it is concluded by the works of Heuvel et al. [9] and Molloy et al. [10] that a constant number of colours is enough for the colouring of the square of each such planar graph. Hence we do not analyze the performance of our algorithm on such graphs.

The rest of this paper is organized as follows: in Section 2 we present our algorithm $MDsatur$. In Section 3 we present the main Lemmata that are used in the proofs of our Theorems. In Section 4 we present our Theorems and a detailed sketch of their proofs. Finally, in the last Section we conclude our work and we give some points for future work.

2 Presentation of the Colouring Algorithm *M*Dsatur

In this Section, we present the colouring algorithm $MDsatur$. In our opinion this algorithm is a novel approach in the colouring of squares of planar graphs. The inspiration for the design of $MDsatur$ originates from the behaviour of the colouring algorithms $Dsatur$ (of Brélaz) and $NoChoice$ (of Turner) on random [11] and k-colourable random graphs [12], respectively. These algorithms have very good and optimal performance on the above families of graphs. This happens, because each vertex of these graphs belongs to a clique of size almost maximum and maximum, respectively. Also the chromatic number of these graphs is strongly related to the size of their maximum clique.

For the case of our interest, it is easily concluded from the known results, that the chromatic number of the square of any planar graph G is strongly related to the size of its maximum clique. More precisely, it is known by Molloy and Salavatipour that $\chi(G^2) \leq 1.66\Delta(G) + 24$ see [10]. It is easily verified that $\omega(G^2) \geq \Delta(G) + 1$. Hence, $\chi(G^2) \leq 1.66\omega(G^2) + 24$.

$MDsatur$ selects the next vertex that will be coloured by taking into account additional requirements, than $Dsatur$. We add requirements R2 and R3 in our algorithm in order to achieve to colour *more locally* and *more compactly* a given graph as it is explained in an example, see Appendix A. Suppose that we apply $MDsatur$ to graph G^2. Let i be the current point of this algorithm in its application on G^2. Then, the current uncoloured vertex of G^2, let it be v, which manages to become the next vertex that will be coloured (i.e $O_i = v$) satisfies the following requirements in turn: R1) has the maximum value on its degree of saturation in this point (as in $Dsatur$), R2) has the oldest coloured neighbour among the oldest coloured neighbours of each uncoloured vertex with maximum value on its degree of saturation (i.e. sits in the leftmost position in O) and R3) is the 'closer' neighbour of vertex $CurrLast$. Observe that both requirements $R2$ and $R3$ limit the number of the potential uncoloured vertices of G^2 which claim to be the *ith* element in order O (O_i), compared to the corresponding vertices during the application of algorithm *Dsatur* on the same graph.

Algorithm: MDsatur
Input : a graph $G = (V, E)$.
Output : a colouring of G.
Begin

1. (a) Set each vertex of G as uncoloured,
   and set a dynamic order on them, called O, as empty.
   (b) Select any vertex of G, let it be v.
   Add it in the beginning of O (i.e. in $O_1 = v$), and colour it with colour 1.
2. **For** $i = 2$ **to** $n = |V|$ **do**
   (a) Set X to be the set of the current uncoloured vertices which
   have the maximum value on their *degree of saturation* (see req. R1).
   (b) **For** each vertex $v \in X$ **do**
   Find which coloured neighbour of v is coloured first
   (i.e. its oldest coloured neighbour) (see req. R2).
   (c) Select from X the vertex v which has the oldest coloured neighbour, denoted by ON, and also has the maximum number of common neighbours with the last coloured vertex (namely O_{i-1}) (see req. R3).
   If more than one of the vertices of X satisfies these requirements,
   then select at random one of them with equal probability.
   If none of the uncoloured neighbours of ON, which are in X,
   has common neighbours with vertex O_{i-1},
   then select at random one of them.
   Set $O_i = v$.
   (d) Colour v with the smallest of its allowable colours.
   If none of the colours used is allowable for this vertex, insert a new colour.
3. Return the colour of each vertex.

End

3 Main Lemmata

In this section we present the main Lemmata used in the proofs of our Theorems. With these Lemmata we mainly characterise the minimal planar square roots of any clique of size at least 8. The proofs of all these Lemmata are presented in the Appendix D to J.

Lemma 3.1 *(***Connectivity Property (CP)***) Suppose that algorithm $MDsatur$ is applied on a connected graph G. Let S be the subgraph of G that has already been coloured, in any point of*

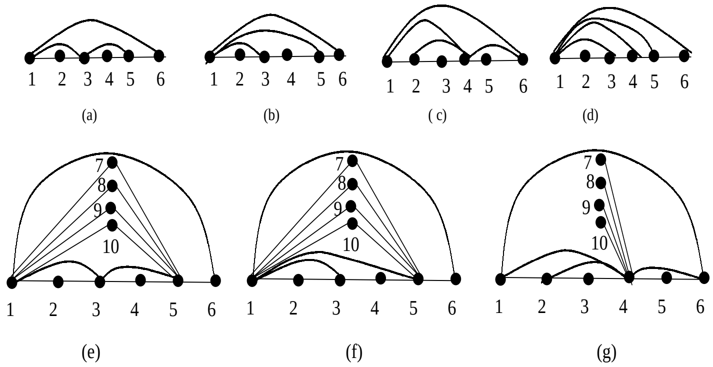

Figure 1: The graphs a, b, c and d correspond to the minimal planar square roots of clique K_6, when its vertices are joined in a cycle standing in the exterior face of these graphs, in at least one of their planar embeddings. The graphs e, f and g correspond to the minimal planar square roots of clique K_x (i.e. for $x = 10$) when only six vertices of this clique lie on the exterior face in an embedding of each of these square roots in the plane.

$MDsatur$ during its application on G. Then S is a connected graph.

We remark here that the Connectivity Property of $MDsatur$ differentiates it from all the other known colouring algorithms designed for colouring squares of planar graphs [9, 2, 5, 10]. This property consists one of the basic features of $MDsatur$ which provides its very good performance especially on squares of planar graphs.

Note 3.2 *In the following when we are referring to the location of a vertex of graph G^2, we mean its location in an embedding of G in the plane without edge crossings.*

Lemma 3.3 *Suppose that G is any minimal planar square root of clique K_6 with all the vertices of this clique joined in a cycle on the exterior face, in a planar embedding of G. Then the graph G is one of the graphs depicted in Figure 1 a , b, c, d.*

Lemma 3.4 *Suppose that $G = (V, E)$ is a planar graph with $|V| \geq 7$, such that a clique of size at least seven, denoted by CL, is a subgraph of G^2. Let EM be an arbitrary planar embedding of G. If at least seven of CL's vertices lie on the same face in EM, then the square root of CL, denoted by $SQRT\_CL$, is the graph $K_{(1,(|CL|-1))}$ plus optional edges between consecutive vertices of $SQRT\_CL$ in EM.*

Lemma 3.5 *Let G be a minimal planar square root of clique K_x, for $x \geq 8$, such that only six of the vertices of K_x lie on the exterior face in an embedding, EM, of G in the plane. Then the graph G is one of the graphs depicted in Figure 1 e, f, g.*

In the Appendix we prove a set of Corollaries, which are refereed to the minimal planar pseudo square roots of the clique K_x, for $x \geq 6$. According to these Corollaries the star graph $K_{1,x-1}$ is a subgraph of these roots of the clique K_x. Hence, our assertions for the square roots of this clique can be used and for its pseudo square roots. Our assertions for the pseudo square roots are omitted, since they can be derived in a way similar to the corresponding proofs about square roots.

In the following when we need a planar square root of the clique K_x, with less than seven of its vertices on the exterior face (i.e. $s < 7$) we explore in detail the case where $s = 6$. The other cases, i.e. $s \leq 5$, can be proved with similar arguments straightforward, so are omitted.

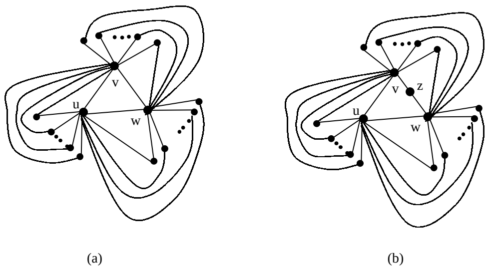

Figure 2: The structures of a planar graph which has a clique of size greater than its degree plus three in its square.

Lemma 3.6 *The size of the maximum clique in the square of any planar graph G is at most $1.5\Delta + 1$, if $\Delta \geq 8$.*

The graph of Fig. 2b forms a clique in its square. If each of the vertices u, v, w has degree equal to the maximum degree of this graph, Δ, then the square of this graph is a clique of size $1.5\Delta + 1$. This instance shows that the result of Lemma 3.6 is tight.

Corollary 3.7 *Let G be a planar graph. If $\omega(G^2) > \Delta + 3$, then G has as subgraph one of the graphs depicted in Fig. 2.*

4 Theorems

In this section we present our Theorems with a detailed sketch of their proofs. Their full proofs are presented in the Appendix B and C. The first of the following Theorems characterizes the worst case graph instances for our algorithm, when it is applied on squares of planar graphs. With the second Theorem we provide the performance of algorithm *MDsatur* on the class of graphs that we are interested for. We also prove that our results are tight.

Let A be a *greedy* colouring algorithm without backtracking and G be a graph. Namely, at each point A colours an uncoloured vertex of G with an allowable already used colour. If such a colour does not exist, then it uses a new one. We denote the number of colours used from algorithm A to colour graph G by $X_A(G)$. Let H^* be an uncoloured subgraph of G in an intermediate point of the application of A on the graph G.

Proposition 4.1 *If the neighbours of each vertex of the graph H^* have all the colours currently used by algorithm A, then A has to increase the number of colours used at least by $X_A(H^*)$.*

The increase on the number of colours used by algorithm A continues whenever there is a subgraph of graph G with the properties of graph H^*. We present this recursive process more concretely as follows: We denote by H the union of the graphs which lead algorithm A to increase the number of colours used in each point. Namely, $H = H_0 \bigcup H_1 \bigcup ... \bigcup H_x$, where H_i $(0 \leq i \leq x)$ is an uncoloured subgraph of graph G at the current point of the application of algorithm A on graph G, such that each of its vertices has neighbours in G that have already been coloured by $\chi_A(H_0) + \chi_A(H_1) + ... + \chi_A(H_{i-1})$ distinct colours. $\chi_A(G)$ obviously is equal to $\sum_{i=0}^{x}(\chi_A(H_i))$. We call the graph H_i as the *graph* H *of level* i.

Definition 4.1 *The minimal worst case squares of planar graphs are the smallest subgraphs in the square of any planar graph, where our algorithm uses the maximum number of colours.*

Lemma 4.2 *Consider planar* r*-regular graphs. Then* $r \leq 5$.

Theorem 4.3 *The minimal worst case squares of planar graphs for the colouring algorithm* $MDsatur$ *are the squares of the graphs obtained from any planar* r*-regular graph, by replacing each of its edges, i.e.* e*, with a set of new vertices and by joining these vertices with* e*'s endpoints.*

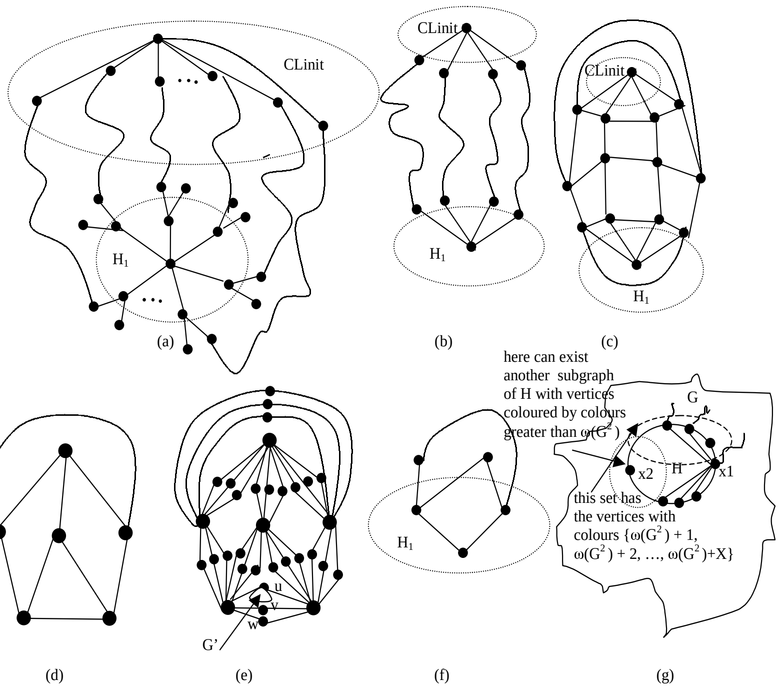

Figure 3: The graphs of this figure are used in the proof of Theorem 4.3 to make its presentation more clear. We emphasize on the graphs d) and e) only. More precisely, the graph d) is a 3-regular planar graph (Ga) and the graph e) is the graph G, that is obtained from graph Ga when $|Si| = 3$ for each $i \in \{a, b, c\}$.

Sketch of proof: Let G be any planar graph and EM be an embedding of G in the plane. To prove the assertion we are based on three critical points. The first one is that the subgraph of G^2 which has been coloured during the application of algorithm $MDsatur$ on this graph is a connected graph, by Lemma 3.1. The second comes from Proposition 4.1. Our algorithm is a greedy non-backtracking algorithm. Hence, it increases the number of colours which it uses to colour G^2, whenever this graph has an uncoloured subgraph H_i', such that the neighbours of each vertex of H_i' have all the colours currently used. The term H_i' denotes the subgraph of graph G^2 on the vertices of the subgraph H_i of graph G. Observe that graph H_i^2 is a subgraph of graph H_i'. For clarity reasons we use term H' to refer to the graph H of Proposition 4.1 in respect to graph G^2 and term H to refer to the subgraph of graph G on the vertices of graph H'. We prove that the graph H' is split in three levels at most (i.e. level 0, 1, 2) in the graphs of our interest. For the graph H' of the last level (i.e. level 2) we prove that $\chi_{MDsatur}(H_2') = 1$. Namely, our algorithm colours graph H_2' using one new colour only. This fact allows us to infer that $MDsatur$ initially colours a subgraph of G^2 with a large number of colours. I.e. H_0' is a maximum clique of G^2. Also, it leads us to infer that the graph H_1' is this one which forces algorithm $MDsatur$ to increase as much as possible the number of colours used. Finally, for the worst cases, we want algorithm $MDsatur$ to use enough colours,so that the ratio $\frac{\chi_{MDsatur}(G^2)}{\chi(G^2)}$ is maximum. It is obvious that $\chi(G^2) \geq \Delta(G) + 1$. So, the above ratio is essentially maximized when we bound $\chi_{MDsatur}(G^2)$ by a function of $\Delta(G)$.

The strong relation between the chromatic number and the clique number of graph G^2 (i.e. $\chi(G^2) \leq 1.66\omega(G^2) + 24$ [10]) lead us to split the rest of the proof of this Theorem in three parts. In the first two of them (A, B) we assume that $\chi(G^2) = \omega(G^2)$. Thereat we explore in detail all the cases arising based on the structure of each of the graphs H_i for $0 \leq i \leq 2$. More precisely, in Part A we assume that H_1' is a clique and we study all the cases arising from each possible minimal square root of clique $K_{|H_1|}$. We remark here, that the structure of these square roots is known by Lemmata 3.3, 3.4 and 3.5. In Part B, we assume that graph H_1' is a $\chi(G^2)$-colourable graph that is not a clique. In this case we prove that $MDsatur$ does not use more than $\omega(G^2)$ new colours in order to colour graph H_1'. Finally, in Part C we assume that $\chi(G^2) \geq \omega(G^2) + 1$. There we show what happens for each possible structure of graph H_1, either it is an $\omega(G^2)$-colourable graph or it is a $\chi(G^2)$-colourable graph. The proof of each of these Parts is comprised by a case analysis. In the following we briefly present these cases in each of the Parts A, B, C. Before going-over the rest of this proof we have to define a few more symbolisms. First of all, we denote the maximum clique in G^2 by MCL and by s its size. Then, we denote a square root of MCL by S. Finally, the first clique coloured by $MDsatur$ is denoted by $CLinit$.

