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# Load Balancing of Unit Size Tokens and Expansion Properties of Graphs *

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## ABSTRACT

Diffusive schemes have been widely analyzed for parallel and distributed load balancing. It is well known that their convergence rates depend on the eigenvalues of some associated matrices and on the expansion properties of the underlying graphs. In the first part of this paper we make use of these relationships in order to obtain new spectral bounds on the edge and node expansion of graphs. We show that these new bounds are better than the classical bounds for several graph classes. In the second part of the paper, we consider the load balancing problem for indivisible unit size tokens. Since known diffusion schemes do not completely balance the load for such settings, we propose a randomized distributed algorithm based on Markov chains to reduce the load imbalance. We prove that this approach provides the best asymptotic result that can be achieved in $l_1$- or $l_2$-norm concerning the final load situation.

## 1. INTRODUCTION

Load balancing is a very important prerequisite for the efficient use of parallel computers. Many parallel applications have work loads with a dynamic behavior, i.e., they can change dramatically during runtime. Therefore, the load has to be balanced in order to achieve an efficient use of the processor network. Obviously, we can ensure an overall benefit of the computation only if the balancing scheme itself is highly efficient.

One example for the importance of efficient load balancing schemes appears in parallel finite element simulations where a geometric space is discretized by a mesh. Then, the mesh is partitioned in as many sub-regions as processors of the parallel machine. Each processor performs the computation on mesh elements in each sub-region independently. Depending on the application, the mesh refines and coarsens during the computation and therefore unbalance between processors occurs. In such situations the load has to be rebalanced. The application is stopped and although the application behaves dynamically we can concentrate on solving a static load balancing problem. For a selection of applications, studies on and references to the problem of parallel finite element simulations we refer to [14].

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In this paper we consider the following problem. We are given a network consisting of $n$ processors, where each processor of the network has a collection of unit-size jobs (called tokens) and can only access the load information of its direct neighbors. Now, the goal is to balance the load by transmitting tokens along edges, in iterations, according to some load balancing algorithm. Formally, if node $i$ contains work load $w_i$ we shift a balancing flow over the edges of the network, in iterations, so that after termination each node $i$ has the load $\left\lfloor \frac{\sum_{j=1}^n w_j}{n} \right\rfloor$ or $\left\lceil \frac{\sum_{j=1}^n w_j}{n} \right\rceil$. We assume that no load is generated or consumed during the balancing process, i.e., we consider a static load balancing scenario.

In most papers, the load balancing problem of arbitrary dividable load items is considered. For such settings, the goal is to obtain the balanced work load

$$\overline{w_i} = \frac{\sum_{j=1}^n w_j}{n},$$

for each node. Highly popular classes of load balancing algorithms are the so called diffusion schemes [10, 5]. There, the flow moved along each edge is dependent on the gradient in the loads across it. These schemes are performed locally, i.e., each step of the algorithm can be implemented in a local manner so that each node consults only its neighbors and avoids expensive global communication.

Diffusion algorithms were introduced by Cybenko in [10]. If we denote by $w_i^k$ the load of node $i$ after $k$ iteration steps and by $y_{i,j}^{k-1}$ the flow over edge $\{i,j\}$ in step $k$, then

$$\forall \quad e = \{i,j\} \in E: \; y_{i,j}^{k-1} = \alpha_{i,j}(w_i^{k-1} - w_j^{k-1}); \text{ and} \quad (1)$$

$$w_i^k = w_i^{k-1} - \sum_{\{i,j\}\in E} y_{i,j}^{k-1}, \quad (2)$$

where all $\alpha_{i,j}$ have to satisfy some conditions (see next section). Equation (2) can be rewritten in matrix notation as $w^k = Mw^{k-1}$, where the matrix $M$ is called the *diffusion matrix* of the network on which the load balancing algorithm proceeds (see also next section). This algorithm is called *first order scheme* (FOS). Improvements of FOS are the so called *second order schemes* (SOS), which converge faster than FOS by almost a quadratic factor [25, 11].

The convergence rate of diffusion algorithms is strongly related to the expansion properties of the underlying graph. In graphs with large edge or node expansion, information can flow more easily and therefore the total load of the network can be spread faster over the whole system.

In the past, several papers studied the relationship between the expansion and the spectrum of some associated matrices of a graph [2, 3, 23, 24, 8, 20, 30]. Tanner considered regular graphs and observed that a small second largest eigenvalue of the adjacency matrix of a graph implies good expansion properties for it [30]. In [2, 4, 3], the authors analyzed arbitrary graphs and related the second smallest eigenvalue of the *Laplacian* of such graphs (for the exact definition see next section) to their expansion properties. The paper [3] considers graphs with high connectivity properties and their applications. In [2], Alon developed a new spectral bound on the vertex expansion of graphs by stating a discrete version of the Cheeger inequality from the theory of Riemannian manifolds [6]. The bound of Tanner concerning regular graphs was improved by Kahale in [20]. He also constructed almost Ramanujan graphs, for which his bound is asymptotically tight.

