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Radiocoloring Graphs via the Probabilistic Method *

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Abstract

We employ here the Probabilistic Method, a way of reasoning which shows existence of combinatorial structures and properties to prove refute conjectures.

The radiocoloring problem (RCP) is the problem of assigning frequencies to the transmitters of a network so that transmitters of distance one get frequencies that differ by at least two and any two transmitters of distance one get frequencies that differ by at least one. The objective of an assignment may be to minimize the number of frequencies used (order) or the range of them (span). Here, we study the optimization version of RCP where the objective is to minimize the order. In graph theory terms the problem is modelled by a variation of the vertex graph coloring problem. We investigate upper bounds for the minimum number of colors needed in a radiocoloring assignment of a graph G.

We first provide an upper bound for the minimum number of colors needed to radiocolor a graph G of girth at most 7.

Then, we study whether the minimum order of a radiocoloring assignment is determined by local conditions, i.e. by the minimum order radiocoloring assignment of some small subgraphs of it. We state a related conjecture which is analogous to a theorem of Molloy and Reed for the vertex coloring problem [12]. We then investigate whether the conjecture contradicts a Theorem of Molloy and Reed for the vertex coloring when applied on the graph G^2 obtained by G. The graph G^2 has the same vertex set as G and there is an edge in G^2 between any two vertices iff their distance in G is at most 2. We prove that there is no case where the conjecture stated here contradicts the Theorem of Molloy and Reed.

Areas: Graph Theory, Probabilistic Method, Coloring, Radiocoloring, Upper Bounds

1 Introduction

In modern networks, Frequency Assignment Problems (FAP) have important applications in efficient bandwidth utilization, by trying to minimize the number (or the range) of frequencies used, in a way that however keeps the interference of nearby transmitters at an acceptable level. Problems of assigning frequencies in networks are usually abstracted by variations of coloring graphs. An important version of Frequency Assignment Problems is the Radiocoloring Problem (RCP). The optimization version of RCP studied here, consists

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in assigning colors (frequencies) to the vertices (transmitters) of a graph (network), so that any two vertices of distance at most one get colors that differ by at least one and every two vertices of distance two get integers that differ by at least one. The objective here is to minimize the number of distinct colors used (order), and the corresponding problem is called *min order RCP*.

1.1 Relevant work

Both optimization versions of RCP (miniming the span or the order) have been proved to be $\mathcal{NP}$-complete even for the class of planar graphs [9, 14, 6]. In the last years a number of researchers studied the approximability of variations of RCP. While the problem for general graphs is inapproximable within a factor of $O(n^{1-\epsilon})$ [4, 10], when restricting to some classes of graphs, it becomes easier. For the case of planar graphs [9, 4] the authors provide an efficient approximation algorithm for the min order RCP of *planar* graphs achieving an approximation ratio tending to 2 and 1.8 (for graphs of large degree ($\Delta(G) \geq 749$)), respectively. This result was recently improved by a work of [13] providing an 1.66-approximation algorithm for the radiocoloring problem on planar graphs. The work of [6] presents efficient approximations for some other interesting families of graphs including outerplanar, bounded treewidth, permutation and split graphs. For radiocoloring other important graph topologies (hierarchical, periodic), see [1, 11]. Also, the problem of coloring the square of any planar graph (which is equivalent to min order RCP) is investigated in [2, 3].

However, all these works studied the problem using tools from graph theory. Another interesting and yet new research line to study graph coloring problems is via probabilistic tools. A very recent book of Molloy and Reed presents in a very comprehensive way this research line, studing lower and upper bounds on many fundamental graph theory problems such as the ordinary vertex coloring, edge coloring and list coloring.

Here, we continue this line of research; we study the radiocoloring problem as far as its approximability is concerned as well as the critical parameters that determine a radiocoloring assignment.

1.2 On the use of the Probabilistic Method

The Probabilistic method is a case of reasoning, which eventually can give existence proofs of combinatorial structures and properties, via a very indirect way. This way essentially consists of imposing a probability space onto the structure of examination and then using the power of probability. In its most basic technique we wish to show the existence of a combinatorial structure satisfying a certain property. To do this, we create (using an abstract random experiment) an appropriate probability space whose sample points correspond to the combinatorial structures under examination. We then try to show that, in this space, the probability of the desired property is positive (non-zero). This means (by the definition of probability) that at least one point in the space satisfies the desired property, i.e. that there exists at least one combinatorial structure that has the desired property. Thus, we provide a *non-constructive* proof of existence. The Probabilistic Method consists of several techniques, ranging from simple to more complex ones. We demonstrate the power of the method through its use in some very non-trivial examples, where both the Lovász Local Lemma and Talagrand's inequality come into play. For an excellent survey of the Probabilistic Method see [5], and also [15].