Part A: Here H_1' is a clique and $\chi(G^2) = \omega(G^2)$. It is obvious, that $\omega(G^2) \geq \omega(H_1')$. It is also easily understood, that the harder case comes when the size of H_1' is maximum. The cases that will be studied in this part are determined from each possible structure of graph H_1 in respect to each possible structure of graph S. More precisely:

In Case A1 we assume that graph H_1 has as subgraph the star graph $K_{1,|H_1|-1}$. In Subcases A1.1 and A1.2 we also assume that each vertex of graph H_1' has neighbours in graph $G^2 - H_1'$ that have at most $\omega(G^2)$ distinct colours. I.e. algorithm $MDsatur$ colours graph H_0 with $\omega(G^2)$ colours. In these subcases we focus this proof on the way that the vertices of graph H_1 are joined with the rest of graph G, in order for their neighbours to have as much as possible distinct colours before the colouring of H_1'. We specify how many neighbours has each vertex of H_1 in $G - H_1$. We also determine the influence of the fact that nearby vertices of H_1, in EM, can have a common neighbour in the above graph. We prove that if some vertices of graph H_1, let them u, v, have in common more than one neighbour in graph $G - H_1$, then this case does not lead to worst case graph instances for our algorithm. As it is shown in Fig. 3f the graph G

is split in at least two parts (i.e. $G1, G2$). We prove that it is not possible to be colour both of these components of graph G, in such a way that the neighbour of the vertices of graph H'_1 have all the colours currently used. The proof of the assertion is based on the Connectivity Property (CP) of $MDsatur$. Hence, we focus our attention on the case where nearby vertices of H_1 have in common at most one neighbour in graph $G - H_1$.

To find the worst case instances of squares of planar graphs, which are obtained under these conditions we first prove that: any four vertices (or more) of graph G cannot simultaneously belong in more than two cliques of G^2 of size at least eight each, when each of these cliques has at least three distinct vertices. In the worst case instances of squares of planar graphs we prove that each vertex of graph G belongs to two cliques of maximum size in G^2.

In the case where nearby vertices of graph H_1 have one common neighbour in graph $G - H_1$, then seeing abstractly the resulting planar graph G we observe that it is a planar r-regular graph (we called it Ga) whose edges were replaced by a set of new vertices and these vertices were joined to the endpoints of the corresponding edge. We call Sa, Sb, Sc, Sd, Se the sets of nearby vertices which have replace the r edges incident to a vertex of Ga. An example is shown in Fig. 3 d, e. Observe that, if each vertex of graph Ga has maximum degree in G, then each vertex of G belongs to two cliques of maximum size in G^2.

For this case we prove that graph H' cannot have more than two levels. We prove it by showing that the increase of the number of neighbours of each vertex that was inserted in Ga to get graph G, does not influence the number of colours used by $MDsatur$ to colour G^2. Observe that any such vertex has degree two in G (i.e. vertex u in Fig. 3e). We prove that even if we add new vertices in G (i.e the vertices of H'_2) and we join them with the vertices which have degree two in G, $MDsatur$ does not use more colours to colour the new G^2.

We remark here, that if the edges of Ga remain in G in addition to the set of vertices which corresponds to each of them, then $MDsatur$ uses at most 26 more colours to colour G^2, against the number of colours that it uses to colour the graph G^2 when the edges of Ga are not edges of G. This is true because, the value of r is bounded by five when Ga is a planar graph see Lemma 4.2.

The graphs of Fig. 3 c,d shown two planar 4-regular graphs. Observe that in the square of the graph G that is obtained from the graph of Fig. 3c the vertices which are incident to the same vertex of graph Ga (i.e. x) and belong to nearby sets, i.e. Si, Sj $(i, j, k \in \{a, b, c, d, e\})$ have common neighbours in the graph G^2 vertices which are not incident to the vertex x. Let that these neighbours of the vertices of sets Si, Sj belong to the set Sk. It is easily seen, that in this case the colouring of the vertices of set Sk influences the degree of saturation of the distance two neighbours of the vertices of both sets Si, Sj. Thus, it is more possible in this case to colour more neighbours of the vertices of graph H'_1 before the colouring of H'_1, instead of the case where the vertices of sets Si, Sj do not have common neighbours other than the vertices incident to the vertex x. An example of such a case is shown in the graph of Fig. 3 b.

Let G be the graph of Fig. 3 a. In this case the vertices of the graph H_1 do not have a common neighbour in $G - H_1$. It is easily understood, that the graph G^2 is not harder to colour by $MDsatur$, compared to the squares of the graphs obtained by planar regular graphs, as it is explained above.

In subcase A1.3 we study the case where the neighbours of the vertices of H_1^1 have more than $\omega(G^2)$ colours. This is an interesting case, because here we have to see graph H' as a graph that is split in more than two levels. We prove that its maximum possible number of levels is three. Actually in the last level we prove that our algorithm needs only one extra colour to colour H'_2. Thus, for simplicity reasons we omit the graphs obtained by this way, since they do not influence the performance of our algorithm by more than a constant value, that is at most one.

In Case A2 we work on the cases where the graphs H_1 and/or S are not star graphs. These cases are much easier than the cases studied in Case A1, because of the structure of these graphs that is known by Lemmata 3.3 3.5. It is easily seen that there are much more faces in graphs S and H_1 in this case, in respect to Case A1 where both have only one face (are star graphs). Hence we prove, that it is not possible in this case for more vertices of H'_1 to have neighbours in $G^2 - H'_1$ coloured with distinct colours. This is proved, because of the CP of our algorithm.

In Case A3 we consider the cases raised when the graph H_1^2 is a clique of size bigger than $\Delta(G)+3$. The structure of such graphs is presented in Lemma 3.6 and is shown in Fig.2. The structure of H_1 makes impossible the case where there are neighbours of more than 6 vertices of H'_1 in $G^2 - H'_1$ which have been coloured, again because of the CP. Hence, this case does not lead to worst case instances of squares of planar graphs for our algorithm.

Part B: In Part B we show what happens when graph H_1^2 is not a clique. Here we conclude that, in the worst case, graph H_1 has more that one star graphs as subgraph on $\Delta+1$ vertices each. However, the colours assigned to the vertices of any one of them do not influence the colours of the others. Thus, the number of colours used by our algorithm to colour the squares of the graphs obtained in this case is not bigger than that used in the colouring of the squares of the graphs obtained in Part A. Here, with similar arguments we also prove that we don't have to consider more than two levels of graph H'.

Part C: Here we assume that $MDsatur$ firstly colours a $\chi(G^2)$-colourable graph instead of the maximum clique of G^2. This is the graph H_0. The colours of the vertices of this graph which have value bigger than s (reminder: s is the size of the maximum clique in G^2) influence the colouring of H_1^2 if these vertices are at distance at most two from each vertex of H_1. Knowing that graph $K_{3,3}$ is not a subgraph of any planar graph we conclude that the vertices of H_0, which have big colours, have to be joined in a star graph. The above restrictions lead to graphs very similar to the graphs obtained from the previous Parts. ∎

Theorem 4.4 *Let G be a planar graph with $\Delta(G) \geq 8$. Algorithm $MDsatur$ colours G^2 using at most $1.5\Delta(G)+26$ colours.*

Sketch of proof: The proof of the assertion is based on Theorem 4.3. Let G^2 be a worst case graph instance for algorithm $MDsatur$. According to Theorem 4.3, the graph G was a planar r-regular graph (Ga), which is modified in such a way that each of its edges, i.e. e, is replaced by a set of new vertices and each of these vertices is joined to $e's$ endpoints. We denote by $\{a,b,c,d,e\}$ the sizes of these sets of new vertices. We consider five values on these sizes only in the case where $r=5$. In the other cases we consider sets with only r different sizes, respectively. We denote by Si, where $i \in \{a,b,c,d,e\}$ a set of size i of new vertices which have been replaced the same edge of graph Ga. For technical reasons and proof's clarity we handle all the possible cases obtained from each value of r, separately.

We will use symbols $S_Ha, S_Hb, S_Hc, S_Hd, S_He$, when we refer to the vertices of graph H_1, which belong to sets of sizes a,b,c,d,e, respectively. We note that $a+b+c+d+e=\Delta$. We denote by a_b, a_c, a_d, a_e the vertices which belong to sets of sizes b,c,d,e, respectively, and are distance two neighbours of the vertices of set S_Ha and do not belong in graph H_1. Analogously, we denote the distance two neighbours of the vertices of the sets S_Hb, S_Hc, S_Hd, S_He which do not belong to the graph graph H_1. We denote by $N_G(H)$ the set of the neighbours of each vertex of the graph H in the graph G.

The proof of this theorem is also split in two main Parts. In the first one, Part A, we assume that $MDsatur$ has already coloured the graph G^2 except of the graph H_1 with the distance one and distance two neighbours of the vertices of H_1. We note that $\Delta + 1$ colours are enough for the colouring of the subgraph of G^2 that has been already coloured. This is true, because the maximum clique in G^2 has size $\Delta + 1$ (see Part A of Theorem 4.3). We call *old colours* the colours $\{1, 2, 3, ..., \Delta(G) + 1\}$. If we assume that the vertices which have been coloured use more than $\Delta + 1$ distinct colours, then there are more old colours available for some vertices of graph H'_1. Hence, their colouring will not increase the number of colours used by $MDsatur$ to colour G^2. Hence, we examine in detail the case where the neighbours of each vertex of H_1 in $G^2 - H'_1$ have only colours with value at most $\Delta + 1$.

By the way that our algorithm grows the order O, it is easily verified that it colours all the distance two neighbours of the vertices of graph H_1 which belong in a set Si of non-maximum size, before the colouring of their neighbours which belong in sets Si of maximum size. Let $a > b \geq c \geq d \geq e$. The rest of the proof of this part is based on the order that the sets of size a are coloured. Namely, if sets i_a are coloured first, last or in-between to the colouring of the vertices of set $S_H a$, where $i \in \{b, c, d, e\}$.

In the last part, Part B, we examine what happens if the distance two neighbours of the vertices of set $S_H i$ which belong in a set of size a have been coloured before the colouring of set $S_H a$.

In the case where $r = 3$ we show that the total number of colours used in these worst case instances is $\leq \Delta(G) + max\{b, c\} + 10$. Since, $a + b + c = \Delta(G)$ and $a > b, c$. This implies that this number is at most $1.5\Delta(G) + 26$. The case of $r = 2$ is almost trivial and is mentioned in the full version of this proof.

In the cases where $r = 4$ and $r = 5$ we show that the total number of colours used by $MDsatur$ in the worst case instances is $\leq \Delta(G) + b + d + 26$. This number is bounded from above from $1.5\Delta(G) + 26$, because $a + b + c + d + e = \Delta(G)$ and $a > b \geq c \geq d \geq e$. In the case where $r = 5$ we need a detailed analysis to prove the above bound. This case achieves the biggest value of the constant, i.e. 26.

Due to lack of space, we present here the cases obtained in Part A of $r = 3$ only.

We begin presenting some useful propositions which hold in Part A.

Proposition 4.5 *Let $S_H i$ be the first set with vertices from graph H_1 where $MDsatur$ used new colours (i.e. with value bigger than $\Delta + 1$). If this set is not set $S_H a$ and this set is still uncoloured, then set $S_H a$ is always the next set with vertices from graph H_1 that will be coloured by our algorithm.*

Proposition 4.6 *Let that the set of vertices $S_H a$ has already been coloured. Then the sets i_a ($i \in \{b, c, d, e\}$), which correspond to the uncoloured sets with vertices from graph H_1 (i.e. $S_H i$), remain uncoloured until the colouring of sets $S_H i$.*

Proposition 4.7 *After the colouring of set $S_H a$ the uncoloured sets with vertices from graph H_1 are coloured in increasing order of their sizes.*

Proposition 4.8 *Always there are enough old colours for the colouring of the first set with vertices from graph H_1 that is coloured by $MDsatur$.*

Proposition 4.9 *The 'closer' neighbour of each inner vertex of a set Si, in an embedding of G in the plane, ($i \in \{a, b, c, d, e\}$) is any other vertex of this set.*

Proposition 4.10 *The 'closer' neighbour of each outer vertex of set Si, in an embedding of G in the plane, ($i, j \in \{a, b, c, d, e\}$) is any other vertex of this set, or an outer vertex of its nearby set Sj.*

As it is easily understood from the above discussion, the most critical set with vertices from graph H_1 is the set of maximum size, i.e. set S_Ha. So, we first choose the order in which this set is coloured, in respect to the order of the sets S_Hi (i.e. if it is coloured first, second, ...). Then, we examine in detail all the potential cases obtained from all the combinations among the order of each of the sets S_Hi, for $i \in \{b, c, d, e\}$. At this point we define some useful variables, that will make the presentation of our results more clear. So, we denote the cardinality of the set of new colours which are used by $MDsatur$ to colour the first set with vertices from graph H_1 by NC_1. Similarly, we define the variables NC_2, NC_2, NC_3, NC_4 and NC_5. The sum of the new colours, which are used by our algorithm to colour the graph H_1^2 is denoted by SUM. Finally, we use the variable $\#OC_i$ to refer to the number of the old colours that have already been used from the neighbours of the vertices of set S_Hi which do not belong in the graph H_1. We note here, that the number of new colours needed for the colouring of a set of size i (i.e. $\{a, b, c, d, e\}$) can be computed by $\#OC_i - \Delta + i$. We recall that $C(x)$ denotes the colour that has been assigned to vertex x. When x is a set of vertices, then $C(x)$ denotes the set of colours that have been assigned to the vertices of set x.

Now we are ready to study in detail the Part A when $r = 3$. We begin with the case where the set S_Ha is coloured first. Namely before the other sets with vertices from graph H_1, i.e. S_Hi when $i \in \{b, c\}$. By Proposition 4.8 we get that $NC_1 = 0$ in every case.

Case r3-1: Let S_Hc be the second set of H_1 that is coloured. Then, $\#OC_c \leq |c_b| + |S_Ha|$ implying that $NC_2 = 0$. This is true, because the set c_a remains uncoloured at least until the colouring of the vertices of set S_Hc, due to Proposition 4.6. The last set of H_1 which is coloured is set S_Hb. If $C(S_Hc) \notin C(b_c)$, then $\#OC_b \leq |b_c| + |S_Ha| + |S_Hc|$ implying that $NC_3 \leq |S_Hc|$. Hence, $SUM = NC_1 + NC_2 + NC_3 \leq |S_Hc|$.

In the case where the second set with vertices from graph H_1 coloured is set S_Hb (instead of S_Hc) the result the same and it is easily proved (in the same way).

Observation 4.1 *The biggest value of the variable SUM comes from the biggest possible value of the variable NC_r in each case.*

Let that S_Hj is the last set with vertices from graph H_1 which is coloured from $MDsatur$. Then, according to the above observation we assume in the following that as many as possible from the old colours which have been assigned to the vertices of sets S_Hi, where $i \in \{a, b, c, d, e\}$ and $i \neq j$, have not also been assigned to the distance two neighbours of the vertices of set S_Hj which are not vertices of graph H_1.

Case r3-2: In this case we assume that set S_Ha is coloured second among the other sets S_Hi. Hence, we have to look at all the cases obtained from each potential choice for the first set with vertices from graph H_1 that has been coloured.

Case r3-2-1: Let that the first set of H_1 which has been coloured is set S_Hb. Then, if $C(S_Hb) \notin C(a_b \bigcup a_c)$, this implies that $NC_2 = |S_Hb|$. For the last set that will be coloured, namely set S_Hc, the following holds: if *old colours* $\bigcap C(S_Ha \bigcup S_Hb) \notin C(c_b)$ this implies that

$\#OC_c \leq |c_b|+|S_Hb|+(|S_Ha|-|S_Hb|)$. Hence, $NC_3 = 0$. The last term in the above sum indicates that the number of old colours used in the colouring of set S_Ha. Hence, $SUM \leq |S_Hb|$.

Case r3-2-2: If the first set which has been coloured is the set S_Hb and $C(S_Hb) \in C(a_b \bigcup a_c)$, then $NC_2 = 0$. When the last set of H_1 (i.e. S_Hc) will be coloured, if $C(S_Ha \bigcup S_Hb) \notin C(c_b)$, then $\#OC \leq |c_b| + |S_Hb| + |S_Ha|$. Hence, $NC_3 \leq |S_Hc|$. This is because $|c| \leq |b|$. So, $SUM \leq |S_Hc|$.