Concerning the edge expansion, the classical bounds were stated in [23]. Analyzing these bounds, we observe that there is a quadratic gap between the lower and upper bound. Mohar improved the lower bound by computing a strong discrete version of the Cheeger inequality in [24].

In Section 3, we use diffusion algorithms to state new spectral bounds on both, edge and node expansion of general graphs. Furthermore, we describe the graphs for which these new bounds are better than the classical bounds.

In the second part of this paper we analyze the load balancing problem of indivisible unit size tokens. This problem has been considered before in e.g. [25, 28]. There, the authors use discretizations of FOS by transmitting the flow $y_{i,j} = \lfloor \alpha_{i,j}(w_i^{k-1} - w_j^{k-1}) \rfloor$ over edge $\{i, j\}$ in step $k$. In [25] it is shown that after $k = \Theta\left(\frac{d}{\lambda_2}\log(Kn)\right)$ steps the error in $l_2$-norm $||e^k||_2 = \sqrt{\sum_{i=1}^n (w_i^k - \overline{w_i})^2}$ is reduced to $O\left(\frac{nd^2}{\lambda_2}\right)$, where $K = \max_i |w_i^0 - \overline{w}_i|$ is the *maximal load imbalance* in the network, $\lambda_2$ denotes the second smallest eigenvalue of the Laplacian of the graph, and $d$ is the maximal vertex degree. Rabani, Sinclair and Wanka improved this bound by showing that $\max_i |w_i^k - \overline{w_i}|$ is $O\left(\frac{d^2 \cdot \log(n)}{\lambda_2}\right)$ after $k = \Theta\left(\frac{d}{\lambda_2}\log(Kn)\right)$ iterations [28].

In Section 4, we first consider SOS and present an upper bound on the final error $||e^k||_2$, provided that the load at any node is sufficient to supply the demand of load to be moved on its incident edges in each iteration. Then, we develop a randomized algorithm which reduces $||e^k||_2$ to $O(\sqrt{n})$ in $k = \Theta\left(\frac{d}{\lambda_2}(\log(K) + \log(n)\log(\log(n)))\right)$ iteration steps (with high probability), if $\overline{w}_i$ exceeds some certain threshold. We will show that $O(\sqrt{n})$ is the best asymptotic result that can be achieved in $l_2$-norm concerning the final load distribution.

We also show that our new algorithm works as well in static as in dynamic environments. Dynamic networks are modelled by graphs, where the set of nodes is fixed but edges between two nodes can fail or reappear while the algorithm proceeds. Similar models of dynamic networks have been used for the analysis of the so called token distribution problem (see e.g. [15]). However, in contrast to diffusion, the token distribution problem only allows one token to be sent over an edge during an iteration step.

The new algorithm we present in this paper is mainly based on a random walk approach. Random walks on graphs have been analyzed before in several papers (e.g. [19, 29, 21, 16]). Notably, Jerrum and Sinclair used the rapid mixing property of Markov chains to obtain polynomial time approximation algorithms for several counting and generation problems [19, 29]. Gilman analyzed an alternative procedure introduced by Aldous in [1] and showed that this procedure (sometimes in a modified form) requires fewer number of steps than the standard procedure [16].

It is worth mentioning that recently in [9] a diffusion based algorithm on static, dynamic and asynchronous networks was given. Using this algorithm the best balance can be achieved. However, it is hard to compute any bound on the runtime of this algorithm.

The outline of the paper is as follows. In the next section, we give an overview of the basic definitions and the main theoretical background. Then, we derive new spectral bounds on the node and edge expansion of graphs. In Section 4, we study the final load discrepancy of SOS, we introduce our randomized strategy, and we analyze its behavior and run time. We conclude by referring to some further research directions.

## 2. DEFINITIONS AND BACKGROUND

Let $G = (V, E)$ be an undirected graph with $V$ being the set of vertices and $E$ being the set of edges. We denote with

$$i(G) = \min_{\substack{X \subset V \\ |X| \leq |V|/2}} \frac{E(X, \overline{X})}{|X|}$$

the edge expansion of $G$, where $E(X, \overline{X})$ represents the number of edges connecting vertices of $X$ with vertices of $\overline{X}$. Furthermore, let

$$\rho(G) = \min_{\substack{X \subset V \\ |X| \leq |V|/2}} \frac{|N(X)|}{|X|},$$

be the vertex expansion of $G$, where $N(X)$ is the set of vertices belonging to $\overline{X}$, which are adjacent to some vertices of $X$. As described in the introduction, the expansion properties of a graph are strongly related to the spectrum of its Laplacian defined as follows. For a graph $G = (V, E)$, with

$|V| = n$, the $n \times n$ Laplacian $L = \{l_{v,w}\}$ has the form:

$$l_{v,w} = \begin{cases} \deg(v), & \text{if } v = w \\ -1, & \text{if } v \neq w \text{ and } \{v,w\} \in E \\ 0, & \text{otherwise} \end{cases}$$

The eigenvalues $0 = \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$ of $L$ are nonnegative. They have pairwise perpendicular eigenvectors and the normalized eigenvector with eigenvalue 0 is $\frac{1}{\sqrt{n}}(1,1,1,...,1)^t$. For connected graphs the multiplicity of this eigenvalue is 1 (see e.g. [26]).