1.3 Summary of Results

In this paper we utilize the probabilistic method to provide two basic results for the radiocoloring problem:

- We prove that the minimum number of colors needed to radiocolor a graph with girth at least 7 and sufficiently large maximum degree $\Delta(G)$ is at most $(1 - \frac{2}{3e^6})\Delta^2$.
- We state a conjecture on when the minimum number of colors needed to radiocolor a graph G is determined by local conditions, i.e. the number of colors to radiocolor some small subgraphs of it. We investigate whether the conjecture contradicts a Theorem of Molloy and Reed for the vertex coloring when applied on the square graph G^2, obtained by G. The graph G^2 has the same vertex set as G and there is an edge between any two vertices of G^2 iff their distance in G is at most 2. We prove that there is no case where the conjecture stated here contradicts the Theorem of Molloy and Reed.

We believe these results constitute some fruitful ground for further research in order to improve the results obtained here as well as for developing upper and lower bounds for the radiocoloring problem, exploiting the powerful tools of the probabilistic method.

2 The radiocoloring problem

The Problem of Frequency Assignment in radio networks (FAP) is a well-studied, interesting problem. It is usually modelled by variations of graph coloring. The interference between transmitters is usually modelled by the interference graph $G(V, E)$, where the set V corresponds to the set of transmitters and E represents distance constraints. The set of colors represents the available frequencies. In addition, the color of each vertex in a particular assignment gets an integer value which has to satisfy certain inequalities compared to the values of colors of nearby nodes in the interference graph G (frequency-distance constraints). We here study a variation of FAP, called the Radiocoloring Problem (RCP), that models co-channel and adjacent interference constraints.

The Radiocoloring problem is defined as follows:

Definition 2.1 Radiocoloring Problem: *([8, 9]) Given a graph $G(V, E)$, find a radiocoloring assignment of G, i.e. a coloring function $\lambda : V \to N^*$ assigning integers (colors) to the vertices of the graph G such that $\lambda(u) \neq \lambda(v)$ if $d(u, v) = 2$ and $|\lambda(u) - \lambda(v)| \geq 2$ if $d(u, v) = 1$, where $d(u, v)$ is the distance between u and v in G.*

There are several variations of RCP, corresponding to different objectives of an assignment. Two important variations are to optimize the number of distinct frequencies used (order) or the range of frequencies used (span). The *order* of a radiocoloring assignment is $|\lambda(G)|$ and the of span of such an assignment is $max_{v \in V} \lambda(v) - min_{u \in V} \lambda(u) + 1$. In this work we study the optimization version of the Radiocoloring problem, where the objective is to minimize *the number* of frequencies used:

Definition 2.2 Min order RCP: *Given a graph $G(V, E)$, find an assignment of G, i.e. a coloring function $\lambda : V \rightarrow N^*$ assigning integers (colors) to the vertices of G such that $|\lambda(u) - \lambda(v)| \geq 2$ if $d(u, v) = 1$ and $\lambda(u) \neq \lambda(v)$ if $d(u, v) = 2$, where $d(u, v)$ is the distance between u and v in G, that uses a minimum number of colors. The number of different integers in such an assignment, is called the* minimum order *of RCP on G and is denoted by $\lambda_{order}(G)$, i.e. $\lambda_{order}(G) = |\lambda(V)|$.*

The min order RCP is related to a fundamental problem concerning the square of a graph G:

Definition 2.3 *Given a graph $G(V, E)$, the square of the graph G, G^2 is the graph having the same vertex set V and an edge set $E' : \{u, v\} \in E'$ iff $D(u, v) \leq 2$ in G.*

The related problem tries to color the square of a graph G, G^2, with the minimum number of colors, denoted as $\chi(G^2)$, so that neighbour vertices have different colors.

Observation 2.1 *The min order RCP of G, is equivalent to the problem of vertex coloring of G^2, i.e. $\lambda_{order}(G) = \chi(G^2)$.*

This is so because from any coloring of G^2 we can get a radiocoloring of G of the same order by doubling the color (integer) assigned to each vertex. On the other hand, every radiocoloring of G is a valid coloring of its square.

In the sequel, we consider the problem of finding $\chi(G^2)$, given a graph G. From such an assignment we can get a radiocoloring assignment of the same order as explained above.

3 Radiocoloring graphs with girth at least 7

Recall that a graph of *girth at least x* is a graph the smallest cycle of which has size x. Also, a triangle-free graph is a graph containing no triangles.