If the first set that was coloured was set S_Hc, then is easily seen that $SUM \leq |S_Hc|$.

Case r3-3: Now let us prove the case where the set S_Ha is coloured last. In this case we have to look at all the possible cases obtained from each potential choice for the first and the second set with vertices from graph H_1 that have been coloured. Note that the colours that have been assigned to the first set with vertices from graph H_1 have also been assigned to the vertices of set $a_b \bigcup a_c$. Otherwise, the set S_Ha would be the second set coloured.

Case r3-3-1: If the first set of H_1 which has been coloured is set S_Hb then the second is set S_Hc. If $C(S_Hb) \notin C(c_a \bigcup c_b)$, then $NC_2 \leq |S_Hc|$. In this case for the colouring of set S_Ha are needed zero new colours. So, $NC_3 = 0$. This is true, because the c colours which have been assigned to the vertices of set S_Hc are all new.

On the other hand, if $C(S_Hb) \in C(c_a \bigcup c_b)$, then there are no used new colours in the colouring of set S_Hc (namely $NC_2 = 0$), but the $|S_Hc|$ new colours will be used in the colouring of set S_Ha (namely $NC_3 \leq |S_Hc|$), due to Observation 4.1.

Hence, in both of the above cases $SUM \leq |S_Hc|$.

Case r3-3-2: The case where set S_Hc was first coloured and then set S_Hb is very similar to the above. The value of SUM on this case is at most $|S_Hb|$.

As we can see from the above cases the maximum number of colours used by our algorithm, in the case where $r = 3$, is $\Delta(G) + |S_Hb| + 10$. From the initial assumption we know that $a > b$. This implies that $|S_Hb| = b \leq \Delta(G)/2$. Hence, the total number of colours used is $\Delta(G) + 0.5\Delta(G) + 10$. So, the Theorem is proved for this case of r.

■

5 Conclusions

We have experimentally evaluated the performance of *MDsatur* in [1] and we observe that it uses $\Delta(G) + c$ colours in order to colour almost all the worst case graph instances, when $a = b = c = d = e$. *MDsatur* uses less than $1.3\Delta(G)$ colours in order to colour worst case graph instances when $a = 5x$, $b = 4x$, $c = 3x$, $d = 2x$ and $e = x$. In this case the sets of colours used in order to colour the sets Sa, Sb, Sc, Sd, Se, are mixed very much (may be this is the worst case, when $\chi(G^2) = \Delta(G) + 1$). We prove that our algorithm *MDsatur* colours the square of any planar graph G with at most $1.5\Delta(G) + c$ colours. We conjecture that this is not a tight bound in the case where $\chi(G^2) = \Delta(G) + 1$. So, this is an open problem for this case. We also conjecture that *MDsatur* colours any planar graph with at most 5 colours.

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APPENDIX

A Example

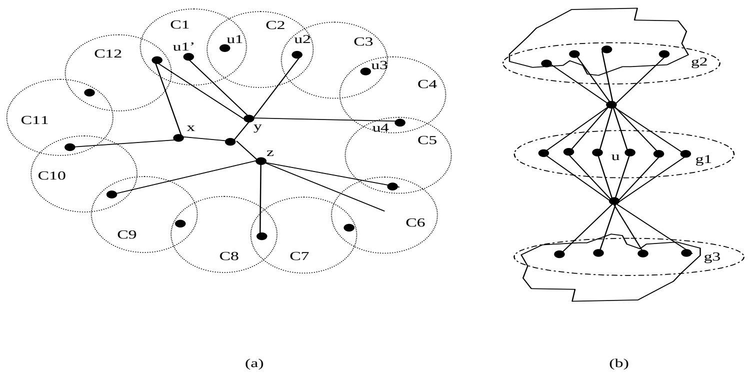


Figure 4: The graphs which are depicted in this figure are used in order to explain why $MDsatur$ colours *more locally* and *more compactly* a given graph, than the algorithm $Dsatur$.

At this point, lets just get across an intuitive idea of why the algorithm $MDsatur$ gives better results than the original algorithm $Dsatur$. This is achieved, because $MDsatur$ colours the given graph *more locally* based on the last vertex that has been coloured. Also, the subgraph of the given graph that has already been coloured at any point of its application on this graph is *more compactly* (dense) than the corresponding subgraph obtained from $Dsatur$. We explain how these properties of our algorithm are achieved using an example and we mention their consequences on the colouring resulted.

Let G be a planar graph and suppose that $MDsatur$ is applied on the graph G^2. Also suppose that in an intermediate point of this application the following hold: $\{C1, C2, C3, C4, ...\}$ is a sequence of cliques that have been partially coloured, and $\{u1, u2, u3, u4, ...\}$, is a sequence of uncoloured vertices. Each vertex uj ($j = \{1, 2, 3, ...\}$), belongs in both cliques Cj and $C(j + 1)$ as it is shown in Fig. 4a. We also assume that the value of the degree of saturation of all these vertices is the same at the current point of our algorithm. Observe that at the application of the algorithm $Dsatur$ on this graph under these conditions, it is possible to colour the above vertices sequentially before the colouring of other vertices which belong to the above cliques.

This fact, quite possibly yields an increase on the number of colours used, because it is possible to have vertices (like the vertices x, y, z in the figure), such that their degree of saturation (when they will be coloured) is mainly computed based on the colours of non-adjacent vertices.

To avoid the above situation we add requirement R2 in $MDsatur$. Let as call ON_{u1} (ON_{u2}) the neighbour of vertex $u1$ ($u2$) that has been coloured first (i.e. the oldest coloured neighbour of vertex $u1$). If $u1$ is the next vertex to be coloured, then this implies that vertex ON_{u1} was coloured before vertex ON_{u2} (due to R2). If $ON_{u1} \in C1$, then the next vertex that will be coloured by $MDsatur$ cannot be the vertex $u2$. This is true, because there is at least one vertex with maximum value on its degree of saturation which has a neighbour coloured before vertex ON_{u2}. Such a vertex is $u1'$. Thus $MDsatur$ colours next the vertex $u1'$. Observe that vertex $u1'$ extends the same clique as the predecessor vertex coloured (namely $u1$). If $ON_{u1} \notin C1$, then unfortunately it is possible the vertex $u2$ to be coloured before the vertex $u1'$. By requirement R2 we try to minimize the effect of this situation.

Finally, requirement R3 is used in order to colour in turn nearby vertices of the same clique (in a planar embedding of a square root G of the given graph G^2). Namely we try to locally colour these vertices. As we have said before, the most 'closer' neighbour of the last vertex coloured lies near to its location in a planar embedding of the graph G. The graph of Fig. 4b (we call it G) is representative for the explanation of this fact. Suppose that all the vertices of G have the same value on their degree of saturation in G^2. Let u be the next vertex that will be coloured. We mention that all the vertices of sets $g1, g2, g3$ are neighbous of vertex u in G^2. Hence, the colouring of u influences the value of the degree of saturation of each other vertex. Because of requirement R3 our algorithm colours after the colouring of the vertex u a vertex from set $g1$, because only vertices of this set have the maximum number of common neighbours with the last vertex that has been coloured.

We note here that we want to find the closer neighbour of any vertex in EM. However, we find a set of vertices which are closer to this specific vertex than other vertices. We also mention here that $MDsatur$ most of the times (also in the worst case instances) sequentially colours the vertices of this set. These is one of the reasons why we do not have to find the most closer neighbour of a vertex.

B Proof of Theorem 4.4

Proof: This proof completes the sketch of Section 4. Here we prove in detail the cases obtained from each potential value of r (except for $r = 3$) i.e. $r = 2, 4, 5$ in Parts A and B. For Part B we also prove the case were $r = 3$.

We begin with the proofs of the Propositions presented in Section 4, which are not trivial.

Proof of Proposition 4.5: The assertion is true, because the vertices of $S_H a$ have bigger value on their degree of saturation, than any vertex of any other set of size a (denoted by Sa). Recall that the only sets which are still uncoloured are the sets i_a (namely b_a, c_a, d_a, e_a) and some sets with vertices from H_1, i.e. $S_H i$.

Proof of Proposition 4.6: This follows again, because the vertices of $S_H i$ have bigger value on their degree of saturation, than the vertices of sets i_a.

Part A:

We proceed with the proof of the case where $r = 2$. The case where $r = 1$ is trivial and is omitted.

Case of r=2:

It is well known, that the only 2-regular graph is the cycle [4]. It is also well known that each cycle of odd size needs three colours in order to be coloured [4]. Using analogous arguments to that used in the proof of the above statement, it can be proved that G^2 needs at most $1.5\Delta(G)+5$ colours in order to be coloured. Asymptotically this bound is tight and is achieved, when all of the edges of the initial cycle, that is the graph Ga, are replaced by sets of vertices of the same size. Now, we will explain why our algorithm does not needed more colours in order to colour this graph.

Let x be the first vertex of Ga that is coloured (see, Fig. 5a). Subsequently, $MDsatur$ colours all the neighbours of x, which belong in the set that lies to its left or to its right in EM. W.l.o.g. we assume that the neighbours of x to its left are a (i.e. belongs in a set Sa, we denote this set by S_xa). The neighbours of x to its right we assume that belong in a set Sb. We denote this set by S_xb. W.l.o.g. we assume that $MDsatur$ chooses a vertex from the set S_xa to colour next. Then, it also colours all the other neighbours of x which belong in this set, because all these have the maximum number of common neighbours in G^2 with the last vertex that has been coloured (see Proposition 4.9). In the following, $MDsatur$ chooses a neighbour of x from the set S_xb. Observe that the vertices of this set (S_xb) have the same value on their degree of saturation as the other distance two neighbours of the vertices of set S_xa, which are not neighbours of x, i.e. the vertices of the set S_ub. u is a neighbour of vertex x in the graph Ga, which sits to the left of x in EM. The choice of a vertex from set S_xb as the next vertex that will be coloured by our algorithm comes from step 3.c. At this step $MDsatur$ takes into account a procedure similar to the Breath First Search. Then, $MDsatur$ continuous colouring the vertices of this set until all the neighbours of x become coloured. Subsequently, $MDsatur$ colours a vertex that is a distance two neighbour of x and lie to its left (i.e. a neighbour of the vertex x in the graph Ga, i.e. vertex u). This is true, because of the procedure of step 3.c that is similar to Breath First Search. In the following the algorithm colours the rest of the graph in the same way. Thus, $MDsatur$ uses three sets of colours, let us call them A, B, C to colour this graph. The cardinalities of these sets are a, b, c, respectively and each of them has size at most $\Delta/2$ in the worst case instances. The vertices of the last set that is coloured have neighbours that have already been coloured with $a + b$ distinct colours. That is way these vertices receive new colours (see Figure). So, it colours almost optimally any such graph. Actually it uses at most five more colours, which are used in order to colour the vertices of the cycle Ga.

We proceed with the case where $r = 4$. At this point we present two propositions, which are concluded easier after the detailed proof of the case where $r = 3$.

Proposition B.1 *Only during the colouring of sets S_Ha and S_Hb is possible to have b new colours used by $MDsatur$. More precisely for the second case, this happens only if at least two sets with vertices from H_1 different from S_Ha have been coloured before the set S_Hb. Also the old colours that have been used for the colouring of these sets have to be more than the size of one of these sets.*

Proposition B.2 *It is not possible to use a new colours during the colouring of any set of H_1^2 by $MDsatur$.*

In order to hold the Theorem on this case, the value of SUM must be comprised at most two components. We recall that $a > b \geq c \geq d$ and that $MDsatur$ have already used $\Delta(G) + 1$ distinct colours, before the colouring of any set of H_1^2. The final number of colours used by $MDsatur$ is $\Delta(G)+1+SUM$. Then more precisely, if one of the components of SUM is the b, then the other must be the d. In this case $b \leq \frac{D(G)-c-d}{2} \leq \frac{D(G)-2d}{2} = \frac{D(G)}{2} - d$, because $d \leq c$. This implies that, $b + d \leq D(G)/2$. On the other hand, if the second component were c, then $SUM = b + c \leq 0.66\Delta$. The following cases show that this case is not possible. In the case where b isn't a component of SUM, then the Theorem follows, because the term a can't be a component of the sum due to Proposition B.2 and each of the other terms is appeared only ones in the SUM.

Case r = 4

Let U, V, W, X be four variables. Each of these variables represents a set with vertices from H_1. If we don't assign explicit value to any of them, then it can have any value. Of course the value of each of them has to be different from the value of the others. We also use the variables Z_a, Z_b, Z_c, Z_d (and Z_e when $r = 5$) in order to refer to the old colours which have been assigned to the neighbours of the vertices of sets S_Ha, S_Hb, S_Hc, S_Hd (and S_He), respectively, in the graph $G^2 - H_1^2$. W.l.o.g. we assume that the first set with vertices from H_1 which will be coloured by $MDsatur$ is assigned to variable U. The second set with vertices from H_1 that will be coloured is indicated by V. The third and the fourth such set is indicated by W and X, respectively.

At this point we present an important proposition, that concerns the value of the variables NC_i, i.e. if they have to receive or not their maximum value, in order to use $MDsatur$ the maximum number colours to colour G^2. Let that $MDsatur$ has already coloured some sets with vertices from H_1. By Proposition 4.5 it is easily concluded that the number of colours used by $MDsatur$ to colour G^2, when $X = S_Ha$ (and/or $Y = S_Ha$ in the case where $r = 5$) is at most $\Delta(G) + 1 + |W| + |Y|$. In the following we prove in detail that this number is at most $1.5\Delta(G)+26$. For the cases where $S_Ha \neq X, Y$ the analysis of the performance of our algorithm is split in more subcases not so easily proved, as it is shown in the following. Hence, we assume that $MDsatur$ has already coloured two such sets (i.e. U, V) and if $S_Ha \neq U, V$, then $S_Ha = W$.

Let that $W = S_Ha$. Let that $NC_2 < |U|$. This means that even if $old\ colours \bigcap C(U \bigcup V) \notin C(Z_X)$, then $NC_4 \leq |U| + |V| - NC_2$. Namely at most $|V|$ plus $|U| - NC_2$. This means that the term $|U|$ is split in more than one parts. By Observation 4.1 we know that $C(U) \notin C(X)$ ($C(U) \notin C(Y)$ if $r = 5$). Hence, in the computation of NC_r are shown the following terms (among others). $|U| - NC_2-$ the parts of the other NC_i that concerns the parts where $|U|$ is split. NC_r has as component one of these parts, if the union of them is still less than $|U|$. In the case where $r = 5$, observe that the term $|U|$ can be part of NC_4. In this case in the computation of NC_5 the following terms are shown (among others). $|U| + |X| - NC_4$. Hence, NC_5 does not have one of the parts where the variable $|U|$ is split.

Hence, the bound on the total number of colours used by $MDsatur$ is the same either variables NC_i have their maximum value or not. Observe that each of the terms a, b, c, d, e has coefficient at most one when it is a positive term of the SUM.

We come to this conclusion and in the case where $S_Ha \neq W$ with similar arguments. The above arguments also hold and for the parts where is split each other term, i.e. $|V|, |W|$.

Proposition B.3 *The bound on the total number of new colours used by $MDsatur$ to colour the graph H_1^2 is the same either the variables NC_i have their maximum value or not.*

So, in the following we assume that these variables have the maximum possible value under the conditions that have to be satisfied in each case.

Case r4-1: $U = S_Ha$ (we recall that $NC_1 = 0$ always). This implies that whichever is the set V (i.e. $V = S_Hb$ or $V = S_Hc$ or $V = S_Hd$) $NC_2 = 0$. By Proposition 4.7 $X = S_Hb$. $NC_4 \leq |V + W| - NC_3$. Then $SUM = NC_3 + NC_4 \leq |V + W| = c + d$

Case r4-2-1: Here $V = S_Ha$. If $U = S_Hc$ or $U = S_Hd$ then $X = S_Hb$ (because of Proposition 4.7). Since, $V = S_Ha$ the vertices of the sets W_a, X_a remain uncoloured until the colouring of sets W, X due to Proposition 4.6. Hence, from the above discussion it is easily verified, that $NC_2 + NC_3 \leq |U|$. This is true, because either $NC_2 = |U|$ and $NC_3 = 0$ or $NC_3 \leq |U| - NC_2$. Hence, $NC_4 \leq |U| + |W| - NC_2 - NC_3$. So, $SUM \leq |U| + |W| = c + d$.