Several spectral bounds have been developed before on the expansion properties of graphs (e.g. [2, 3, 8, 23, 24]). Well-known bounds on the node expansion of graphs are

$$\frac{\rho^2(G)}{4 + 2\rho^2(G)} \leq \lambda_2 \leq \frac{d_{\max}\rho(G)}{2(1-\rho(G))}, \tag{3}$$

where $d_{\max}$ represents the maximum vertex degree of $G$ [2]. A slightly improved spectral bound have been derived in [3] by stating the inequality: $\rho(G) \geq \frac{2}{T+\sqrt{T^2+1}}$, where $T = \frac{1}{2}\left(1 + \frac{d_{\max}}{\lambda_2}\right)$.

Due to Tanner's result,

$$\rho(G) \geq \frac{2d^2}{\nu^2 + d^2},$$

where $G$ is a regular graph of degree $d$, $\nu$ denotes, in this case, the second largest eigenvalue, according to absolute values, of the adjacency matrix of $G$. Kahale defined the linear expansion of a regular graph to be

$$\rho_\psi(G) = \min_{\substack{X \subset V \\ |X| \leq \psi|V|}} \frac{|N(X)|}{|X|},$$

where $\psi$ is a small positive constant. He proved that for any $\epsilon$ such that $\psi = d^{-1/\epsilon}$ it holds

$$\rho_\psi(G) \geq \frac{d}{2}\left(1 - \sqrt{1 - \frac{4d-4}{\tilde{\nu}^2}}\right)(1 - O(\epsilon)),$$

where $\tilde{\nu} = \max\{\nu, 2\sqrt{d-1}\}$ [20]. Using the bound above, the linear expansion of Ramanujan graphs is asymptotically at least $d/2$. In [20], constructions of almost Ramanujan graphs, for which this bound is asymptotically tight, are also presented.

Similar bounds for the edge expansion have also been developed [23, 24, 8]. The classical bounds have the form:

$$\frac{i^2(G)}{2d_{\max}} \leq \lambda_2 \leq 2i(G)$$

[24]. An improvement of the lower bound consists of the inequality $i(G) \leq \sqrt{\lambda_2(2d_{\max} - \lambda_2)}$. We observe that there is a quadratic gap between the lower and upper bound.

For the graph $G = (V, E)$ with $|V| = n$ and $|E| = N$ we define $A \in \{-1, 0, 1\}^{n \times N}$ as being the node-edge *incidence matrix* of $G$. $A$ contains a row for each node and a column for each edge. Each column has exactly two non-zero entries "1" and "$-1$" for the two nodes incident to the corresponding edge. A vector $x \in \mathbb{R}^N$ is called a *balancing flow* on the edges of $G$ iff $Ax = w - \overline{w}$. It expresses the fact that the flow balance at each node corresponds to the difference between its initial load and the mean load value, i.e. after shipping a load of exactly $x_e$ via each edge $e \in E$, the load is globally balanced.

The diffusion scheme described in equation (2) performs iterations on the nodes of $G$ and solves $Ax = w - \overline{w}$ by using communication with adjacent nodes only. As described in the introduction, this can be rewritten in matrix notation as $w^k = Mw^{k-1}$ with $M = I - \tilde{L}$, where $\tilde{L}$ contains $\alpha_{i,j}$ at position $(i,j)$ of edge $e = \{i,j\}$, $\sum_{e=\{i,j\}\in E} \alpha_{i,j}$ at diagonal entry $i$, and 0 elsewhere. If we assume that the nodes are able to access only local information, we can set $\alpha_{i,j} = \frac{1}{c \cdot \max\{d_i, d_j\}}$, where $d_i$ is the degree of node $i$ and $c \in (1,2)$ is a constant. Then $M$ has the eigenvalues $\mu_i = 1 - \tilde{\lambda}_i$, where $\tilde{\lambda}_i$ is the $i^{th}$ eigenvalue of $\tilde{L}$ and any $\alpha_{i,j}$ has to be chosen such that $1 = \mu_1 > \mu_2 \geq \ldots \geq \mu_n > -1$. Since $G$ is connected, $\mu_1 = 1$ is a simple eigenvalue to the eigenvector $(1,1,\ldots,1)^t$. We denote the second largest eigenvalue of $M$, according to absolute values, by $\gamma = \max\{|\mu_2|, |\mu_n|\} < 1$ and call it the *diffusion norm* of $M$ [11].

Several modifications of the first order scheme have been discussed in the past. One of them is the *second order scheme* (SOS) [25], which has the form

$$w^1 = Mw^0 \text{ and } w^k = \beta M w^{k-1} + (1-\beta)w^{k-2},\ k = 2, 3, \ldots$$

with $\beta$ being a fixed parameter, whereby the fastest convergence is archived for $\beta = \frac{2}{1+\sqrt{1-\gamma^2}}$.