Theorem 3.1 *There is a Δ_0 such that if G is a regular graph of girth at least 7, with $\Delta(G) \geq \Delta_0$ then $\chi(G^2) \leq (1 - \frac{2}{3e^6})\Delta^2$.*

In [12], Molloy and Reed presented a similar Theorem considering the problem of vertex coloring. Their method inspired the proof of our theorem. Their theorem is the following:

Theorem 3.2 *There is a Δ_0 such that if G is a triangle-free graph, with $\Delta(G) \geq \Delta_0$ then $\chi(G) \leq (1 - \frac{2}{3e^6})\Delta$.*

Remark. The graph G^2 has maximum degree at most Δ^2. We take this degree if we take the sum of degrees of the distance one and two neighbours of a maximum degree vertex in G. This sum is at most $\Delta(\Delta-1)+\Delta=\Delta^2$.

Consider the square of a graph G of girth at least 7. One might think that we could apply the Theorem 3.2 of Molloy and Reed on the square of a graph of girth at least 7, obtaining the same result as in our theorem. However, this is not possible since the square of a graph of girth at least 7 is not a triangle-free graph. To see why note that for any distance two path in G, (u,v,w), in G^2 there is a triangle defined by u,v,w.

In the proof of Theorem 3.1, we will use the famous Lovász Local Lemma [5, 7], which we state below:

Lemma 3.3 The Lovász Local Lemma *[7]: Consider a set $\mathcal{E}$ of (typically bad) events such that for each $A\in\mathcal{E}$*
(a) $\mathbf{Pr}(\mathrm{A})\le \mathrm{p}\le 1$, and
(b) A is mutually independent of a set of all but at most d of other events.
If $4\cdot p\cdot d\le 1$ then with positive probability, none of the events in $\mathcal{E}$ occur.

3.0.1 A proof of Theorem 3.1

Proof: We set $C=\lfloor\frac{\Delta^2}{2}\rfloor$. We assign for each vertex w a color chosen uniformly at random from set $\{1,2,\ldots,C\}$. If w is assigned the same color as a distance one or two neighbour of it we uncolor it, otherwise we say *w retains its color.*

We prove that with positive (non-zero) probability, every vertex has at least $\frac{\Delta^2}{2e^6}+1$ colors which appear more than once in its distance one and two neighbourhood. Therefore, if $(1-\frac{1}{2e^6})\Delta^2$ colors are available, the vertex must avoid to take at most $\Delta^2-\frac{\Delta^2}{2e^6}-1$ colors, so there is one available color for it from the set of $(1-\frac{1}{2e^6})\Delta^2$ colors. So, we can color vertices that have lost their colors by a greedy algorithm.

For each vertex v of G, we are interested in the number of colors which are assigned to at least two distance one or two neighbours of v and are retained by at least two of these vertices.

In order to simplify our analysis, we consider the random variable:
$\mathbf{X_v}$ *which counts the number of colors which are assigned to at least two distance one or two neighbours of v and retained by* all *of these vertices.*

For each vertex v, we let:
$\mathbf{A}_v$ *be the (bad) event that X_v is less than $\frac{\Delta^2}{2e^6}+1$.*

Let $\Omega=\{A_v|v\in V(G)\}$.

To prove, using the Probabilistic Method, that the desire partial radiocoloring described above exists, we only need to show that with positive probability none of these bad events occurs.

Note that A_v depends only on the color of vertices which are joined to v by a path of length at most 4. Thus setting,
$S_v=\{A_w\mid v,\ w\ \textit{are joined by a path of length at most}\ 8\}$.

We see that A_v is mutually independent of $\Omega-S_v$. But $|S_v|<\Delta^8$. So, as long as no A_v has probability greater than $\frac{1}{4\Delta^8}$ we are done, because:

applying the Local Lemma with $\mathbf{Pr}(\mathrm{A_v}) \leq \mathrm{p} = \frac{1}{4\Delta^8}$ and $|S| \leq d = \Delta^8$, we have

$$4 \cdot p \cdot d = 4\frac{1}{4\Delta^8} \cdot \Delta^8 \leq 1$$

Thus, by the Lemma with positive probability none of the (bad) events of Ω happens.