Case r4-2-2: $V = S_Ha$ and $U = S_Hb$. If $C(S_Hb) \in C(a_b \bigcup a_c \bigcup a_d)$, then $NC_2 = 0$. Hence, $SUM \leq c + d$.

Case r4-2-3: $V = S_Ha$ and $U = S_Hb$. If $C(S_Hb) \notin C(a_b \bigcup a_c \bigcup a_d)$, then $NC_2 = |S_Hb|$. In this case in the counting of the number of old colours, which have already been assigned to the neighbours of the vertices of the sets S_Hc or S_Hd which have already been coloured and belong in the graph H_1^2, the following terms are components of this sum: $|S_Hb|+|S_Ha|-|S_Hb| = |S_Ha|$. Hence, at least one of the sets S_Hc and S_Hd is coloured without new colours. So, the other is coloured using at most so many new colours as the size of the smallest of these sets. So, $SUM \leq b + d$.

Case r4-3 Here $W = S_Ha$.

Case r4-3-1: If $C(U) \notin Z_V$, then $NC_2 = |U|$.By Proposition 4.5 it is concluded that $C(U) \in C(a_b \bigcup a_c \bigcup a_d)$. Hence, if *old colours* $\bigcap C(V) \notin Z_a$, then $NC_3 \leq |V| - NC_2$. Thus, when the set X will be coloured $\#OC_X \leq |U| + |V| - NC_2 + |S_Ha| - |V| + NC_2 + Z_x$. Hence, $NC_4 = min\{|U|, |X|\}$.

So, $SUM \leq NC_2 + |V| - NC_2 + min\{|U|, |X|\} = |V| + min\{|U|, |X|\}$. Thus, $SUM \leq |S_Hb| + |S_Hd|$.

Case r4-3-2: If $NC_2 = |U|$ (i.e. $C(U) \notin Z_V$) and $NC_3 = 0$, namely, *old colours* $\bigcap C(U \bigcup V) \in Z_a$. Then, when the set X will be coloured $\#OC_X \leq |U| + |V| - NC_2 + |S_Ha| + Z_X$. Hence, $NC_4 \leq min\{|V|, |X|\}$.

So, $SUM \leq |U| + min\{|V|, |X|\}$. Thus, $SUM \leq |S_Hb| + |S_Hd|$.

Case r4-3-3: Let $NC_2 = 0$, namely $C(U) \in Z_V$. Hence, $NC_3 = |V|$. Then, when the set X will be coloured $\#OC_X \leq |U| + |V| + |S_Ha| - |V|(= NC_3) + Z_X$. Hence, $NC_4 \leq min\{|U|, |X|\}$.

So, $SUM \leq |V| + min\{|U|, |X|\} \leq |S_Hb| + |S_Hd|$.

Case r4-3-4: if $NC_2 = NC_3 = 0$, then when the set X will be coloured, we have that $\#OC_X \geq |U| + |V| + |S_Ha| + Z_X$. This implies that $NC_4 \leq min\{(|U| + |V|), |X|\}$. Hence,

$SUM \leq min\{|U| + |V|, |X|\} \leq |S_Hb|$.

Case r4-4: Here, $X = S_Ha$. This means that the sets U and V have been coloured using only old colours, otherwise set S_Ha would not be the last set of H_1 that is coloured. When X is coloured, $\#OC_X \leq |W| - NC_3 + Z_X$. So, $SUM = NC_3 + |W| - NC_3 = |W| \leq |S_Hb|$.

With all this cases we prove the Part A when $r = 4$.

Now we make some remarks about the last case, where $r = 5$. In this case, we have to do a more detailed analysis for the performance of $MDsatur$ on squares of planar graphs. This is because it is possible to stand up against the following sum $c + d + e$, which is at most $\frac{3}{5}\Delta(G)$. In this analysis we are mainly based on the fact that the colouring algorithm $MDsatur$ colours first a clique of size $\Delta(G) + 1$, denoted by $CLinit$, and then in the worst case the rest of the vertices of G except of the vertices of the graph H_1 and the sets i_a. An example of a 5-regular planar graph is shown in Fig. 5b. Studying carefully this case we can easily seen, that the neighbours, in $G^2 - H^2$, of each pair of sets with vertices from the graph H_1, i.e. S_Hi, S_Hj, can't be coloured by our algorithm with the same colours, even in the case where the cardinalities of all the sets of vertices, have the same value (i.e. a = b = c= d= e). This is true, because we assumed that the vertices $N_{G^2-H_1^2}(H_1)$ have only old colours, otherwise there are sets of H_1 (i.e. S_Hi) that have more old colours available for their colouring, so the increase on the number of colours used by $MDsatur$ is minimized (so we omit such situations). Thus, we assume by way of contradiction, that $C(N_{G^2-H_1^2}(S_Hi)) == C(N_{G^2-H_1^2}(S_Hj))$, for $1 \leq i, j \leq 5$ and $i \neq j$. Lets go to check if the current colouring is valid (namely none pair of adjacent vertices in G^2 have

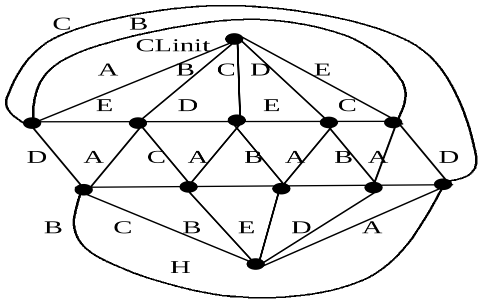


Figure 5: a) a 2-regular planar graph. b) A 5_regular planar graph (Ga) with a colouring of the graph G that is obtained from this Ga graph, using the sets of colours A, B, C, D, E of sizes a, b, c, d, e, respectively. c), d) 4-regular planar graphs.

the same colours). It is easily verified that, if this assumption holds in a colouring of the square of any planar graph G (that is obtained from any planar r-regular (Ga) as described above), then $C(N_{G^2-CLinit}(CL_i)) = C(N_{G^2-CLinit^2}(CL_j))$, where $CL_i, CL_j \in CLinit$ ($i \neq j$ are sets of vertices of $CLinit$ analogous to sets S_Hi, S_Hj). This is checked going backwards, starting from the neighbours of the distance two neighbours of the vertices of H_1 (in G) until the distance two neighbours of the vertices of $CLinit$. This yields that all the vertices of clique $CLinit$ have the same colour, or a set of new colours would be used for its colouring. This is a contradiction, because we assumed that graph $G^2 - H_1^2$ has been coloured only by old colours. Hence, we conclude the following Proposition.

Proposition B.4 *The difference on the colours assigned to the neighbours of the vertices of each pair of sets with vertices from the graph H_1 (in $G^2 - H_1^2$) is at least as large as the smallest set.*

Case r = 5

In this case we make also use of some variables. Let them be U, V, W, X, Y similar to the case where $r = 4$. Y denotes the last set of H_1 that is coloured.

Case r5-1: $U = S_Ha$. This case is trivial, in the score of Propositions B.2 and 4.6. By Prop. 4.6 sets b_a, c_a, d_a and e_a remain uncoloured until the colouring of the graph H_1^2. Thus, $NC_2 = 0$. So, set S_Ha can be omitted and the rest four sets S_Hb, S_Hc, S_Hd and S_He can be seen as the four sets of the case where $r = 4$. Thus,

$SUM \leq |S_Hc| + |S_He|$.

Case r5-2-1: $V = S_Ha$. If $U \neq S_Hb$, then $Y = S_Hb$ due to Proposition 4.7. Then when the last set of H_1 is coloured $\#OC_Y \leq |U| + |V| - NC_2 + |W| - NC_3 + |X| - NC_4 - |S_He| + Z_b$. Hence $SUM \leq NC_1(= 0) + NC_2 + NC_3 + NC_4 + |U| - NC_2 + |W| - NC_3 + |X| - NC_4 - |S_He| = |U| + |W| + |X| - |S_He| \leq c + d$.

Case r5-2-2: $V = S_Ha$. If $U = S_Hb$ and $C(S_Hb) \in Z_a$ then S_Ha is coloured without any new colour. Thus, by virtue of Proposition B.4.

$SUM \leq |S_Hc| + |S_Hd|$.

Case r5-2-3: $V = S_Ha$. If $U = S_Hb$ and $C(S_Hb) \notin Z_a$ then $NC_2 = |S_Hb|$. Hence, $NC_3 = 0$. Then, the sets X and Y have to be coloured using at most d new colours. This is true because, $W = S_He$, $X = S_Hd$ and $Y = S_Hc$ by Proposition 4.7. Then $NC_5 = d - e + e$, if $NC_4 = 0$, otherwise if $NC_4 = d - e$, then $NC_5 = 0$. Hence,

$SUM \leq |S_Hb| + |S_Hd|$.

Case r5-3-1: $W = S_Ha$. If $V = S_Hb$ and $C(S_Hb) \notin Z_a$, then $NC_3 = |S_Hb|$. If $U \neq S_Hc$ then if $U = S_He$, then $X = S_Hd$ and $Y = S_Hc$. In this case $NC_X = 0$, because of Proposition B.4. Thus, $NC_5 \leq d - e$. In the case where $U = S_Hd$, then $X = S_He$. For the same reason $NC_4 = 0$ and $NC_5 \leq d - e$.

If $U = S_Hc$ then it is easily verified, that $NC_4 + NC_5 \leq d$.

So, $SUM \leq |S_Hb| + |S_Hd|$.

Case r5-3-2: $W = S_Ha$. If $V = S_Hb$ and for the colouring of the set S_Ha are not needed new colours (namely $C(V) \in Z_a$)), then by Proposition B.4 it comes obviously that

$SUM \leq |S_Hc| + |S_Hd|$.

Case r5-3-3: $W = S_Ha$. If $U = S_Hb$ then as in the previous cases it is easily follows that,

$SUM \leq |S_Hc| + |S_Hd|$.

Case r5-3-4: $W = S_Ha$. If $U, V \neq S_Hb$, then $Y = S_Hb$ due to Proposition 4.7. Then if the sets S_Hc and S_Hd are coloured using only old colours then

$SUM \leq |S_Hb|$.

Otherwise $SUM \leq |S_Hc| + |S_Hd|$.

Case r5-4-1: $X = S_Ha$. Hence, $NC_1 = 0, NC_2 = 0$. So, $SUM \leq |W| + |Y|$. If the sets W, Y are not the sets S_Hb, S_Hc, then the theorem straightforward follows. On the other hand, if $NC_3 = 0$, then $SUM \leq b + d$, even when $NC_4 = |W|$. Otherwise, if $NC_3 = |W|$, then $NC_4 = 0$. Thus, $NC_5 \leq d$

So, $SUM \leq b + d$.

Case r5-5-1: $Y = S_Ha$. This case is trivial because the sets U, V and W are coloured without new colours. So, even if are used $|X|$ new colours for the colouring of set X these colours will be deleted from the new colours of S_Ha. Thus,

$SUM \leq |S_Hb|$

Part B

We proceed with a detailed proof of Part B for each value of r. In this part we examine what happens if the neighbours of the vertices of sets S_Hi which belong in sets of big size (i.e. a, b, ..) are coloured before the colouring of H_1^2. In this Part we consider also the order in which are coloured the sets with the vertices of the graph H_1^2 and we try to colour as much as possible of their neighbour in $G^2 - H_1^2$, before the colouring of H_1^2.

Case r = 2 This case is trivial.

Case r = 3

Here we present a crucial observation on which this proof is based.

Observation B.1 *When the vertices of the set $N_{G^2 - H_1^2}(S_Hi)$ (i.e. the neighbours of the vertices of set S_Hi in the graph $G^2 - H_1^2$) become coloured, for $i \in \{b, c, d, e\}$, then MDsatur colours next the vertices of the set S_Hi.*

By Observation B.1 we conclude, that the vertices of the sets S_Hb and S_Hc can have at most a coloured neighbours in the graph $G^2 - H_1^2$, before the colouring of vertices from H_1^2. If

$a \leq b + c$ and the vertices of Z_a have been coloured, then the set $S_H a$ is coloured first (among the vertices of the graph H_1^2). Using the previous way of presentation of our results we have that:

$U = S_H a$ ($NC_1 = 0$). If $V = S_H b$ then $\#OC_b \leq |V_a| + |S_H a| - c$ (by Proposition B.4) $\leq a + b + c - c \Rightarrow NC_2 \leq b - c$. Hence, $SUM \leq b$ in this case even if $NC_3 = c$ ($W = S_H c$).

For the case where $r = 3$ it is easily verified, based on the above part of the proof, that the value of $SUM \leq b$, either the vertices of the set $S_H c$ have been coloured before the vertices of the set $S_H b$, or the vertices of the sets b_c and c_b have been coloured instead of the vertices of the sets b_a and c_a, before the colouring of the vertices of $S_H a$.

If $a > b + c$ then the vertices of the sets $S_H b$ and $S_H c$ are coloured, before the colouring of $S_H a$. In this case the assertion of the Theorem holds, because $SUM \leq b + c$. In this case $b + c \leq 0.5\Delta(G)$.

We proceed with the proof of Part B when $r = 4$. With similar arguments is proved the case where $r = 5$, so we omit it.

The proof of the assertion in this case is based on Observation B.1. According to this at least one of the sets with neighbours of each set with vertices from graph H_1 i.e. $S_H i$, for $i \neq a$, have to remain uncoloured until the colouring of $S_H i$. In order to force $MDsatur$ to increase as much as possible the number of colours that it uses to colour G^2 we assume that it remains uncoloured the smallest set with neighbours of the vertices of each set $S_H i$. It is easily understood, that this leads to the maximum increase on the number of colours used by our algorithm. In other words we say, that at most $a + c$ of the neighbours of the vertices of $S_H b$ have already been coloured, also at most $a + b$ from the neighbours of the vertices of $S_H c$ have been coloured and $a + b$ from the neighbours of the vertices of $S_H d$.

Now, let us consider what happens when $a + b$ from the neighbours of the vertices of $S_H d$ have already been coloured. We assume w.l.o.g. that this set with vertices from H_1 has first two sets of its distance two neighbours coloured. In the following we examine two cases.

Case 1: In the first one we assume that each pair of nearby sets with vertices from H_1 has a common set of neighbours in $G^2 - H_1^2$. An example of a 4-regular planar graph (i.e. Ga) that gives a graph G where the above conditions hold is shown in Fig. 5c. In this case it is easily verified, that either the vertices of the set d_c have degree of saturation bigger than $a + b$ or the vertices of at least one of the sets d_a and d_b had degree of saturation at most $a + b$ when they had selected by $MDsatur$ to be coloured.

If the first case is true, then it is impossible the set d_c to remain uncoloured until the colouring of the vertices of $S_H d$. This is true because the maximum value on the degree of saturation of the neighbours of the vertices of the graph H_1 is at most $a + b + c$ when the graph H_1^2 is still uncoloured. This yields, that in such cases some sets of vertices of H_1 are coloured before the colouring of $S_H a$. More precisely the vertices of the first two sets $S_H i$, which have at least two sets of neighbours in the graph $G^2 - H_1^2$ coloured, are coloured using only old colours. Observe that the last two sets $S_H i$ that are coloured have only one set of neighbours already coloured in the graph $G^2 - H_1^2$. Hence, even if the last sets of H_1 that are coloured are the sets $S_H b$ and $S_H a$ (if these sets are the sets $S_H c, S_H d$ then the proof of the assertion is obvious), then $MDsatur$ uses at most b new colours to colour graph G^2. Namely, it uses $\Delta(G) + b$ colours in total. I.e. it uses at most $1.5\Delta(G) + 26$ colours. This is true because the third set with vertices from H_1 is also coloured by old colours only.

The last pair with vertices from H_1 that we have to consider is the $\{S_H b, S_H c\}$. In this case $SUM \leq b$, because the third and the fourth set with vertices from H_1 that are coloured have at most one set of its neighbours in $G^2 - H_1^2$ already coloured.