We denote by $e^k$ the "error" after $k$ iteration steps, where $e^k = w^k - \overline{w}$. The convergence of diffusion schemes depend on how fast the system is $\epsilon$-balanced, i.e., the final error $||e^k||_2$ is no more than $\epsilon||e^0||_2$, for a fraction $\epsilon$. By using simple calculations and the results of [25] we obtain the following lemma.

LEMMA 1. *Let $G$ be a graph and $L$ be its Laplacian. Let $M = I - \tilde{L}$ be the diffusion matrix of $G$ and let $\beta = \frac{2}{1+\sqrt{1-\gamma^2}}$. Then, FOS and SOS require $\Theta(\frac{d}{\lambda_2} \cdot \ln(1/\epsilon))$ and $\Theta(\sqrt{\frac{d}{\lambda_2}} \cdot \ln(1/\epsilon))$ steps, respectively, to $\epsilon$-balance the system.*

Due to Lemma 1, SOS converges faster than FOS by almost a quadratic factor. However, by using SOS we have an added difficulty: in an iteration step the sum of the loads transferred from some node $i$ may exceed $w_i$. If we consider diffusion on static networks, we can use the so called two phase model to avoid this problem [11]. In the first phase, a balancing flow is determined by using diffusion, without moving any load across the edges of the network. In the second phase, the load is moved according to the balancing flow computed in the first phase. The drawback of this two phase model is that it can not be applied for load balancing on dynamic networks. However, even if in each iteration step the sum of loads transferred from some node $i$ does not exceed $w_i$, we still can not show that SOS converges in dynamic systems. Therefore, in the rest of the paper we mainly consider FOS for the analysis of diffusion schemes on dynamic networks.

Since the matrix $M$ is double stochastic, FOS can also be interpreted as a Markov process, where $M$ is the corresponding transition matrix and the balanced load situation is the related stationary distribution. Using this Markov chain approach, the following lemma can be stated [13, 22].

LEMMA 2. *Let $G$ be a graph and we assume that a token performs a random walk on $G$, according to the diffusion matrix $M$. Then, a constant $a$ exists such that, after $a(\frac{d\cdot\log(n)}{\lambda_2})$ steps, the probability that the token lies on some node $i$ is $p_i \in \left[\frac{1}{n}-\frac{1}{n^4}, \frac{1}{n}+\frac{1}{n^4}\right]$.*

# 3. NEW BOUNDS ON THE EXPANSION OF GRAPHS

As described in the introduction, there is a strong relationship between the convergence of diffusion schemes and the expansion properties of graphs. In this section, we use diffusion to improve the known spectral bounds on the expansion of several graphs. Theorem 1 describes the relationship between spectra and expansion of arbitrary (weighted and unweighted) graphs. In the case of weighted graphs

$$\rho(G) = \min_{\substack{X\subset V\\ vol(X)\le vol(V)/2}} \frac{vol(N(X))}{vol(X)},$$

where $vol(X)=\sum_{i\in X} c_i$ is the volume of the set $X$ and $c_i$ denotes the weight of node $i$.

THEOREM 1. *Let $G=(V,E)$ be a graph with node weights $c_i\in I\!R$, $i\in V$. Let $\rho(G)$ be its vertex expansion and let $\lambda_2$ and $\lambda_n$ the second smallest and largest eigenvalue of the weighted Laplacian $LC^{-1}$ of $G$, where $C$ contains the weights $c_i$ as diagonal entries and $0$ elsewhere. If $\kappa(G)=\lambda_2/\lambda_n$, then*

$$\rho(G)\ge \frac{2\kappa(G)}{1+\kappa^2(G)}.$$

PROOF. Let $U\subset V$ be a set with $vol(U)\le vol(V)/2$. We are given the following load situation in $G$. The vertices $1,\ldots,p\in U$ have load $w_i = c_i\frac{\sum_{i\in V}c_i}{\sum_{i\in U}c_i}$ and all other nodes have load 0. It has been shown in [12] that diffusion can also be used on weighted graphs, where $\overline{w}=\frac{\sum_{i\in V}w_i}{\sum_{i\in V}c_i}(c_1,\ldots,c_n)$. We consider the diffusion matrix $\mathcal{M}=I-\alpha LC^{-1}$ with $\alpha=\frac{2}{\lambda_2+\lambda_n}$. Then (see [12]),

$$||C^{-1/2}(\mathcal{M}w-\overline{w})||_2^2 \le \gamma^2||C^{-1/2}(w-\overline{w})||_2^2. \tag{4}$$