In order to prove that $\mathbf{Pr}(\mathrm{A_v}) < \frac{1}{4\Delta^8}$ we will provide a lower bound of the expectation $E(X_v)$ of order Δ^2 and then we will prove that X_v is highly concentrated around its expected value:

Lemma 3.4

$$E(X_v) \geq \frac{2\Delta^2}{3e^6}$$

Lemma 3.5

$$\mathbf{Pr}\left\{|\mathrm{X_v} - \mathrm{E(X_v)}| > \log \Delta^2 \cdot \sqrt{\mathrm{E(X_v)}}\right\} < \frac{1}{4\Delta^8}$$

These two lemmas complete the proof because,

$$\begin{aligned}
|X_v - E(X_v)| &> \log \Delta^2 \cdot \Delta \quad (for\ X_v < E(X_v)) \Leftrightarrow \\
E(X_v) - \log \Delta^2 \cdot \Delta &> X_v \Leftrightarrow \\
E(X_v) - \log \Delta^2 \cdot \Delta &> \frac{2\Delta^2}{3e^6} - \log \Delta^2 \cdot \Delta \\
&> \frac{\Delta^2}{2e^6}\left(\frac{4}{3} - \frac{2e^6 \log \Delta^2}{\Delta}\right) \\
&> \frac{\Delta^2}{2e^6} + 1
\end{aligned}$$

for sufficiently large Δ. Thus,

$$\mathbf{Pr}\left\{\mathrm{X_v} < \frac{\Delta^2}{2\mathrm{e}^6} + 1\right\} \ < \ \frac{1}{4\Delta^8}$$

3.1 Proof of Lemma 3.4

Proof: For each vertex v we define $\mathbf{X}'_v$ *the number of colors assigned to* exactly *two vertices in the distance one and two neighbourhood of v and are retained by both these vertices.* Note that $X_v \geq X'_v$.

A pair of vertices in the distance one and two neighbourhood of v will both retain the same color α which is assigned to no other vertex of $S = N_{1,2}(v) \cup N_{1,2}(u) \cup N_{1,2}(w) - v - u - w$. Thus, $|S| \leq 3\Delta^2 - 3 \leq 6C - 3$.

For any color, the probability that this occurs is at least

$$\left(\frac{1}{C}\right)^2 \cdot \left(1 - \frac{1}{C}\right)^{3\Delta^2 - 3} > \frac{4}{\Delta^4} \cdot \frac{2}{3e^6} \tag{1}$$

There are C choices for color α.

How many such pairs u, w exist in the distance one and two neighbourhood of v?

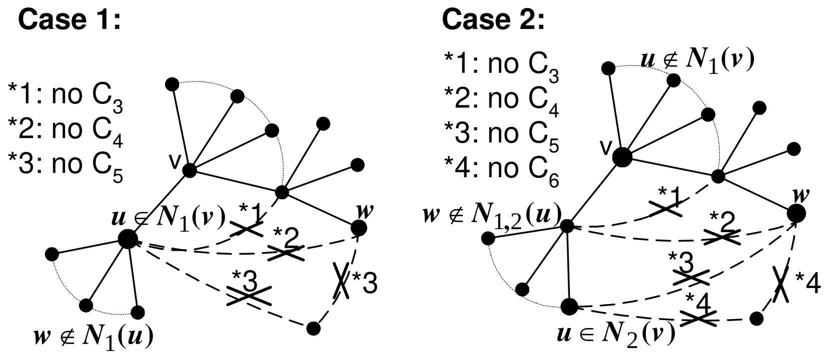

Figure 1: The possible cases for a pair of distance one or two neighbours of v to have the same color. In case 1, $u \in N_1(v)$ and in case 2, $u \in N_2(v)$.

Claim 3.6 *For any vertex of a regular graph G with girth at least 7, there are $\Delta^4-\Delta^3-\Delta^2$ possible pairs of vertices in any distance one and two neighbourhood of v that can have the same color.*

Proof: Note first that if $u \in N_1(v)$ then $w \notin N_1(u)$ and $w \notin N_1(v)$. So, in this case there are at most $\Delta^2 - \Delta - \Delta$ choices for vertex w. Since, there are Δ choices for such vertex u, in this case we have: $\Delta \cdot (\Delta^2 - 2\Delta) = \Delta^3 - 2\Delta^2$ possible pairs u, w. Note that all these pairs are possible because we assumed that the graph G has no 3,4,5-cycles (see Figure 1(case 1)).

Otherwise, in the case where $u \in N_2(v)$ there are $\leq \Delta^2 - \Delta$ choices for vertex w, since $w \notin N_{1,2}(u)$. Also, there are $\Delta^2 - \Delta$ choices for such vertex u, since $u \notin N_1(v)$. Thus, in this case we have $(\Delta^2 - \Delta) \cdot (\Delta^2 - \Delta) = \Delta^4 - 2\Delta^3 + \Delta^2$ possible pairs u, w. Note that all these pairs are possible because we assumed that the graph G has no 3,4,5,6-cycles (see Figure 1(case 2)).