Now, let us study the case where the vertices of at least one of the sets d_a and d_b had degree of saturation $a + b$ at most when the had been coloured. In this case the neighbours of the vertices of the set d_c which do not belong in the sets $d_a \bigcup d_b$ are uncoloured (otherwise holds the above case). Observe that the vertices of one of these sets of neighbours of the vertices of d_c are also neighbours of the vertices of one the sets d_a or d_b. Otherwise the vertices of one of these sets of neighbours of the vertices of d_c have common neighbours with the vertices of one of the sets d_a or d_b (i.e. the set X in the figure 5c). If the set X is uncoloured then the vertices of set d_a had degree of saturation at most $b + c$ when they were coloured. On the other hand, if the vertices of set X were coloured, then some other set that joins the subgraph of G that joins the set S_Hd with the vertices of the $CLinit$ with the subgraph of G that joins the set S_Hb with the vertices of the $CLinit$ is uncoloured. Otherwise the neighbours of the vertices of set d_c would be coloured. From the colouring of this set and then there are sets of vertices that are coloured when their degree of saturation is less than $a + b + c$. Then, either the vertices of set d_a or X were coloured with degree of saturation at most $b+c$. This implies that none uncoloured vertex had degree of saturation more than $a + b$ at that step of $MDsatur$. Hence, in this case the set S_Hd (that is the first set with vertices from the graph H_1 that has two sets of its neighbours coloured, by the assumption in the beginning of Case 1) is coloured after the colouring of $a+b$ of its neighbours. It is clear that the vertices of the set S_Hd receive old colours only. Subsequently, $MDsatur$ colours the vertices of one more set S_Hi under the same conditions. I.e. it has at most $a + b$ neihbours in $G^2 - H_1^2$ coloured. The vertices of this set also receive old colours only. The last two sets are coloured with less than b new colours by $MDsatur$ even when these sets are the sets S_Ha and S_Hb or S_Hb and S_Hc. This is proved with similar arguments as in the previous case.

Case 2: In this case nearby sets with vertices from graph H_1 do not have common neighbours in the graph $G^2 - H_1^2$. The graph Ga in this case have the structure of the graph that is shown in Fig. 5d. In order to achieve to keep uncoloured one set with neighbours of the vertices of each of the sets S_Hi, we assume that there are uncoloured parts of the graph $G^2 - H_1^2$. The influence of this fact is that in intermediate points our algorithm colours vertices which have only two sets of their neighbours already coloured. This is true, because these vertices have neighbours in the uncoloured part of the graph $G^2 - H_1^2$. This is similar to Case 1. However, here the vertices which have been coloured under this condition are not neighbours of the vertices of graph H_1. In this case, the subgraphs of G that join the vertices of the clique $CLinit$ (of G^2) with the sets of vertices of the graph H_1 are almost independent. Hence, when does not exist a vertex with degree of saturation more than $a + b$ then $MDsatur$ colours a vertex which has the maximum value on its degree of saturation. I.e. the vertices of set X in the figure. Subsequently, $MDsatur$ colours vertices which belong in the same subgraph of G, i.e. the vertices of the set Y. Then the vertices of set y have degree of saturation at least $b + c + d$, so are coloured at this point. Subsequently the degree of saturation of the vertices of sets x and z becomes $b + c + d$, so these sets are coloured at this point.

Following the same way $MDsatur$ achieves to colour two sets with distance two neighbours of vertices of a set of H_1. So, the first set of vertices of H_1 that has degree of saturation $a + b$ is coloured. By Proposition 4.9 we conclude that the vertices which belong in the same set of H_1 with the last vertex that has been coloured, are also coloured. The vertices of the first set of H_1 that have been coloured received old colours only. Continuing with a similar way as in Case 1, the vertices of the second set of graph H_1 that are coloured also receive old colours.

The last two sets of H_1 are coloured by $MDsatur$ with at most b new colours. This is proved with similar arguments as in the previous cases. Hence, the total number of colours used by $MDsatur$ to colour G^2 is $1.5\Delta + 26$. ■

C Proof of Theorem 4.3

In the proof of Theorem 4.3 we make use of the following result, which is proved after the proof of this Theorem.

Lemma C.1 *Suppose that S is a planar square root of the clique K_x (denoted by CL), for $x \geq 8$. It is not possible to extent S in any way, so that any five of its vertices to belong to more than two cliques in S^2 of size at least eight, call them $C1, C2$, while the maximum clique in S^2 remains the K_x and each of the cliques $C1, C2$ has at least three distinct vertices.*

C.1 Proof of Theorem 4.3

Proof: The proof of the assertion is split in three Parts, A, B and C, as they have been defined in the sketch of this proof in Section 4.

Part A: We recall that in this part of the proof we assume that H_1' is a clique of size $\omega(G^2)$. We assume that H_0' is also a clique of the same size. We proceed by examining all the cases obtained from each potential size of these cliques and each possible structure of their minimal square roots.

We begin with the case where the maximum clique of G^2 has the least possible size, which is $\Delta(G) + 1$. We recall that our aim is to achieve each vertex of H_1^2, i.e. v, to have the biggest possible value on its degree of saturation before its colouring. We begin with the cases where the neighbours of v in $G^2 - H_1^2$ form a clique. We denote another two vertices of the graph H_1 by u, w.

Case A1, $S, H_1 = K_{1,(x-1)}$ **and the vertices of** $N_{G^2-H_1^2}(v)$ **form a clique in** G^2**:**

We split the proof of this case in three subcases based on $|N_{G-H_1}(v)|$, that is the number of v's neighbours in $G - H_1$, and on CN. $CN = |N_{G-H_1}(v) \bigcap N_{G-H_1}(w)|$, where v, w are two consecutive vertices of H_1 in the planar embedding EM of G.

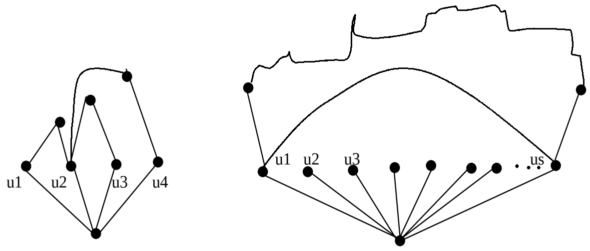
Case A1.1 $|N_{G-H_1}(v)| = 1$, $CN \leq 1$**:**

We denote by v' the only neighbour of the vertex v in the graph $G - H_1$. Then, either there are neighbours of v' in $G - H_1$ which are also neighbours of another vertex w', such that $w' = N_{G-H_1}(w)$, or $N_{G^2-H_1^2}(v) \bigcap N_{G^2-H_1^2}(w) = \emptyset$. The way that is used by our algorithm to grow the order O and the fact that it has the CP are the keypoints where we are based to prove that the graphs where $N_{G^2-H_1^2}(v) \bigcap N_{G^2-H_1^2}(w) \neq \emptyset$ are harder to colour by $MDsatur$, than the graphs where $N_{G^2-H_1^2}(v) \bigcap N_{G^2-H_1^2}(w) = \emptyset$.

Case A1.1.1, $N_{G^2-H_1^2}(v) \bigcap N_{G^2-H_1^2}(w) \neq \emptyset$**, CN = 1:**

In this case observe that the vertices of the graph H_1 are split in a number of sets, i.e. x, and the vertices of each of these sets have a common neighbour in the graph $G - H_1$. At this point we note that $MDsatur$ colours the vertices of at least one of these sets with old colours (i.e. colours with value less than $\chi_{MDsatur}(H_0')$). The rest of the vertices of the graph H_1^2 will be coloured with new colours by $MDsatur$ only if all the neighbours of the x vertices of graph $G - H_1$ which are incident to the vertices of the graph H_1, have been coloured by $MDsatur$ before the colouring of the vertices of this graph. Namely $MDsatur$ achieves to colour all the vertices (except of the vertices of H_1) of the x maximum cliques where belong the vertices of H_1.

At this point it is time to make the following crucial propositions that are useful in the way of construction the graph G.



(a) (b)

Figure 6: An example for Proposition C.3.

Proposition C.2 *There are always available old colours, for the first set with vertex from H_1 that is coloured.*

Proposition C.3 *Let $u1, u2, u3, u4, u5$ be any five sequential vertices of H_1 (we recall that H_1 is a star graph). If $N_{G-H_1}(u2) \bigcap N_{G-H_1}(u1) \neq \emptyset, N_{G-H_1}(u2) \bigcap N_{G-H_1}(u3)$ (or $u4$) $\neq \emptyset, N_{G-H_1}(u2) \bigcap N_{G-H_1}(u5) \neq \emptyset$, then the vertices of $N_{G-H_1}(u3) \bigcup N_{G-H_1}(u4)$ cannot have neighbours in the rest of the graph $G - H_1$ which lies out of the face created by the join of each of the vertices $v1, v3, v4, v5$ to the vertex $v2$.*

This is follows straightforward, because G has to remain a planar graph. An example is shown in Fig. 6a.

Corollary C.4 *There is always available an old colour for the colouring of at least one of the vertices $u3$ and $u4$.*

Let F be a bounded face in G. Then, there is a cycle that fixs its boundaries. Let us denote by bi, be the first and the last vertex on this cycle. In other words, bi, be are two adjacent vertices on the boundary of F. Let that the vertices on the boundaries of this face are joined to a common vertex. Then, the fact that no more than one of the vertices, which lie on the boundary of this face, can be joined to both vertices bi, be leads to the following corollaries. This fact follows straightforward from the arguments used in the proof of Lemma 3.4 and is based on the fact that the graph $K_{3,3}$ is not a subgraph of any planar graph.

Suppose that the conditions of Proposition C.3 hold and also that vertices bi, be correspond to the vertex $u1$ and to the most right vertex, from a sequence of vertices, see Fig. 6b.

Corollary C.5 *If there are s vertices in a sequence between bi and be, then there is always available an old colour for the colouring of at least $s - 1$ of these vertices when the current subgraph of G^2 that has been coloured lies outside F.*

This is true, because each vertex from the boundary of F has to be joined to both of the vertices bi, be in order to have neighbours that have already been coloured by $X_{MDsatur}(H_0^2)$ different colours. Otherwise, if these vertices are joined to one of the vertices bi, be, then their neighbours which have been coloured are at most the neighbours of this vertex. Then the assertion is proved, because the degree of each of the vertices bi, be have to be at most Δ. So, Δ colours are enough for the colouring of its neighbours. An example is shown in Fig. 6 b.

Corollary C.6 *The square of any planar graph G which has as subgraph of graph H_1 the graph described in Proposition C.3 is not an instance of the minimal worst case graphs for algorithm $MDsatur$, when it is applied on squares of planar graphs.*

Based on Proposition C.3 and the corollaries on it we conclude, that it is meaningful to consider only the cases where the neighbours of each vertex u', which is the neighbour of the vertex u in $G - H_1$, are consecutive vertices of H_1. Also, the vertices of H_1 have not be joined with the vertices of the graph $G - H_1$ nested. Namely, consecutive vertices of H_1 are split in x sets and all the vertices of each of these sets are joined to the same vertex in $G - H_1$. The fact that $N_{G^2-H_1^2}(v) \bigcap N_{G^2-H_1^2}(w) \neq \emptyset$ for two vertices v, w of H_1 which belong in two nearby sets with vertices from H_1 implies that the vertices v' and w' have common neighbours in G. By Lemma H.1 it can be easily proved, that when the sizes of the above described sets are bigger than four, then the vertices of each such set can simultaneously be joined, to at most two vertices in G (recall that all these vertex are almost joined to one vertex, i.e. the root of H_1). Thus, graph G becomes harder colourable, when all the vertices of each set (with at least four vertices) are joined to two vertices of maximum degree. Meaning that they belong together to two cliques of maximum size.

We denote the sets of vertices where the vertices of H_1 are split by $S_Ha, SH_b, ..., S_Hx$ (in this case x indicates the x^{th} letter in the alphabet). Also, suppose that the neighbours of each vertex of degree Δ, i.e u', are grouped in the same number of sets, namely x, with sizes similar to that of sets $S_Ha, S_Hb, ..., S_Hx$. We denote these sets by $S_{u'}a, S_{u'}b, ..., S_{u'}x$. The vertex that corresponds to these sets is omitted, when there is no danger for confusion and when we are speaking roughly and there is no reason to specify this vertex. We note here, that the sizes of these sets of vertices can be different among them. In the following we will explain why the above assumptions are allowable.

Seeing the resulting graph G, that is constructed from the above procedure, abstractedly, namely seeing each set of vertices Si, for $i \in \{a, b, ...\}$ as an edge, we observe that the resulting graph, let us denote it by Ga, is a x-regular planar graph. An example of graph G is shown in Fig. 3e, for $x = 3$. In Fig. 3d is shown the corresponding 3-regular planar graph, Ga. The maximum value on x is five, when the graph, Ga, is planar by Lemma 4.2.

Observe that, each of the vertices inserted in Ga during the replacement of an edge (in order to get the graph G), let it be v, has degree two (see Fig. 3 c). The chromatic number of G^2 remains the same even when $\Delta(G) - 2$ new vertices are inserted in G as neighbours of v (these vertices play the role of the graph H_2). This is true, because each of the v's new neighbours, let it be v'', sits in a face bounded by the initial neighbours of v (in G) and another vertex (a distance two neighbour of v, let it be w). This yields, that the vertex v'' can be joined to at most two vertices of G, that are the vertices v and w (two of the distance one neighbours of v already have maximum degree). Following the way of reasoning in Lemma H.1 and the above way of construction of the graph G, we conclude that the new neighbours of v can belong to a graph similar to graph G, we denote it by G' (see Fig. 3 e). Observe that the distance two neighbours of the vertex v'' in $G - G'_s$ are at most six (when vertices $(u, v, w) \in P_3$).

Suppose that there are as much as possible copies of graph G' as subgraphs of graph G (i.e. as much as the number of vertices of degree two). It is easily understood, that the graph G in this case is not easier colourable, than the graphs G^2, which does not have graphs G' as subgraphs. This is true, because if it were possible to be coloured the graph $G^2 - H_1^2$ before the colouring of any vertex of H_1, then this would make more possible the case where the neighbours of each vertex of H_1, that have already been coloured, have more different colours, than in the case where do not exist graphs G' in G. However, this is not the case for our algorithm, because of its connectivity property. According to this property the subgraph of G^2 that has already been coloured in every intermediate point of $MDsatur$ is a connected graph. So, it is impossible to become coloured more than one graph G' before the colouring of at least one vertex of H_1. It is easily seen, that after the colouring of the first six vertex of H_1, it is impossible $MDsatur$ to colour any other graph G' before the colouring of all of the vertices of H_1. The above argument leads to the conclusion, that the existence of graphs G' in G do not actually make graph G harder to be coloured. Their existence may increase by at most six the number of colours needed by our algorithm. Hence, for simplicity reasons and proof's clarity these graphs ($G' = H_2$) are omitted from G^2.

At this point we will explain, why the assumption that all the sets S_ui, for each vertex of maximum degree u ($i \in \{a, b, c, d, e\}$), have the same size is permissible. It is easily understood, that the case where there are neighbours of u which belong to the same set S_ui and are not joined to two vertices of maximum degree in G, does not give a worst case graph for our algorithm. On the other hand, it is permissible two sets to be integrated (or the sizes of the sets S_ui to be different among different vertices, instead of u) provided that the vertices which belong to any set also simultaneously are joined to two vertices of maximum degree in G. Observe that if $|S_ui| = X$, then the vertices of a neighbour of vertex u, i.e. v, are split in some sets and the size of one of them i.e. S_vi has to be X. It is not straightforward how to achieve this. Actually, we are not sure if this situation is possible. However, this not so clear situation does not influence the progress of this work, because our algorithm is considered for the value of the degree of saturation of each uncoloured vertex (in each point of its application on G^2) and not in which sets belong the neighbours of such a vertex that have already been coloured. Thus, even if it is possible this situation do not gives harder to colour graphs for our algorithm.

Case A1.1.2, $N_{G^2-H_1^2}(v) \bigcap N_{G^2-H_1^2}(w) = \emptyset$**, CN = 1:** An example of the graph Ga that is obtained in this case is shown in Fig. 3 b. It is easily verified that the graph G that is obtained from such graphs Ga are not harder to colour by $MDsatur$, than the graph obtained in Case A1.1.1. This is true, because for the graphs obtained in Case A1.1.1 the colouring of a set of vertices influences the value of the degree of saturation of vertices which belong to two nearby sets i.e. S_ui, S_uj and not only one as in the case of the graphs obtained in this case (Case A1.1.2). Hence, $MDsatur$ is not possible to colour more neighbours of the vertices of the graph H_1, which belong to the graph $G^2 - H_1^2$, in the graphs obtained in Case A1.1.2, than that in the graphs obtained in Case A1.1.1. So, $MDsatur$ colours the graphs obtained in Case A1.1.1 with more colours, than the graphs obtained in Case A1.1.2.