Now, $||C^{-1/2}(w-\overline{w})||_2^2=\frac{Z}{P}(Z-P)$, where $Z=\sum_{i\in V}c_i$ and $P=\sum_{i\in U}c_i$. Simple calculations show that $||C^{-1/2}(\mathcal{M}w-\overline{w})||_2^2$ is minimized if any node $i\in U\cup N(U)$ has the load $c_i\frac{Z}{Q}$, where $Q=\sum_{i\in U\cup N(U)}c_i$. Then, $||C^{-1/2}(\mathcal{M}w-\overline{w})||_2^2 = \frac{Z}{Q}(Z-Q)$. This implies together with inequality (4) that

$$\frac{Z}{Q}(Z-Q)\le\gamma^2\frac{Z}{P}(Z-P).$$

Therefore,

$$ZP-PQ\le\gamma^2(ZQ-PQ)$$

which leads to

$$Q\ge\frac{ZP}{P+\gamma^2(Z-P)}.$$

Thus,

$$\begin{aligned}\rho(G) &= \frac{Q-P}{P} \ge \frac{Z}{\gamma^2Z+(1-\gamma^2)P}-1\\ &= \frac{Z-P}{P+\gamma^2(Z-P)}(1-\gamma^2)\\ &\ge \frac{Z-P}{Z/2+\gamma^2Z/2}(1-\gamma^2) = 2\cdot\frac{1-\gamma^2}{1+\gamma^2}\cdot\frac{Z-P}{Z}.\end{aligned}$$

Noting that $\frac{1-\gamma^2}{1+\gamma^2}=\frac{2\kappa(G)}{1+\kappa^2(G)}$, we obtain the theorem. □

By setting $c_i=1$ for any $i\in V$, we obtain the theorem for unweighted graphs.

Rewriting inequality (3), we get

$$\rho(G)\ge\frac{2\lambda_2}{d_{\max}+2\lambda_2}.$$

Now, since $\lambda_n\in\left[\frac{n}{n-1}d_{\max}, 2d_{\max}\right]$, the bound in Theorem 1 shows a better performance than the bound above whenever $d_{\max}\ge\frac{(\lambda_n-\lambda_2)^2}{\lambda_n}$.

In [8], Chung considered spectral bounds on the Laplacian $C^{-1/2}LC^{-1/2}$, where $c_i=d_i$ for any $i\in V$ ($d_i$ is the degree of node $i$). It can be easily shown that the eigenvalues of $LC^{-1}$ coincide with the eigenvalues of $C^{-1/2}LC^{-1/2}$ and therefore, by using the result of Theorem 1 we can improve the upper bound of [8] on the second smallest eigenvalue of $C^{-1/2}LC^{-1/2}$.

The diffusion based approach can also provide new bounds on the edge expansion of graphs.

THEOREM 2. *Let $G=(V,E)$ be an unweighted graph, let $i(G)$ be its edge expansion and let $\lambda_2$ and $\lambda_n$ be the second smallest and largest eigenvalue of the Laplacian of $G$. Then it holds that*

$$i(G)\ge\min\left\{1,\frac{\lambda_2\lambda_n}{2(\lambda_n+\lambda_2-2)}\right\}.$$

PROOF. Again, the vertices $1,\ldots,p\in U\subset V$ have load $w_i=\frac{n}{p}$ and all other nodes have load 0, where $p\le n/2$ and $U$ is the subset for which $i(G)=\frac{E(U,\overline{U})}{p}$ holds. Using FOS with diffusion matrix $M=I-\alpha L$, where $\alpha=\frac{2}{\lambda_2+\lambda_n}$, we have

$$||Mw-\overline{w}||_2^2\le\gamma^2||w-\overline{w}||_2^2. \tag{5}$$

Now, $||w-\overline{w}||_2^2=\frac{n}{p}(n-p)$. Let $t=E(U,\overline{U})$ and we assume that $t\le p$ (if $t\ge p$ then it holds that $i(G)\ge1$). We know that FOS sends over each edge of the set $E(U,\overline{U})$ the amount of load $\frac{n}{p}\alpha$. Now, we assume that there are $t_1$ nodes in $U$ and $t_2$ nodes in $\overline{U}$ incident to these $t$ edges. Then, $p-t_1$ nodes have load $\frac{n}{p}$, $n-p-t_2$ nodes have load 0, $t_1$ nodes contain total load $t\frac{n}{p}(1-\alpha)$ and $t_2$ nodes contain total load

$t\frac{n}{p}\alpha$. Using simple calculations, $||Mw - \overline{w}||_2$ is minimized if $t_1 = t_2 = t$. It follows that

$$\begin{aligned} ||Mw - \overline{w}||_2^2 &\geq (p-t)(\frac{n}{p}-1)^2 + (n-p-t) \\ &+t(\frac{n}{p}(1-\alpha)-1)^2 + t(\frac{n}{p}\alpha - 1)^2 \\ &= \frac{1}{p}((n-p)^2 + p(n-p)) \\ &+\frac{t}{p^2}(-(n-p)^2 - p^2 + n^2(1-\alpha)^2 \\ &-2(1-\alpha)np + p^2 + n^2\alpha^2 - 2n\alpha p + p^2) \\ &= \frac{n}{p}(n-p) + \frac{t}{p^2}(-n^2 + 2np + n^2 \\ &-2n^2\alpha + n^2\alpha^2 + n^2\alpha^2 - 2np) \\ &= \frac{n}{p}(n-p) + t\frac{n^2}{p^2}(2\alpha^2 - 2\alpha) \end{aligned}$$

Using inequality (5), we obtain

$$\begin{aligned} t &\geq (1-\gamma^2)\frac{n}{p}(n-p)\frac{p^2}{n^2} \cdot \frac{1}{2\alpha - 2\alpha^2} \\ &= \frac{p}{n}(n-p)\frac{4\lambda_2\lambda_n}{(\lambda_2+\lambda_n)^2} \cdot \frac{(\lambda_2+\lambda_n)^2}{4(\lambda_2+\lambda_n-2)}. \end{aligned}$$

Noting that $p \leq n/2$ the theorem follows. □

We observe that the bound of Theorem 2 provides a better performance than the classical bound if $\lambda_2 \leq 2$.