Summing up, in total we have $(\Delta^3 - 2\Delta^2) + (\Delta^4 - 2\Delta^3 + \Delta^2) = \Delta^4 - \Delta^3 - \Delta^2$ possible pairs u, w. ∎

By the above results, the Linearity of Expectation, Inequality 1 and the fact that $e^{-1/C} < 1 - \frac{1}{C} + \frac{1}{2C^2}$ we get

$$\begin{aligned}
E(X'_v) &\geq C \cdot (\Delta^4 - \Delta^3 - \Delta^2) \cdot \left(\frac{1}{C}\right)^2 \left(1 - \frac{1}{C}\right)^{6C-3} \\
&> (\Delta^4 - \Delta^3 - \Delta^2) \cdot \frac{2}{\Delta^2} \cdot \frac{2}{3e^6} \\
&> \Delta^4 \cdot \left(1 - \frac{1}{\Delta} - \frac{1}{\Delta^2}\right) \cdot \frac{2}{\Delta^2} \cdot \frac{2}{3e^6} \\
&> \frac{2\Delta^2}{3e^6}
\end{aligned}$$

for C sufficiently large. ∎

3.2 Proof of Lemma 3.5

Proof: Instead of proving the concentration of X_v directly, we split it into two variables and compute the concentration of them. The first of these is
$\mathbf{AT_v}$ (assigned twice): counts the number of colors assigned to at least two distance one or two neighbours of v.

The second one is:
$\mathbf{Del_v}$ (deleted): counts the number of colors assigned to at least two distance one or two neighbours of v but removed by at least one of them.

We note that $X_v = AT_v - Del_v$, and so to prove the Lemma it suffices to prove the following concentration bounds on these two variables, for $t = \frac{1}{2}\Delta \log \Delta^2$,

Claim 3.7

$$\mathbf{Pr}\left\{|\mathrm{AT_v} - \mathrm{E}(\mathrm{AT_v})| > \frac{1}{2}\Delta \log \Delta^2\right\} < 4\mathrm{e}^{-\Omega(\ln^2 \Delta^2)}$$

Claim 3.8

$$\mathbf{Pr}\left\{|\mathrm{Del_v} - \mathrm{E}(\mathrm{Del_v})| > \frac{1}{2}\Delta \log \Delta^2)\right\} < 4\mathrm{e}^{-\Omega(\ln^2 \Delta^2)}$$

To see that these two claims imply the Lemma, we observe that by Linearity of Expectation, $E(X_v) = E(AT_v) - E(Del_v)$. Therefore, if $|X_v - E(x_v)| > \Delta \log \Delta^2$, then setting $t = \frac{1}{2}\Delta \log \Delta^2$, we must have either
$|AT_v - E(AT_v)| > t$ or $|Del_v - E(Del_v)| > t$, (recall that $|X_v - E(x_v)| = |AT_v - Del_v - E(AT_v) + E(Del_v)|$.

We are interested to compute the probability of $|X_v - E(X_v)| > \Delta \log \Delta^2$. If this event occurs, then $|AT_v - E(AT_v)| > t$ or $|Del_v - E(Del_v)| > t$. Thus, by the Subadditivity of Probabilities, the probability of this to happen is (recall that $t = \frac{1}{2}\Delta \log \Delta^2$),

$$4e^{-\Omega(\ln^2 \Delta^2)} + 4e^{-\Omega(\ln^2 \Delta^2)} < 2\frac{1}{8\Delta^8} < \frac{1}{4\Delta^8}$$

3.2.1 Proof of Claim 3.7

Proof: The value of AT_v depends only on the Δ^2 assignments of the distance one and two neighbourhood of v. Furthermore, changing any of those assignments can affect AT_v by at most 2 as this change can affect whether the old and whether the new colors are counted in AT_v. Therefore, applying the Simple Concentration Bound with $c = 2$ and $t = \frac{1}{2}\Delta \log \Delta^2$ we have

$$\mathbf{Pr}\left\{|\mathrm{AT_v} - \mathrm{E}(\mathrm{AT_v})| > \mathrm{t}\right\} < 2\mathrm{e}^{-\frac{\mathrm{t}^2}{8\mathrm{c}^2\Delta^2}} \Leftrightarrow$$

$$\mathbf{Pr}\left\{|\mathrm{AT_v} - \mathrm{E}(\mathrm{AT_v})| > \frac{1}{2}\Delta \log \Delta^2\right\} < 2\mathrm{e}^{-\frac{\left(\frac{1}{2}\cdot\Delta \ln \Delta^2\right)^2}{8\Delta^2}} < 2\mathrm{e}^{-\frac{\ln^2 \Delta^2}{4\cdot 4\cdot 8}} < 4\mathrm{e}^{-\Omega(\ln^2 \Delta^2)}$$