Case A1.1.3, $|N_{G-H_1}(v)| = 1$**, CN = 0:** An example of the graph G in this case is shown in Fig. 3 a. It is easily seen, that in this case $MDsatur$ would colour the vertices of the graph H_1^2 with new colours (i.e. have value bigger than $\chi_{MDsatur}(H_0^2) = \Delta + 1$) only if it achieves to colour the neighbours of all the vertices of H_1 in $G^2 - H_1^2$ with $|H_1| = \Delta + 1$ copies of each of the colour used in the colouring of the graph H_0^2. However, $MDsatur$ is not possible to achieve this, even in the case where $N_{G^2-H_1^2}(v) \bigcap N_{G^2-H_1^2}(w) \neq \emptyset$ (v, w are vertices of H_1). This is because of the CP of our algorithm. Having in mind the fact that $MDsatur$ applies at step 3.c. a procedure similar to Bread First Search, we conclude that it is not colour vertices which belong in nearby cliques, if there are still uncoloured vertices which belong to the clique where belong the last vertex that has been coloured (a similar situation is explained in the example in Appendix A). Based on these arguments we show that the graph obtained in this case is not harder to colour by $MDsatur$ than the graph obtained in Case A1.1.1.

Case A1.2, $|N_{G-H_1}(v)| \geq 2$**, CN** ≥ 2**:** This case follows straightforward studying the graph of Fig.3f, which represents the subgraph of G on the root of H_1 with another two vertices of this graph (let them u, v) plus two vertices of $G - H_1$ which are both neighbours to the shown vertices of H_1 except of its root. Observe that these vertices of H_1 have nested neighbours in G. Proposition C.3 also concerns a similar situation. It is proved that the vertices of H_1 which lie between the vertices u, v receive old colours by $MDsatur$. Also, the colouring of the vertices u, v is based only on the colours of the neighbours of one of their common neighbours in $G - H_1$ (i.e. this which lie on the outer of these faces). Because of this, the join of the vertices u, v to two vertices in $G - H_1$ do not lead to a graph G whose square is harder to colour by $MDsatur$, than the squares of the graphs obtained in Case A1.1

Case A1.3.1, $|N_{G-H_1}(v)| \geq 2$**,** $CN \leq 1$**,** $|C(N_{G^2-H_1^2}(v))| = \chi(G^2)$**:** This case is very similar to the Case A1.1 Thus, we skip the details of its proof.

Case A1.3.2, $|N_{G-H_1}(v)| \geq 2$**,** $CN \leq 1$**,** $|C_(N_{G^2-H_1^2}(v))| > \chi(G^2) - 1$**:** This case is one of the most important. The statement of this case, in other words, it says that the vertices of H_1 have more than one neighbour in the graph $G - H_1$ and their neighbours in $G^2 - H_1^2$ that have already been coloured have more than $\chi(G^2){-}1$ colours. Observe that $|N_{G-H_1}(v)| \geq 2$, otherwise the second part of this statement couldn't hold. W.l.o.g. we assume that $|N_{G-H_1}(v)| = 2$. We denote by $v1, v2$ the neighbours of v in $G - H_1$. We also assume that the neighbours of $v1$, in $G - H_1$, have colours from $\{1, 2, ..., \omega(G^2)\}$, which have been used from our algorithm to colour the clique $CLinit$. So, the neighbours of $v2$ will have as much as possible from the colours over the above, that are already used from $MDsatur$, in the currently subgraph of G^2 that has been coloured. At this point we split the rest of the proof of this case in two subcases, based on the number of extra colours (those which have value larger than $\omega(G^2)$) that have been assigned on the neighbours of $v2$ (we denote this number by N).

Case A1.3.2.1. $N \leq \omega(G^2) - 1$**:** We begin with the case where the subgraph of G on the vertices $v2 \bigcup N(v2)$ play the role of graph H_1^2. H_0^2 obviously is the $CLinit$. Until now (during the subcases of Case A1), we achieved to colour graph H_1^2 with the larger number of extra colours, when this graph is the graph H_1^2 of Case A1.1.1. It is clear, that we are interested only for the cases where the graphs H_j^2, for $1 \leq j \leq max\ \{level\}\ - 1$, are coloured with as much as possible extra colours, in order to be coloured H_{j+1} with colours, which have the bigger possible values. This argument is stronger when the number of levels is very small as in our case, where we prove that this is less than four. As we prove in Parts B, C the graph H_1 in these cases has similar structure as the graph H_1 in Case A1,1,1, hence we examine in detail only this case.

We recall here, that the graph H_1^2, in the Case A1.1.1, is the graph that is obtained from a x-regular planar graph by replacing its edges with sets of new vertices (see Case A1.1.1 for details). At this point it's time to make the following crucial remark. *There is a cycle in every regular graph.* The consequence of this remark, which is essential for our study, is that clique $CLinit$ lies inside a bounded face, let us denote it by F. The boundaries of F are delimited by vertex $v2$ and two vertices of H_1, which are neighbours in this graph and their *nearest common ancestor* in a BFS tree of G, which starts from any vertex of $CLinit$, is the root. This in combine with the Connectivity Property implies, that it is not possible to be coloured both graphs H_0^2 and H_1^2, which correspond to each vertex of H_2. In the following we show, that even if it is possible to be coloured a part of each of the graphs H_0^2 and H_1^2 (which correspond to each vertex of H_2^2), then the graph H_2^2 is coloured by $MDsatur$ with the already used colours, plus one.

For proof's clarity we denote by $h_{2,1}$ the first vertex of H_2^2 that is coloured by our algorithm and by $h_{2,2}, h_{2,3}, ..., h_{2,\omega(G^2)-1}$ the rest of its vertices correspondingly. We also denote by $H_{0,i}^2$ and $H_{1,i}^2$ the graphs which correspond to graphs H_0^2, H_1^2 and have the distance one and two neighbours of $h_{2,i}$ in $G^2 - H_1^2 - H_0^2$. Suppose that firstly $C1$ vertices of $H_{0,1}^2$ are coloured. Then, are coloured $C2$ vertices of $H_{1,1}^2$ with new colours (these are the colours $\{\omega(G^2)+1, ..., \omega(G^2)+C2\}$). Observe that if $C1 = \omega(G^2)$ and $C2 = 0$, then the next vertex that will be coloured by $MDsatur$ is the vertex $h_{2,1}$. This is true, because the value on the degree of saturation of each vertex of $H_{1,1}^2$ is less than $\omega(G^2) - 1$, otherwise its vertices with so big value on their degree of saturation would have already been coloured. If $C1 < \omega(G^2)$ and $C2 > 0$, then $MDsatur$ continues to colour vertices of $H_{1,1}^2$, while the value on their degree of saturation is less than $C1 + C2 + C3$. Where $C3$ denotes the number of colours used in the colouring of $H_{1,i}^2$, with values less than $\omega(G^2)$ and are not used in the colouring of $H_{0,i}^2$. It is easily verified, that the maximum value on $ds(h_{2,1}) = C1 + C2 + C3$, when $C3 = \omega(G^2) - C1$, hence $ds(h_{2,1}) \leq \omega(G^2) + C2$. Thus,$h_{2,1}$ is coloured by a new colour.

Now lets go to see how $MDsatur$ colours vertex $h_{2,2}$. Here, we also assume that $C1 < \omega(G^2)$ and $C2 > 0$, the value of $C3$ is less than $\omega(G^2) - C1$, because vertex $h_{2,1}$ has been coloured by a colour that is not used in the colouring of none of the graphs $H_{0,i}^2$ and $H_{1,i}^2$. Hence vertex $h_{2,2}$ is coloured, before the colouring of all the vertices of $H_{1,2}^2$, by a colour with value less than $\omega(G^2)$. This situation holds for the first $C3$ vertices of H_2^2. It is clear that the value of $C3$ is well defined when the value of $C1$ is the same on each graph $H_{0,i}^2$. Observe that this assumption is allowable. On the other hand, we obviously assume that $C2$ has its larger possible value. When $C3$ also has its larger possible value, then we conclude that $C1 = C2$, because $C1 + C3 = \omega(G^2)$ and $C2 + C3 = \omega(G^2)$.

As we explained above, we can assume that the $C1$ vertices of graph $H_{0,i}^2$ are coloured before the colouring of $H_{1,i}^2$. On the other hand, the vertices of $H_{1,i}^2$ are coloured, just before the colouring of the vertex $h_{2,i}$, otherwise it must be possible to be coloured more than one vertex of H_2 in the same time.

Based on the fact that $C1 + C3 = \omega(G^2)$, we conclude that the graph $H_{1,i}^2$, for $i > C3 + 1$, is coloured after the vertex $h_{2,i}$, because the value on the degree of saturation of their vertices is less that the corresponding value of $ds(h_{2,i})$, that is already $\omega(G^2)$. Hence, $\omega(G^2) - C3 - 1 = C2 - 1$ vertices of H_2 are coloured with the colours $\{\omega(G^2) + 1, ..., \omega(G^2) + C2 - 1\}$. Observe that we achieved to colour the whole graph H_2 adding only one new colour. Hence, for simplicity reasons we will omit graphs which have the structure of the graphs presented in this case, since they don't actually give harder to colour graphs for $MDsatur$. We omit them, also because the graph $H_{1,i}^2$ is not possible to have the ability to be coloured before the colouring of vertex $h_{2,i}$, as we explain in the beginning of this case.

Case A1.3.2.2 $N > \omega(G^2) - 1$**:** This case is impossible, because we could not create a graph G^2 that is coloured by $MDsatur$ with more than $2\omega(G^2)$ colours. This is easily concluded from all the cases in the proof of this Theorem.

Case A2: S, H_1 **are not star graphs**

We proceed with the case where the graph S isn't the star graph. This means that the square root of the maximum clique, in G^2, isn't the star graph. Then it is easily seen, that no more than six of its vertices lie in only one face in a planar embedding of S, by Lemma 3.5. According to this lemma the structure of each such square root of $K_{\omega(G^2)}$ is shown in the graphs that are depicted in Figure 1 e, f, g. As it is easily understood, the vertices which lie inside the cycle on vertices $\{1, 2, 3, 4, 5, 6\}$, have at least two common neighbours in the above cycle, or one vertex of MCL is joined to each other. In Lemma H.1 it is shown, that it is not possible for these vertices to belong to at least two cliques of size $\Delta(G) + 1$ each (with at least three distinct vertices each), except of the case where the second clique belongs to graphs like $G' = H_2$. However, for the same reasons, as in the Case A1, we omit the graphs G'. Thus, the graphs that are obtained from this case are not harder to colour than the graphs obtained when H_1 is the star graph. As we note above the graph H' in this case cannot have more than two levels. Namely does not exist a graph H_2.

Note C.7 *If the number of vertices on the exterior face of H_1 are less than six, then similar results hold, as in the case where there are six vertices on this face.*

Case A3: $s \geq \Delta + 4$

As we have proved in Lemma 3.6 the size of the maximum clique in the square of any planar graph, G, is at most $1.5\Delta(G) + 1$. The subgraph of G that forms the square root of this clique, let it be S, has the structure of the graph presented in Fig. 2, when MCL has size at least $\Delta(G) + 3$. So, if the maximum clique of G^2 has size bigger than $\Delta(G) + 3$, then there are three vertices, let them u, v, w, which form a cycle in S (i.e. C_3) (or a C_4 with the vertex z, see Fig. 2b), such that each of the neighbours of each of them has to be a neighbour of one of the other two. As it is easily seen from the figure, does not exist more than three vertices in one face (except of the vertices u, v, w). Thus, using the above arguments we can prove, that in such case it is not possible a vertex of G to belong in more than one clique of maximum size (even of size at least Δ), except if the second of them is in the graph G'. As explained above the graphs G' are omitted. Observe that the maximum number of levels of graph H' in this case is only one.

Part B: H_1^2 is not a clique

We recall that our aim in this Part is to show which are the worst case instances of squares of planar graphs for $MDsatur$, when H_1^2 is not a clique. In other words, we want to study in the case where the graph H_1^2 is a $\chi(G^2)$-colourable graph, if $MDsatur$ is possible to increase the number of colours, that it needs in order to colour G^2, in respect to the number of vertices on the exterior face of H_1.

We assume for contradiction, that there are X vertices from H_1 coloured, with colours $\{\omega(G^2) + 1, ..., \omega(G^2) + X\}$. In order to be coloured the rest (or some of the rest) of the vertices of H_1 with new colours, they must have coloured neighbours, in $G^2 - H_1^2$, with the colours $\{1, 2, ..., \omega(G^2)\}$. They also have to be at distance at most two from each of the X vertices of H_1^2 which have the X largest already used colours. With similar arguments as in the proof Lemma 3.6 (because of the fact that the graph $K_{3,3}$ is not a subgraph of any planar graph), we prove that one of the vertices of H_1, which have the X largest colours, has to be joined to each of the others (we call this vertex $x1$). Hence, it is possible $\Delta - X$ of the rest of the vertices of H_1 to be at distance at most two from each of these vertices. This is true, because the degree of vertex $x1$ has to be at most Δ. Fig. 3 g impress this case. Thus, because of the bound on the degree of $x1$, we conclude that the rest of the vertices of H_1^2 could not receive colour with value greater than the value of the maximum colour needed by $MDsatur$ in order to colour the graphs obtained from Case A1, where the graph H_1 is a star graph on $\Delta+1$ vertices. However, some of the vertices of H_1 which are not adjacent to vertex $x1$ (are the $|H_1|-\Delta-1$ vertices) can receive colours with value bigger than $\omega(G^2)$, because it is possible to exist more than one groups on H_1's vertices like the group on vertex $x1$ with its neighbours. I.e. more than one star graphs with $\Delta+1$ vertices each. As it is explained above the colours assigned to the vertices of one of these star graphs can be used from the vertices of the others in any colouring of G^2. For all these reasons the proof of this case follows. We note here, that the maximum number of levels where the graph H' is split in this case is also two.

Part C, graphs H_0^2, H_1^2 are $\chi(G^2)$-colourable graphs and $\chi(G^2) > \omega(G^2)$

The proof of this part is easily concluded from the proof of Part B. The keypoint here is that the vertices of H_0^2 which have colours bigger than $\omega(G^2)$ have to lie on one face in an embedding of $G - H_1^2$ in the plane. This restriction have to exist, because each of these vertices have to be at distance at most two from each vertex of H_1, in G. Thus, with similar arguments as in the proof of Lemma 3.6 the distance two constrains satisfied by the existence of a vertex which is joined to all the vertices which have to be at distance at most two. We note here, that the degree of this vertex has to be at most Δ. Hence, it is created a star graph, with some vertices from H_0 and some from H_1. In order to receive the vertices of this star graph colours with value bigger than $\omega(G^2)$ from $MDsatur$, then their neighbours have to have already the colours $\{1, 2, ..., \omega(G^2)\}$. For all these reasons the resulted graph in this case is similar to the worst case graph obtained in Part A. Observe also, that if there are vertices of H_1 which are not vertices of the star graph that is discredited above, then the colouring of these vertices is not influenced by the colours of the vertices of the star. This is proved with similar arguments as in the Part B. ■

C.2 Proof of Lemma C.1

Proof: Let EM be a planar embedding of S. If $x \leq \Delta+3$, then we have to deal with two cases, arising based on the number of vertices which lie on the exterior face of S in EM. Let us denote the number of these vertices by s.