# 4. REDUCING THE FINAL LOAD DEVIATION

As described in the introduction, in this section we first study the final load deviation of SOS for unit size tokens. Then, we present a randomized scheme, which reduces the previously mentioned discrepancy to $O(\sqrt{n})$ (in $l_2$-norm), and we analyze the behavior of this new algorithm.

An upper bound on the final deviation of FOS for unit size tokens have been computed in [25], and then, improved in [28]. Muthukrishnan et al. formulated as an open problem to compute the bound for SOS. If we use SOS, it can happen that the sum of the loads transferred from some node $i$ to its neighbors in a step may exceed $w_i$. Therefore, simply rounding down the flow in each step to the nearest integer does not necessarily simulate the algorithm. Muthukrishnan et al. proposed setting so called IOU units at each edge. These IOUs denote the difference between the total flow moved along that edge in the idealized scheme and that of the realistic algorithm, which distributes the available load only. If the IOUs accumulate, then we can not prove any result w.r.t. the convergence of the adapted SOS. Nevertheless, in most of our computer simulations, even if the IOUs emerge they become very small after a few rounds (see also [25]).

If we assume that the IOUs tend to zero after a few iteration steps, then we can use the techniques of Golub [17] to state the following theorem.

Theorem 3. *Let $G = (V, E)$ be a graph and let $w^0$ be the initial load distribution on $G$. Executing SOS on $G$, we assume that after a few iteration steps the IOUs are bounded by some constant on each edge of the graph. Then, after sufficient number of steps $k$, the error $||e^k||_2$ is $O(\frac{\sqrt{n}\cdot d}{1-\gamma})$, where $\gamma$ is the diffusion norm of the corresponding diffusion matrix $M$.*

Proof. As described in the previous section, SOS has the form

$$w^k = \beta M w^{k-1} - (\beta - 1) w^{k-2}, \quad (6)$$

where $\beta \in \frac{2}{1+\sqrt{1-\gamma^2}}$. Since we round the flow such that only integer load values are shifted over the edges, we get

$$\tilde{w}^k = \beta M \tilde{w}^{k-1} - (\beta - 1)\tilde{w}^{k-2} + \delta_{k-1}, \quad (7)$$

where $\tilde{w}^k$ represents the realistic value of the load after $k$ steps and $\delta_k$ is the round-off error. Let $a_k = \tilde{w}^k - w^k$ be the accumulated round-off after $k$ iterations. Substracting (6) from (7) leads to

$$a_k = \beta M a_{k-1} - (\beta - 1) a_{k-2} + \delta_{k-1},$$

where $a_0 = 0$ and $a_1 = \delta_0$. Noting that $a_j \perp (1, \ldots, 1)$ for any $j \in I\!N$ and using the technique of Golub [17], we obtain

$$||a_k||_2 \leq \sum_{j=0}^{k-1} (\beta - 1)^{\frac{k-j-1}{2}} (k-j)\delta,$$

where $||\delta_j||_2 \leq \delta$ for any $j \in \{0, \ldots, k-1\}$. This yields

$$||a_k||_2 \leq \left( \frac{1 - r^{k+1}}{(1-r)^2} - \frac{(k+1)r^k}{1-r} \right) \delta,$$

where $r = \sqrt{\beta - 1}$. Since $\delta \leq \sqrt{n}\cdot d$, the theorem follows. □

Using similar techniques, the same bound can also be obtained for FOS (without any restrictions).

Due to [25, 28], known diffusion schemes do not completely balance the load in a network containing indivisible unit size tokens. Since $\frac{1}{1-\gamma} = O(n^3)$ and $d \leq n-1$, the load discrepancy can be $O(n^k)$ at most, where $k$ is a small constant. In this section we present a randomized strategy, which reduces the final load deviation in $l_2$-norm to $O(\sqrt{n})$. Concerning the lower bound, we state the following proposition.

Proposition 1. *Using any load balancing scheme on a network containing indivisible unit size tokens the expected final load deviation $||e^k||_2$ is $\Omega(\sqrt{n})$.*

To show this, let $G = (V, E)$ be a graph with $n$ nodes and assume that there is a constant $b \in (0, 1)$ such that the total load in the network is $nw + bn$, where $w, bn \in I\!N$. Then, if the load is totally balanced, there are $n - bn$ nodes having load $w$ and $bn$ nodes having load $w + 1$. This implies that $||e^k||_2 = \sqrt{\sum_{i \in V}(w_i - \overline{w_i})^2} = \sqrt{n(1-b)b^2 + nb(1-b)^2} = \Omega(n)$.