■

3.2.2 Proof of Claim 3.8

Proof: The value Del_v depends on the up to Δ^4 assignments made to the vertices of distance at most 4 from v. Because $E(Del_v) \leq \Delta^2 = o(\Delta^4)$ (since in fact Del_v is always at most Δ^2), the Simple Concentration Bound will not apply here. So, we will apply Talagrand's inequality.

Theorem 3.9 Talagrand's Inequality I*[16, 12]: Let X be an non-negative random variable, not identically 0, which is determined by n independent trials $T_1, \ldots, T_n$, and satisfying the following for some $c, r > 0$:*

1. *changing the outcome of any one trial can affect X by at most c, and*
2. *for any s, if $X \geq s$ then there is a set of at most rs trials whose outcomes certify that $X \geq s$,*
   then for any $0 \leq t \leq \mathbf{Med}(X)$,

$$\mathbf{Pr}\left\{|\mathrm{X} - \mathbf{Med}(\mathrm{X})| > \mathrm{t}\right\} \leq 4\mathrm{e}^{-\frac{\mathrm{t}^2}{8\mathrm{c}^2\mathrm{r}\mathbf{Med}(\mathrm{X})}}$$

We first study how Del_v may differentiate by changing the color assignment of a vertex at distance at most 4 from v. By changing the color of such a vertex, up to Δ^2 vertices can be uncolored, since all vertices of its distance one or two may have this color. However, note that this change may differentiate Del_v by at most two since it concerns only two colors; the old color and the new color of the vertex.

Furthermore, if $Del_v \geq s$ there is a set of at most $3s$ color assignments which certify that $Del_v \geq s$. This is because for each of these s colors α counted by Del_v we take 2 vertices of that color in $N_{1,2}(v)$ that are assigned that color, α, and one of their distance one or two neighbours which also received that color (since α is removed by at least one of these two vertices, one of its distance one or two neighbours has also received that color).

Therefore, we can apply Talagrand's inequality with $c = 2$ and $r = 3$ to obtain:

$$\mathbf{Pr}\left\{|\mathrm{Del_v} - \mathrm{E}(\mathrm{Del_v})| > \mathrm{t}' + 60\mathrm{c}\sqrt{\mathrm{rE}(\mathrm{Del_v})}\right\} \leq 4\mathrm{e}^{-\frac{\mathrm{t}'^2}{8\mathrm{c}^2\mathrm{rE}(\mathrm{Del_v})}}$$

having $t > t' + 60c\sqrt{rE(Del_v)} \Leftrightarrow t' < t - 60c\sqrt{rE(Del_v)} \Leftrightarrow$ we get,

$$\begin{aligned}
\mathbf{Pr}\left\{|\mathrm{Del_v} - \mathrm{E}(\mathrm{Del_v}) > \mathrm{t}\right\} &\leq 4e^{-\frac{\left(t-60c\sqrt{rE(Del_v)}\right)^2}{8c^2rE(Del_v)}} \\
&\leq 4e^{-\frac{\left(t-120\sqrt{3E(Del_v)}\right)^2}{96E(Del_v)}} \\
&\leq \frac{4}{e^{\frac{\left(t-120\sqrt{3E(Del_v)}\right)^2}{96\Delta^2}}} \qquad (2)
\end{aligned}$$

We are interested in the minimum value of $t - 120\sqrt{3E(Del_v)}$. Since $t \geq \frac{1}{2}\Delta \log \Delta^2$:

$$\begin{aligned}
\left(t - 120\sqrt{3E(Del_v)}\right)^2 &> t^2 + 120^2 \cdot 3E(Del_v) - 2 \cdot 120t\sqrt{3E(Del_v)} \\
&> t^2 + 120^2 \cdot 3 \cdot \Delta^2 - 2 \cdot 120\sqrt{\Delta^2 \log \Delta^2} \cdot \Delta \\
&> \Delta^2(\log \Delta^2)^2 + 120^2 \cdot 3 \cdot \Delta^2 - 2 \cdot 120\sqrt{\Delta^2 \log \Delta^2} \cdot \Delta \\
&> t^2
\end{aligned}$$