Case 1, $s \geq 7$: In this case the minimal planar square root, S, of the clique K_x is the star graph $K_{1,x-1}$, by Lemma 3.4. According to the statement of this Lemma, we have to properly extent S to have the cliques $C1, C2$ in S^2. Suppose that the vertices of S, except its root, are joined in a path, let us call it P. Let V_5 be a set on five vertices of S. Let $V_5 = \{u, v, w, x, y\}$, these vertices are shown in turn according to their location in EM. The vertices of V_5 belong to a new clique of size at least eight in S^2, i.e. $C1$, iff $|C1|-5$ new vertices are inserted in S and each of these vertices is joined to at least two vertices of V_5 or one of the new vertices is joined to each other vertex of this clique. The second case is obvious. The first case is also easy and it is proved as follows: Since none of the five vertices of V_5 is joined to each of the other vertices of this set, it is impossible any of the new vertices to be joined to only one vertex of V_5, since these vertices have to belong to the clique $C1$ in S^2. Thus, each of the new vertices has to be joined to at least two vertices of V_5, such that their neighbours to be all the vertices of V_5. All the possible pairs of such vertices (let us call these vertices endpoints) are the $\{v, x\}$, $\{v, y\}$, $\{u, x\}$. Observe that there is at least one vertex, from set V_5, between the endpoints of these pairs. Let $e1, e2$ denote the endpoints of the pair of vertices of V_5, that are joined to the first new vertex of the clique $C1$, we denote it by v_1. Then, each other new vertex of $C1$ has to be joined to at least one of these endpoints as it is explained in the following. We denote by v_x

the first new vertex of $C1$, which is joined to a pair of vertices from V_5 different from $(e1, e2)$. This pair has one common endpoint with the $(e1, e2)$, because as it is easily checked, there is a vertex which sits inside the face created from the edges that join v_1 with the endpoints $e1, e2$. So, in order to be this vertex at distance at most two from vertex v_x, one of the edges $\{v_x, e1\}$ or $\{v_x, e2\}$ has to belong to S. W.l.o.g. we assume that this endpoint is the $e1$.

The last point we have to mention for this case is that one of the endpoints, which are joined to the vertices v_1, v_x, sits inside the bigger face created by the join of each of these vertices to their endpoints. Observe that this endpoint is not the common endpoint of vertices $v1, vx$. Hence, each of the other new vertices of $C1$ has to be joined to the common endpoint of the above vertices, either it is located inside or outside the above face.

The vertices of V_5 cannot belong in another clique, i.e. $C2$ with at least three distinct vertices from the vertices of the clique $C1$ and with size at least eight, in an augmented S^2. This is true, because using the above argument all the new vertices of the clique $C2$ have also to be joined to the endpoint $e1$ (as the vertices of $C1$). Otherwise, there are vertices of V_5, which have distance more than two from some vertices of $C2$, in S.

Thus, this Lemma follows straightforward in this case by the above arguments. The case where the vertices of V_5 are not joined in a path follows with similar arguments as in the proof of Lemma 3.4.

Case 2, $s \leq 6$: In this case there is a cycle on at most six vertices, on the exterior face of S (in EM). We call this cycle C. If the five vertices of CL, which we want to belong to more than two cliques simultaneously, are vertices of the cycle C, then this case is similar to Case 1. Otherwise, if these five vertices sit inside C, then by Lemma 3.5 these vertices either share two neighbours in C (let us denote them by $c1, c2$) or there is a vertex of C which is joined to each other vertex of CL. We begin considering the first subcase. Observe that the degree of each of the vertices $c1, c2$ is at least $x - 3$, before the insertion of new vertices in the initial graph S (that is just a square root of CL). We recall that the maximum degree of S must be at most $x - 1$. Thus, it is allowable to be inserted at most two distinct vertices to each of the vertices $c1, c2$. Hence, the vertices which sit inside C can belong to three cliques of size more than eight each, but the cliques in each pair of them have less than three different vertices each. So, this Lemma also follows in this case.

The second subcase follows straightforward from Case 1, if the new vertices of cliques $C1, C2$ lie inside the cycle C. This is because the vertices of CL are joined in a star graph inside the cycle C. Otherwise, if the vertices of $C1$ ($C2$) lie outside (inside) C, then this case is similar to the first subcase of Case 2.

If $x > \Delta + 3$ then this Lemma obviously follows, because of the structure of S, which is unique in this case as it is described in the proof of Lemma 3.6 and it is shown in Fig. 2. In this case observe that no more that four vertices are in the same face in EM. ■

Corollary C.8 *Let G be any planar graph with $\Delta(G) \geq 8$. It is not possible any five vertices of this graph to belong to more than two subgraphs of G^2 with chromatic number $\chi(G^2)$ and at least three distinct vertices each.*

Corollary C.9 *Let G be any planar graph. If there are at least five vertices of G, which belong in more than two cliques in G^2, with more than eight vertices each, then the influence of this fact on the number of colours used by $MDsatur$ to colour G^2 is to increase this number by seven.*

D Proof of Lemma 3.1

Proof: We prove the assertion by contradiction. We assume that S has at least two not empty not adjacent connected components, let us call them $S1$ and $S2$. We also assume w.l.o.g. that $MDsatur$ firstly colours vertices (not necessarily all of them) from component $S1$. Let v be the first vertex of the component $S2$ that is selected from $MDsatur$. Let us examine what happens when v is the next vertex that will be coloured by our algorithm. Vertex v does not have any coloured neighbour, because components $S1$ and $S2$ are not adjacent, by our assumption. Thus, $ds(v) = 0$. Observe that neither any other uncoloured vertex of G has degree of saturation with value greater than zero. This is true, because in such a case one of the vertices with degree of saturation bigger than zero would be the next vertex that would be coloured, instead of vertex v. The value of zero on the degree of saturation of vertex v implies that the component $S1$ has already been coloured, otherwise there would be at least one uncoloured vertex of $S1$ which would have non-zero degree of saturation. Also does not exist any uncoloured vertex that has neighbours in $S1$. This yields a contradiction, because G is a connected graph. Thus, S is a connected graph. ■

E Definitions

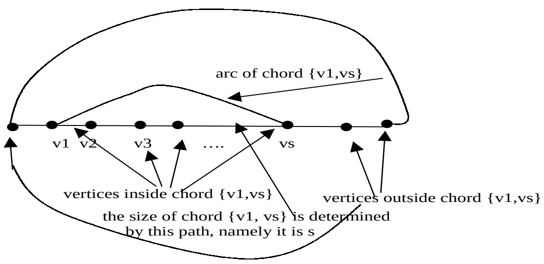


Figure 7: With the graph of this figure we explain more clear the following definitions.

Definition E.1 *A chord is an edge that joins two non-consecutive (non-adjacent) vertices in a path or in a cycle. These vertices are called endpoints of this chord.*

If there is a cycle in G, then we assume that its vertices lie on a horizontal path in its embedding in the plane. We do this assumption in order to make clear which vertices lie inside and outside of a chord.

Definition E.2 *We define as arc of a chord the drawing of the edge that joins the endpoints of this chord in a planar embedding of a graph G. The size of a chord is the length of the path which lie inside of its arc and joining its endpoints.*

Definition E.3 *The vertices which lie on the path, which defines the size of a chord, are called inside vertices of the arc of this chord, or for simplicity reasons as inside vertices. Similarly we define as outside vertices of the arc of a chord the vertices which do not lie on the above path.*

F Proof of Lemma 3.3

Proof: From the statement of this Lemma we know, that there is a cycle in G on the six vertices of the clique K_6, which lies on the exterior face in a planar embedding, EM, of G. We denote this cycle by C. We prove the assertion of this Lemma, by initially assigning the cycle C to be the graph G. Then we augment G to become a minimal planar square root of the clique K_6. Observe that the current graph G^2, which is the C_6^2, isn't the clique on six vertices. Thus, we have to insert appropriate chords in the inside face of cycle C, in order to become a planar square root of K_6. We recall that cycle C has to lie on the exterior face of G in EM. Hence, chords in the exterior face of C are not allowed. We denote by ch the first chord that is inserted in C. In the proof of this Lemma we deal with all the possible cases arising from each potential choice of the chord ch. The rest of this proof is split in three complementary primary cases based on the size of ch. We denote the size of ch by s. It can be easily seen, that in a cycle of size six the size of its maximum chord is four. Let us call the vertices of G in cycle turn as $1, 2, 3, 4, 5, 6$.

Let ch' be a chord of a cycle. Assume that it is not allowed to be inserted chords between the vertices sitting inside the arc of ch' and the vertices sitting outside the arc of this chord. Then,

Proposition F.1 *The vertices which sit inside ch' manages to become distance two neighbours of each vertex sitting outside this chord only via the endpoints of the chord ch'.*

Let that the conditions of Proposition F.1 hold, then the following Corollary follows straightforward.

Corollary F.2 *Each vertex sitting outside chord ch' has to be joined to at least one of the endpoints of the chord ch'. Also each vertex inside this chord has to be joined appropriate to its endpoints.*

Case 1, $s = 2$: The chord ch joins two vertices having distance two in cycle C. Observe that there is only one vertex inside the arc of the chord ch. We denote this vertex by $v1$. Then, $v1$ is at distance at most two from each vertex outside the chord ch iff each of these vertices is joined to at least one of ch's endpoints (by Corollary F.2). Observe that in the current graph G only one vertex is at distance three from the vertex inside the chord ch. This is true, because each of ch's endpoints is already joined to a distinct vertex outside this chord. W.l.o.g. we assume that the chord ch is the $\{1, 3\}$. Then, $v1 = 2$. So, the vertex which is at distance three from the vertex 2 is the vertex 5. Thus, the vertex 5 has to be joined to at least one of the vertices 1 and 3, which are the endpoints of ch. When the vertex 5 is joined to only one of the endpoints of ch, then these two cases are depicted in Fig. 1a and 1b. Seeing abstractedly these graphs, in the first case we can see two consecutive chords in C (namely that the end of one of them is the beginning of the other) joining vertices having distance two in this cycle, each. In the second case, we can see two chords which share one endpoint and one of them has size four and the other has size two, in C.

If the chord ch is not the $\{1, 3\}$, then with similar arguments we conclude to graphs with two chords in the cycle C, having the properties of the chords inserted in C in the case where $ch = \{1, 3\}$.

In the case where the vertex 5 is joined to both endpoints of ch, then the star graph $K_{1,5}$ has to be a subgraph of G.

Case 2, $s = 3$: The chord ch joins two vertices having distance three in C. For example the vertices 1 and 4. In this case each of the two vertices which sit inside chord ch is at distance three from a distinct vertex which sits outside this chord. In our example, the vertex 2 is at distance three from the vertex 5 only and the vertex 3 from the vertex 6 only. Observe that: if the vertex 3 becomes a distance two neighbour of the vertex 6 via the vertex 4 (namely the edge $\{4,6\}$ is inserted in G), then the vertex 2 must become distance two neighbour of the vertex 5 also via the vertex 4. This is true, because the insertion of the chord $\{1,5\}$ would violate the conditions in the statement of this Lemma, so it is not permissible. If the vertex 3 becomes a distance two neighbour of 6 via the vertex 1, then the vertex 2 must become distance two neighbour of the vertex 5 also via the vertex 1, similarly. Here the insertion of the chord $\{2,4\}$ is not permissible. Consequently, the chords $\{2,4\}$ and $\{4,6\}$ (see Fig. 1c.) or $\{1,5\}$ and $\{1,3\}$ (see Fig. 1d.) must be inserted in the cycle C. Observe that, in each of the above cases the graph $K_{1,5}$ is a subgraph of G. It is not difficult to be verified, that this is the case whichever is the chord ch (with $s = 3$).

Case 3, $s = 4$: The chord ch joins two vertices which are at distance four in the cycle C. In this case there are only two possible chords that can be assigned to ch. Whichever is the chord ch in this case, the vertex which sits in the middle of the cycle C_5 that is resulted, must be joined to at least one of ch's endpoints. This leads to Case 1. ■

Corollary F.3 *Let G to be a minimal pseudo square root of clique K_6, such that the six vertices of this clique lie on the exterior face in a planar embedding of G. If two of these vertices become distance two neighbours via an auxiliary vertex, then the graph $K_{1,5}$ is a subgraph of graph G.*

The proof of this Corollary easily follows from the arguments used in the proof of the above Lemma.

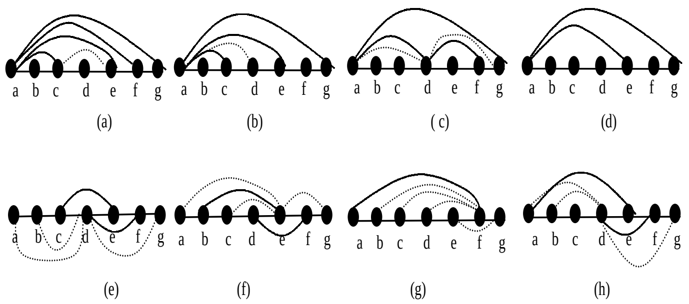


Figure 8: These graphs are used in the proof of Lemma 3.4

G Proof of Lemma 3.4

Proof: We prove the assertion inductively. The basic case is when $|V| = 7$ (let $V = \{1,2,3,4,5,6,7\}$). In this case the vertices of the clique CL are exactly the vertices of the graph G. In other words, in the basis we are looking for the planar square roots of K_7 given that all of its vertices lie on only one face in EM. W.l.o.g. we assume that this face is the exterior face.

Initially, we assume that G is a graph without edges. Then, we properly augment it to become a minimal planar square root of the clique K_x, satisfying the conditions in the statement of this Lemma. For the basis it is obvious that all the vertices of G lie on the exterior face in EM. It is easily seen, that the vertices of this graph remain in one face even when seven of them are joined in a path. The existence or not of such a path in G naturally leads to the split of the proof of this Lemma in two complementary parts. In the first one (Part A) we assume that there is path on seven vertices (i.e. P_7) in G. We denote this path by P. In the second part (Part B) there does not exist such a path in G. The proof of the first part is based on a case analysis. These cases are complementary and arisen from each potential size of the first chord that is inserted in the path P. We denote this chord by ch and its size by s. For simplicity reasons we assume w.l.o.g. that the vertices of P lie on a horizontal line in EM. Hence, the arc of any chord of P lies above this path or below of it in EM. We denote by $\{w, x\}$ ($\{y, z\}$) ordered from left to right the endpoints of the first chord which lies above (below) the path P.

Part A: We begin with the subcase where all the chords of P lie in the same side of this path in the embedding EM. W.l.o.g. we assume that this side is the side above the path. As in the proof of Lemma 3.3 we deal with each potential instance of chord ch. To start with we determine the size of the chord ch and then we explore all the cases obtained from all the possible chords of size s.

We begin with the case where $s = 3$ and we postpone the case where $s = 2$ until the end of this part, because at that point its proof will be more understandable.

Case A1, $s = 3$: Let us denote the vertices which sit inside the chord ch sequentially from left to right by $v1, v2$. Since there are seven vertices in the path P, three of them are outside the chord ch. For contradiction we assume that both ch's endpoints are joined to each of the vertices outside this chord. If there is a vertex in the left (right) of vertex w (x), denoted by $v3$ ($v4$), then this means that the edges $\{v3, w\}$, $\{x, v4\}$ are in G. By the assumption the edges $\{v4, w\}$ and $\{v3, x\}$ are in G also. Observe that one of them i.e. $\{v4, w\}$ or $\{v3, x\}$ must lie below the path P in EM, because has not to exist edge crossing in EM. This is a contradiction, because in Part A we want all the chords of P to lie above this path in EM. Thus, there is at least one vertex outside the chord ch which is joined to only one of its endpoints. W.l.o.g. we assume that the vertex $v3$ is joined only to the endpoint w. This fact obviously forces both vertices which sit inside ch to be joined to the endpoint w. Then, it is easily verified that the vertex $v1$ cannot be joined to the endpoint x, because in Part A does not allowed chords below P in EM. Thus, the vertex w obviously becomes the root of the star graph $K_{1,6}$. This follows because all the vertices outside chord ch have to be joined to w, in order to be at distance at most two from the vertex $v1$ which is joined to this endpoint only. Observe that with similar arguments the cases where $w = 1$ and $z = 7$ are proved, namely whichever is the chord ch when $s = 3$.

Case A2, $s = 4$: This case has two subcases, because $w = 1$ or $w = 2$. The first one (**case A2.1**) comes if $w = 1$ (or conversely when $x = 7$) and is similar to Case A1, in regard to the fact that the two vertices which lie to the right (left) of the endpoint x (w) cannot be joined simultaneously to both of the endpoints w and x. Thus, this case follows from the arguments of the above case. We recall that do not allowed chords below P in EM.

Case A2.2: The second subcase comes when $w = 2$. This condition implies, that there is one distinct vertex which is adjacent only to one of the ch's endpoints. Let us denote the vertices inside the chord ch in turn by $v1, v2, v3$. We proceed with the study of two subcase of this case.