In the following sentences we describe the randomized algorithm that reduces the final error $||e^k||_2$ to $O(\sqrt{n})$. This algorithm can also be used on dynamic networks, however, due to space limitations we describe its behavior only on static topologies. The algorithm consists of three phases. In the first phase, we perform FOS to approximate $\overline{w}$ and to reduce the maximal load imbalance $\max_i |w_i - \overline{w}_i|$ to $O(\frac{d^2 \cdot \log(n)}{\lambda_2})$. Note that the average load $\overline{w}$ can be approximated (within an error of $\pm 1$) by calculating a balancing flow in $G$ without considering the amount of load moved over the edges of the graph.

In the second phase, we perform $\log_2(\ln(n))$ so called *main random walk steps* (MRWS). Let us denote the maximal load imbalance by $q$. We know that each node has at least $\lfloor \overline{w}_i \rfloor - q$ tokens. In an MRWS, we first mark $w_i - (\lfloor \overline{w}_i \rfloor - q)$ tokens on each node, where $w_i$ represents the actual load of node $i$. Then, each marked token executes a random walk of length $a(\frac{d}{\lambda}\ln(n))$ according to the transition matrix $M$, where $a$ is the constant described in Lemma 2. We will show in Lemma 4 that an MRWS reduces the maximal load imbalance to approximately $2\sqrt{2\ln(n) \cdot q}$ (w.h.p.).

In the third phase, we perform $\log_2(\ln(n))$ so called *double main random walk steps* (DMRWS). We first set at any node with less than $\lfloor \overline{w}_i \rfloor$ tokens $\lfloor \overline{w}_i \rfloor - w_i$ so called "negative tokens". We additionally mark at each overloaded node $w_i - \lceil \overline{w}_i \rceil$ tokens. We denote the number of marked tokens by $qn$, where $q \in I\!\!R$ (the number of "negative tokens" is approximately equal to the number of marked tokens). Then, each "negative token" and each marked token performs a random walk of length $a(\frac{d}{\lambda}\ln(n))$, again, according to $M$. Note that if a "negative token" is moved from node $i$ to node $j$, then a real token is transferred accordingly from node $j$ to node $i$. We will show in Lemma 5 that a DRMWS reduces the total number of tokens producing overload in the system by at least a constant factor $c > 2$, provided that $q$ is larger than some certain constant. However, we can only guarantee that this phase is performed correctly if $\overline{w}_i \geq 32(n+1)\ln(n)$.

From the description of the algorithm it immediately follows that the number of iteration steps is bounded by the expression $O\left(\frac{d}{\lambda_2}(\log(n)\log(\log(n)) + \log(K))\right)$. The correctness of the algorithm is proved in the remaining of this section. First, consider the following lemma.

LEMMA 3. *If $p_i \leq 1/n + 1/n^4$ and $q_j \leq 1 - 1/n + 1/n^4$, then it holds that*

$$\binom{m}{k} p_i^k q_j^{m-k} \leq \binom{m}{k} (1/n)^k (1-1/n)^{m-k}(1 + O(1/n^2)),$$

*where $k > m/n$ and $m \geq n$.*

PROOF. First we show by induction that

$$\binom{m}{k} p_i^k \leq \binom{m}{k} \frac{1}{n^k}(1 + \frac{k}{n^3} + \frac{k}{n^5}).$$

Obviously, the claim holds for $k = 1$. We assume that it also holds for $k - 1$. Then

$$\begin{aligned}
\binom{m}{k} p_i^k &= \binom{m}{k-1} p_i^{k-1} \frac{m-k+1}{k} p_i \\
&\leq \binom{m}{k-1} \left(\frac{1}{n}\right)^{k-1} \left(1 + \frac{k-1}{n^3} + \frac{k-1}{n^5}\right) \\
&\quad \cdot \frac{m-k+1}{k} \cdot \frac{1}{n} \left(1 + \frac{1}{n^3}\right) \\
&\leq \binom{m}{k} \left(\frac{1}{n}\right)^k \left(1 + \frac{k}{n^3} + \frac{k}{n^5}\right)
\end{aligned}$$

On the other hand it holds that

$$\begin{aligned}
\left(1 - \frac{1}{n} + \frac{1}{n^4}\right)^{m-k} &\leq \left(1 - \frac{1}{n}\right)^{m-k} \\
&\quad \cdot \left(1 + \frac{2(m-k)}{n^3} + \frac{m-k}{n^5}\right),
\end{aligned}$$

which can also be shown as above. The claim holds for $m - k = 1$. Assuming the claim holds for $m - k - 1$, we obtain

$$\begin{aligned}
\left(1 - \frac{1}{n} + \frac{1}{n^4}\right)^{m-k} &\leq \left(1 - \frac{1}{n}\right)^{m-k-1} \\
&\quad \cdot \left(1 + \frac{2(m-k-1)}{n^3} \right. \\
&\quad \left. + \frac{m-k-1}{n^5}\right) \\
&\quad \cdot \left(1 - \frac{1}{n}\right)\left(1 + \frac{2}{n^3}\right) \\
&\leq \left(1 - \frac{1}{n}\right)^{m-k} \\
&\quad \cdot \left(1 + \frac{2(m-k)}{n^3} + \frac{m-k}{n^5}\right)
\end{aligned}$$

and the lemma follows. ☐

For the next, we assume that $\overline{w}_i \in I\!\!N$ and each node $i$ knows exactly the value $\overline{w}_i$. Lemma 4 and Lemma 5 can also be proved if these assumptions do not hold, however, due to space limitations we show the lemmas under the assumptions mentioned before.