Replacing the above quantity in inequality 2, we get

$$\mathbf{Pr}\left\{|\mathrm{Del_v} - \mathrm{E}(\mathrm{Del_v}) > \mathrm{t}\right\} \;\leq\; \frac{4}{e^{\frac{t^2}{96\Delta^2}}} < \frac{4}{e^{\frac{t^2}{100\Delta^2}}} < 4e^{-\Omega(\ln^2 \Delta^2)}$$

(end of proof of Claim 3.8) ■

(end of proof of Lemma 3.5) ■

(end of proof of the Theorem 3.1) ■

4 Studying whether the RCP is determined by local conditions

In [13] Molloy and Reed proved that:

Theorem 4.1 *There is an absolute constant Δ_0 such that for any $\Delta \geq \Delta_0$ and $k^2+k < \Delta$ if G has maximum degree Δ and if $\chi(G) > \Delta - k$ then G contains a subgraph H such that*

1. $|H| \leq \Delta + 1$
2. $\chi(H) = \Delta - k + 1$

In contrast, they prove that for the rest of the values of Δ the statement does not hold:

Theorem 4.2 *For every $\Delta \geq 2$ and $k^2 + k \geq \Delta$, there are arbitrarily large $\Delta - k + 1$-critical graphs with maximum degree Δ.*

Recall that if G is a x-critical graphs, then $\chi(G) = x$ and every proper subgraph of it can be $x - 1$-colored.

We conjecture that an analogous statement 4.1 holds for the radiocoloring problem.

Conjecture 4.1 *There is an absolute constant Δ_0 such that for any $\Delta^2 \geq \Delta_0$ and $k^2+k < \Delta^2$ if G has maximum degree Δ and if $\chi(G^2) > \Delta^2 - k$ then G contains a subgraph H such that*

1. $|H| \leq \Delta^2 + 1$
2. $\chi(H^2) = \Delta^2 - k + 1$

We know study whether there are cases (values of Δ) for which Theorem 4.2 of Molloy and Reed holds and for the same values of Δ our conjecture may apply. If such cases exist, they will contradict the conjecture.

Assume that conjecture 4.1 is true for a graph G of maximum degree Δ and a given k such that $k^2 + k < \Delta^2$. Consider the case where the graph G is such that the maximum degree of G^2 is less than Δ^2, call it Δ_2, that is $\Delta_2 < \Delta^2$. Observe that such graphs exist;

in such a graph, for each vertex of maximum degree, all its neighbours have degree less than Δ.

Question: *Is it possible the degree Δ_2 (the maximum value of G^2) to be such that the following condition holds?*

(Condition 1:) *there exists a k_2 such that $k_2^2 + k_2 \geq \Delta_2$ and $\alpha = \Delta_2 + 1 - k_2$.*

Observe that if such values exists it means that applying Theorem 4.2 for these values of Δ_2, k_2, we get that:

There ia a graph F of maximum degree Δ_2 and $k_2^2 + k_2 \geq \Delta_2$ which is $\Delta_2 - k_2 + 1 = \alpha$-critical.

If additionally to condition 1, we also have that:

Condition 2: $F = G^2$,

then, the conjecture 4.1 can not be true. This is so because applying Theorem 4.2 for G^2 we get that every proper subgraph of G^2 is $\Delta_2 - k_2 = \alpha - 1$-colorable. In contrast, the conjecture says that G contains a subgraph H such that $\chi(H^2) = \Delta^2 - k + 1 = \alpha$. If there exists values of Δ_2, k_2, so that at least condition 1 is fulfilled then the conjecture will contradict.

Below we prove that there does not exist Δ_2 and k_2 such that the Condition 1 is fulfilled. So, there is the Theorem 4.2 does not contradict with the conjecture as it concerns the values of Δ_2.

4.1 Condition 1

We have the following set of inequalities:

$$\begin{aligned} k^2 + k &< \Delta^2 && (3) \\ \Delta^2 - k + 1 &= \alpha && (4) \\ \Delta_2 &< \Delta^2 && (5) \\ k_2^2 + k_2 &\geq \Delta_2 && (6) \\ \Delta_2 - k_2 + 1 &= \alpha && (7) \end{aligned}$$

We want to compute suitable values of Δ_2, k_2 such that the above requirements to be satisfied.