Case A2.2.1: If the vertex in the middle of the vertices which are sited inside ch, i.e. the vertex $v2$, is adjacent to only one of the endpoints of this chord (w.l.o.g. we assume that this is w), then the vertex w must be joined to the vertex which sits outside ch and is adjacent to the endpoint x. This is the vertex 7. Namely, the chord $\{w, 7\}$ is inserted in P. Observe that the chord $\{x, 1\}$ is not allowed in the graph G, because either it causes an edge crossing in EM, or it lies below the path P. Thus, the vertex w has to become again the root of the needed star graph. This is because the vertices sited inside ch have to be joined to the endpoint w, otherwise their distance from the distinct neighbour of the endpoint w (this is the vertex 1) will be greater than two.

Case A2.2.2: We assume for contradiction that the vertex $v1$ (similar for the vertex $v3$) can be adjacent to both ch's endpoints. In this case the vertex $v3$ cannot be joined to endpoint w. This is because G is planar graph and all the chords have to be above the path P in EM, in Part A. Thus, for the moment the vertex $v3$ is at distance three from the neighbour of the endpoint w which sits outside chord ch (this is the vertex 1). Hence, chord $\{1, x\}$ have to be inserted in G. Until now, vertices $v1$ and $v3$ have became distance two neighbours of each of the vertices outside the chord ch. However, the chords inserted exclude the insertion of the necessary chords, which would decrease the distance between the vertex $v2$ and the vertices which sit outside ch. Hence, we conclude that the vertex $v1$ cannot be joined to both ch's endpoints. So, if $v1$ is only joined to vertex w (or vertex x), then w (x) have to become the root of $K_{1,6}$, for the same reasons as in the Case A1.

Case A3, $s = 5$: For proving this case we make use of the result of case A2.2.2. In this case it is obvious that $w = 1$ or $w = 2$ (so, $x = 7$) and that these two subcases are actually converse between them. Hence, we study only one of them, w.l.o.g. we assume that $w = 1$. We remark here that the chord ch creates a cycle C_6 in G. If the graph G satisfies the conditions in the statement of this Lemma, then the vertices of this cycle belong in a clique of size six in G^2, iff there are at least two consecutive chords of size two in this cycle or a chord of size two with a chord of size four which share a common endpoint. This fact follows from Lemma 3.3 directly. If there are two consecutive chords of size two inside the chord ch, then it can be easily verified that this case leads to a fatal situation similar to that which is obtained in Case A.2.2.2. On the other hand, if there is a chord of size two with a chord of size four inside the chord ch, then the assertion for this case is proved with analogous arguments as that which are used in Case A2.

Thus it is conclude that there is a star graph $K_{1,6}$ as subgraph of G. The same holds even if there is a vertex from C_6 joined to each other vertex.

Case A4, $s = 6$: In this case chord ch is unique and is the only case where all the vertices sit inside this chord or are its endpoints. Let ch' be the second chord inserted in the path P. It is easily checked, that the size of ch' is at most five. Thus, this case immediately follows from the Cases A1-A3 assuming that their's $ch = ch'$.

Case A5, $s = 2$: In this case we have to deal with the case where each chord of P has size two only. The other cases where there is a chord of size two together with a chord of bigger size are examined in Cases A1-A4. In the case that we are interested for observe, that at most three consecutive chords of size two are possible to exist. This case leads to a fatal situation similar to that in case A2.2.2. Observe that this is the case even if there are two such chords on P. Hence, we conclude that it is impossible $G^2 = K_7$ when there are chords of size two only on the path P.

We close Part A with the cases arising when there are at least two chords on the path P, such that in any planar embedding, EM, of G one of these chords lies above the path and one of them below. Let us denote by ch the first chord inserted and w.l.o.g. we assume that this lies above P and by ch' the second chord inserted which lies below this path in EM. We denote the sizes of these chords by s and s', respectively. Before proving the cases arising under these conditions we present a few more auxiliary propositions.

Proposition G.1 *Let ch be a chord which lies above (below) the path P in EM. The vertices which sit inside the chord ch can become distance two neighbours of the vertices which sit outside this chord via its endpoints, or via chords which lie below (above) P and join some of them with vertices outside this chord.*

Proposition G.2 *If one of the endpoints of chord ch' sits inside chord ch, then it must be a vertex that is adjacent to one of the ch's endpoint. Otherwise, less than seven vertices lie on the exterior face of G in EM.*

Observation G.1 *The only case where it is allowable the endpoint of the chord which sits below the path P to be joined to both of the endpoints of the chord ch, obviously is when the $s = 2$.*

Proposition G.3 *The only case where chords which lie below path P in EM are allowed to join vertices which sit to the left and to the right of the endpoints of ch is when the size of the chord ch is two.*

Case A6, s = 2, s' = 2: We begin with the case where each of the two chords ch and ch' joins two vertices having distance two among them in the path P. W.l.o.g. we assume that these chords are the $\{3, 5\}$ and $\{4, 6\}$, respectively. Observe that there are vertices to the left of vertex 3 and vertices to the right of vertex 6 which have to become at most distance two neighbours in G. This requirement cannot be satisfied via vertex 3 (or any other vertex which sits to the left of it in EM). This holds because as it is easily verified, less than seven vertices would be in the same face in EM if vertex 3 is joined to any vertex to the right of vertex 5. Hence, in order to remain the seven vertices of P in one face, the only chords allowed are these which make the vertex 4 or 5 the root of the star graph $K_{1,6}$. Observe that, this is case even when $w = 1$, because vertex 1 is not allowed to be joined to vertices in the right of endpoint x. Hence, the neighbours of the vertices 2, 3 have to be all these vertices. Observe that only one of them can be joined to both of them.

Case A7, s = 3, s' = 2: The graph of Fig.8f shows an instance of this case, when the vertex $w \neq 1$. Let $w = 2$, observe that vertex y ($y = 4$) is not allowed to become a neighbour of vertex 1. Also, vertex 1 is not allowed to have neighbours to the right of the endpoint x and none vertex inside chord ch can either, except of endpoint y. So, each vertex in the left of endpoint y becomes distance two neighbour of each vertex outside chord ch only via vertex $x = 5$. This fact holds because as explained in Case A2 it is not allowable all of them to be joined to both of the ch's endpoints, by inserting chords above the path P only. Observe that the endpoint y cannot be the root. Thus, the graph $K_{1,6}$ should be a subgraph of this square root of clique K_7 and its root is the vertex x. On the other hand, if $w = 1$ (see Fig. 8h), then the root of the star graph can be either the endpoint x or y. It is easily verified in this case, that when we try to do some vertices which sit inside chord ch distance two neighbours of the vertices sited outside this chord via endpoint y (x) and the rest of them via endpoint x (y), then this case trial leads to a fatal situation. This is true because it is impossible to hold the conditions of the statement of this Lemma and each vertex inside chord ch to be joined to both endpoints x and y.

Case A8, s = 4, s' = 2: In this case there are only two vertices outside the chord ch (see Fig.8g). For the same reasons as in the above case the graph $K_{1,6}$ should be a subgraph of graph G with root either the endpoint y or x.

The cases obtained when $s' > 2$ are easily proved with similar arguments.

Part B: Here we have to explore what happens when the seven vertices of G are not joined in a path on seven vertices. We proceed considering two cases as in the proof of Part A. Suppose that the seven vertices of G lie on the same face in EM and we can contact them in turn in the following order $\{1, 2, 3, 4, 5, 6, 7\}$. W.l.o.g. we can assume that these vertices lie on a virtual line. Our aim is to join properly these vertices to become the graph $G^2 = K_7$. Then, the cases arising when the edges inserted, which join these vertices, lie in the same side of the virtual line in EM, it is easily understood that is very similar to the cases where there is the path P and all of its chords lie above P.

On the other hand, if the edges which join the vertices of G lie both above and below of the virtual line in EM, then the cases arising are little more tedious, but is very similar, than the corresponding cases when there is the path P in G.

The generalization for $|CL| > 7$ is straightforward. ■

Corollary G.4 *Let $SQRT\_Kx$ be a minimal planar square root or a minimal pseudo square root of clique K_x, for $x \geq 7$, such that at least seven vertices of this clique lie on the same face, in an embedding of this root in the plane. Let EM be any such embedding of $SQRT\_Kx$. Then the graph $K_{(1,(|x|-1))}$ is a subgraph of graph $SQRT\_Kx$.*

The proof of this Corollary follows from the arguments used in the proof of the above Lemma.

H Proof of Lemma 3.5

We postpone the proof of Lemma 3.5 until the presentation and the proof of usefull result.

Lemma H.1 *Let $G = (V, E)$ be the planar graph $K_{1,d}$ for $d \geq 7$. Let u, v, w, x be any four vertices of the graph G. Let $y1, y2$ be two new vertices (except of the vertices of G). If each of the vertices u, v, w, x is joined to each of the vertices $y1, y2$, then G becomes a non-planar graph.*

Proof: Observe that at least three of the vertices u, v, w, x are joined with the root of the initial star graph. Note that the root is may be one of the four vertices. Thus, the assertion follows straightforward, because the graph $K_{3,3}$ cannot be a subgraph of any planar graph. ■

Proof: of Lemma 3.5

By the statement of this Lemma, we know that only six of the vertices of the clique K_x lie on the exterior face of G in EM. Thus, there is a cycle of size six with these vertices of the clique K_x on the exterior face of G in EM. Let us denote this cycle by C. The remaining vertices of this clique lie inside the cycle C in EM. In the following we consider all the cases obtained from each potential structure of the minimal planar square roots of K_6 when the cycle C_6 is a subgraph on the exterior face of these graphs. We shown how it is possible to be augmented each of these graphs with the rest $x - 6$ vertices of K_x, so that the resulting graphs to be the minimal square roots of K_x.

Let us call the vertices of C in cyclic turn as $\{1, 2, 3, 4, 5, 6\}$. By Lemma 3.3 we know that the minimal planar square roots of K_6, when the cycle C lies on the exterior face in an embedding of each of these graphs in the plane, are the graphs obtained inserting either (a) two consecutive chords of size two each in C, or (b) inserting two chords with one common endpoint such that one of them to have size two and the other to have size four, or (c) one vertex of C to be joined to each of the other. W.l.o.g. we assume that the chords in the first case are the $\{1, 3\}$ and $\{3, 5\}$. The chords in the second case we assume that are the $\{1, 3\}$ and $\{1, 5\}$ and finally for the last case let that the vertex 4 is the root of the star graph $K_{1,5}$.

At this point we will explain how the above graphs are augmented in order to become square roots of the clique K_x, while the conditions of this Lemma hold. The first case (a) is shown in Fig. 1d. Observe that the vertices 2 and 4, which sit inside the arcs of the chords of C, must become distance two neighbours of the new vertices via the endpoints of these chords. These endpoints have to be the vertices 1 and 5 or 3 and 5, because the chord $\{3, 6\}$ is not in C. Otherwise, the vertex 3, which is a common endpoint of the two chords of C_6, has to be joined to each of the new vertices. If this is the case, then either one of the vertex 1, 5 or vertex 6 has also to be joined to each of the new vertices, in order the vertex 6 to be at distance at most two of each of the new vertices.

The second case is shown in Fig. 1e. Thereat each of the new vertices obviously has to be joined to the endpoints 1 and 5. The last case is depicted in Fig. 1f and it is very clear. As it is easily seen, in the graphs which correspond to the first two cases the $x - 6$ new vertices have to be joined to two vertices of cycle C. In the last case, where one of the vertices of C is joined to each of the other vertices, it is obvious that each of the new vertices have to be joined with it. ∎

Corollary H.2 *Let G be a minimal planar square root of clique K_x, for $x \geq 8$, such that only $s \leq 5$ of the vertices of K_x lie on the exterior face in an embedding, EM, of G in the plane and are joined in a cycle C_s. Then, the rest $x - s$ of the vertices of K_x lie inside this cycle and have to be joined to at least two vertices of C, (similar to the case where $s = 6$ see Fig.1 e, f, g) or the star graph $K_{1,x-1}$ is a subgraph of G.*

Observe that $C_s^2 = K_s$ for each $s \leq 5$. The assertion of the above Corollary is easily proved either there are chords in cycle C or not.

I Proof of Lemma 3.6

Proof: We begin this proof with some definitions and notations. Let EM be an embedding of the graph G in the plane. We denote by MCL the maximum clique in G^2 and by x its size. S denotes the subgraph of G on $\Delta + 1$ of the vertices of MCL. We denote by y how bigger is the size of this clique from its lower bound, which is $\Delta + 1$. Namely, $y = x - \Delta - 1$. Finally, we denote the set of vertices of G which belong in MCL and are not vertices of S by Y. We proceed considering the cases obtained from each potential structure of the graph S. The proof of the assertion is mainly based on the fact that the graph $K_{3,3}$ isn't a subgraph of any planar graph.

We begin with the case where S is the star graph $K_{1,\Delta}$ with root the vertex v. Let $u \in Y$. Then the distance among the vertex u and each vertex $v' \in N_S(v)$ is at most two iff $(u, v') \in E$ or both of these vertices have a common neighbour i.e. the vertex w. If holds the first case, where $(u, v') \in E$ for each vertex in $N_S(v)$ the it is obvious that no other vertex of Y could be joined to these vertices. This is true, because the graph $K_{3,3}$ is not a subgraph of any planar graph. Observe that the same fatal situation is hold even if consecutive neighbours of vertex v are joined together to an other vertex and the vertex u to these vertices. If these vertices are more than two. So, we study in detail only the case where the neighbours of vertex v are split in two sets.

A slightly different case comes when the neighbours of v are joined in a path or a cycle. However, this case have the same problems as the above case, where S is the star graph.

We call $V1$ and $V2$ the two sets where the neighbours of vertex v are split. Hence, each vertex of set $V1$ is joined to a distinct vertex from set Y, let it be u. Each vertex from set $V2$ is joined to another one vertex from the set Y, i.e. w. Let S' be the current graph obtained from graph S. The graph S' becomes a clique of size $\Delta + y + 1$, if each other vertex from Y is joined to both of the vertices u and w. We recall here, that the degree of each of the vertices u and w can be at most Δ. Furthermore, $|V1| + |V2| = \Delta$. Thus $y \leq \Delta - max\{|V1|, |V2|\}$, since each vertex of Y is joined to both of the vertices u and w. Obviously, the minimum value of $max\{|V1|, |V2|\}$ is $\Delta/2$. Thus, $y \leq \Delta/2$. Until now we achieved to produce a graph S, such that S^2 is a clique of size 1.5Δ. An instance of the graph S is shown in Figure 2 a.

There is another crucial issue that has to be considered to close this part of the proof of this Lemma. This is that any new vertex which wants to be inserted in MCL, it must be joined to at least two of the vertices u, v, w. So, the degree of the two vertices which are selected to be neighbours of the new vertex is increased by one. It is clear, that the size of the maximum clique is increased at most by 3, when the degree of S' is increased at most by 2. This is achieved when the new vertices uniformly increase the degree of the vertices u, v, w.

In Fig. 2b there is a vertex, which is denoted by z, that joins the vertices v and u and the edge $\{v, u\}$ is omitted. It is easily checked that this is the only case that the size of MCL is increased without an increase of the maximum degree of the graph G.

We proceed exploring what happens if S isn't the star graph. This yields that at most six vertices of clique $K_{\Delta+1}$ lie on a cycle, C, in the exterior face of S, in EM. This is known by Lemmata 3.4 and 3.5. Let S be any of the graphs depicted in Fig. 1. If the vertices of Y lie on the exterior face of S they have to be joined to the vertices of this cycle which are joined to the vertices inside C. This is the only way to achieve these vertices to be at distance at most two from each vertex inside the cycle C. Observe that, the degree of these vertices of the cycle C is at least $\Delta - 2$ before the insertion of any of the vertices of Y in S. This implies that, $y \leq 2$, because the value of max degree has to remain the same. So, the only case having maximum clique in G^2 of size greater that $\Delta + 4$ is the case presented in the beginning of this proof, when the graph S is a star square root of clique $K_{\Delta+11}$. In the case where S is a pseudo square root of this clique, then there is a star graph as subgraph of these roots by the above Corollaries. ■

J Proof of Lemma 4.2

The proof of the assertion comes from the well known fact, that the maximum number of edges, denoted by m, of a planar graph is $m = 3n - 6$, where n is the number of its vertices [4] (Euler formula). It is easily verified, that the number of edges in a x-regular graph are $\frac{nx}{2}$. Hence, $m = \frac{nx}{2} \leq 3n - 6$. This inequality holds only for $x \leq 5$.