We can write

$$k_2 = \Delta_2 + 1 - \alpha \tag{8}$$

Thus, by 6 we get:

$$\begin{aligned} k_2^2 + k_2 &\geq \Delta_2 \Leftrightarrow \\ (\Delta_2 + 1 - \alpha)^2 + (\Delta_2 + 1 - \alpha) &\geq \Delta_2 \Leftrightarrow \\ (\Delta_2 + 1)^2 + \alpha^2 - 2\alpha(\Delta_2 + 1) + \Delta_2 + 1 - \alpha &\geq \Delta_2 \Leftrightarrow \\ \Delta_2^2 + 1 + 2\Delta_2 + \alpha^2 - 2\alpha\Delta_2 - 2\alpha + \Delta_2 + 1 - \alpha &\geq \Delta_2 \Leftrightarrow \\ \Delta_2^2 + (2 - 2\alpha) \cdot \Delta_2 + \alpha^2 - 3\alpha + 2 &\geq 0 \Leftrightarrow \end{aligned}$$

Solving the equation for Δ_2, we get:

$$\Delta_{2_{1,2}} = \frac{-(2-2\alpha) \pm \sqrt{(2-2\alpha)^2 - 4(\alpha^2 - 3\alpha + 2)}}{2} \Leftrightarrow$$
$$\Delta_{2_{1,2}} = \frac{-2+2\alpha) \pm \sqrt{4 + 4\alpha^2 - 8\alpha - 4\alpha^2 + 12\alpha - 8)}}{2} \Leftrightarrow$$
$$\Delta_{2_{1,2}} = \frac{-2+2\alpha \pm \sqrt{4\alpha - 4)}}{2} \Leftrightarrow$$
$$\Delta_{2_{1,2}} = (-1+\alpha) \pm \sqrt{\alpha - 1}$$

So, we get :

$$\left(\Delta_2 - (-1 + \alpha + \sqrt{\alpha - 1})\right)\left(\Delta_2 - (-1 + \alpha - \sqrt{\alpha - 1})\right) \geq 0$$

It can be easily seen that the inequality is true when

(**Case 1**): $\Delta_2 \geq (-1+\alpha) + \sqrt{\alpha - 1}$

(**Case 2**): $\Delta_2 \leq (-1+\alpha) - \sqrt{\alpha - 1}$

We consider both cases:

Case 1 : $\Delta_2 \geq -1 + \alpha + \sqrt{\alpha - 1}$

Thus, using 4, we get that:

$$\Delta_2 \geq -1 + (\Delta^2 - k + 1) + \sqrt{\Delta^2 - k + 1 - 1} \quad (9)$$
$$\Delta_2 \geq \Delta^2 - k + \sqrt{\Delta^2 - k} \quad (10)$$

Recall that by inequality 5, we also require that $\Delta_2 < \Delta^2$. Letting $\Delta_2 = \Delta^2 - k + \sqrt{\Delta^2 - k} + \beta$, $\beta > 0$, we require that:

$$\begin{aligned} \Delta_2 = \Delta^2 - k + \sqrt{\Delta^2 - k} + \beta &< \Delta^2 \Leftrightarrow \\ -k + \sqrt{\Delta^2 - k} + \beta &< 0 \Leftrightarrow \\ \sqrt{\Delta^2 - k} + \beta &< k \Leftrightarrow \end{aligned}$$

The best situation is when $\beta = 0$:

$$\begin{aligned} \Delta^2 - k + \sqrt{\Delta^2 - k} &< \Delta^2 \Leftrightarrow \\ -k + \sqrt{\Delta^2 - k} + \beta &< 0 \Leftrightarrow \\ \sqrt{\Delta^2 - k} &< k \Leftrightarrow \\ \Delta^2 - k &< k^2 \Leftrightarrow \\ \Delta^2 &< k^2 + k \end{aligned}$$

which contradict to inequality 3. So, none of the possible values of this case of Δ_2 fulfills the requirements of inequality 5. We now study the second case for Δ_2:

Case 2 : $\Delta_2 \leq -1 + \alpha - \sqrt{\alpha - 1}$

Replacing Δ_2 in inequality 8, we get that:

$$k_2 = \Delta_2 + 1 - \alpha \quad \leq \quad -1 + \alpha - \sqrt{\alpha - 1} + 1 - \alpha = -\sqrt{\alpha - 1}$$

Since $k_2 > 0$, there is no value of k_2 that satisfy this inequality. So, and for this case of Δ_2 there is no value of k_2 that all requirements of inequalities 3 to 7 are fulfilled.

We conclude that there are no values of Δ_2, k_2 such that the requirements of Condition 1 are satisfied.

5 Conclusions and Open problems

We believe these results constitute some fruitful ground for further research in order to improve the problems studied here as well as for developing upper and lower bound for the radiocoloring problem, exploiting the powerful tools of the probabilistic method. Moreover, we believe that the method have not been exploited yet for many generalizations of the graph coloring problem, henceforth this direction can be addressed in future works.

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