If the maximal load imbalance is more than $32\ln(n)$, then the following lemma holds.

LEMMA 4. *Let $G = (V, E)$ be a graph and denote with $w \in I\!\!R_+^n$ the load in $G$. Let $\sum_{i=1}^n w_i = qn$ be the number of tokens executing random walks of length $a(\frac{d\ln(n)}{\lambda_2})$, where $q > 32\ln(n)$ and $a$ is the constant described in Lemma 2. Then, a constant $c > 2\sqrt{2}$ exists such that, after the random walks, the maximal load imbalance in the system is less than $c\sqrt{q \cdot \ln(n)}$ with probability $1 - o(1/n)$.*

PROOF. Due to Lemma 2, after $a(\frac{d\ln(n)}{\lambda_2})$ steps each token lies on a certain node $i$ with probability $p_i \in [1/n - 1/n^4, 1/n + 1/n^4]$. Then, the probability that on node $i$ are more than $\tau$ tokens is

$$P_{i,\tau} \leq \sum_{j=\tau}^{qn} \binom{qn}{j} (1/n + 1/n^4)^j (1 - 1/n + 1/n^4)^{qn-j}.$$

Due to Lemma 3 we get

$$P_{i,\tau} \leq \sum_{j=\tau}^{qn} \binom{qn}{j} (1/n)^j (1 - 1/n)^{qn-j} (1 + O(1/n^2)).$$

For the next, we only consider

$$\tilde{P}_{i,\tau} = \sum_{j=\tau}^{qn} \binom{qn}{j} (1/n)^j (1 - 1/n)^{qn-j},$$

where $q > 32\ln(n)$. Now, let $\tau = (1 + \frac{\phi}{\sqrt{q}})q$, where $\phi \leq c\sqrt{\ln(n)}$ and $\sqrt{q/4\ln(n)} \geq c > 2\sqrt{2}$. Using a Chernoff type bound [7], we obtain

$$\begin{aligned}
\tilde{P}_{i,\tau} &\leq \left(\frac{1}{1+\phi/\sqrt{q}}\right)^{\left(1+\frac{\phi}{\sqrt{q}}\right)q} \\
&\quad \cdot \left(\frac{1-1/n}{1-\frac{1+\phi/\sqrt{q}}{n}}\right)^{\left(1-\frac{1+\phi/\sqrt{q}}{n}\right)qn} \\
&= \left(\frac{1}{1+\phi/\sqrt{q}}\right)^{\left(1+\frac{\phi}{\sqrt{q}}\right)q} \left(\frac{1}{1-\frac{\phi/\sqrt{q}}{n-1}}\right)^{\left(1-\frac{1+\phi/\sqrt{q}}{n}\right)qn}
\end{aligned}$$

Noting that $\left(1+\frac{1}{x}\right)^{x+\frac{1}{2}} > e > \left(1+\frac{1}{x}\right)^x$ for any $x \geq 1$, we have

$$\begin{aligned}
\tilde{P}_{i,\tau} &\leq \left(1+\frac{\phi}{\sqrt{q}}\right)^{\frac{1}{2}\sqrt{q}\phi\left(1+\frac{\phi}{\sqrt{q}}\right)} \left(\frac{1}{e}\right)^{\sqrt{q}\phi\left(1+\frac{\phi}{\sqrt{q}}\right)} \\
&\quad \cdot e^{\sqrt{q}\phi\left(1-\frac{1+\phi/\sqrt{q}}{n}\right)} \left(\frac{1}{1-\frac{\phi/\sqrt{q}}{n-1}}\right)^{2\sqrt{q}\phi\left(1-\frac{1+\phi/\sqrt{q}}{n}\right)} \\
&\leq e^{\frac{1}{2}\phi^2\left(1+\frac{\phi}{\sqrt{q}}\right)} \left(\frac{1}{e}\right)^{\sqrt{q}\phi\left(1+\frac{\phi}{\sqrt{q}}\right)} \\
&\quad \cdot e^{\sqrt{q}\phi\left(1-\frac{1+\phi/\sqrt{q}}{n}\right)} e^{\frac{2\phi^2}{n-2}} \\
&\leq (1/e)^{\frac{\phi^2}{4} - \frac{2\phi^2}{n-2}}
\end{aligned}$$

A similar bound has also been developed in [27]. It follows that, after the random walks, the overload of each node is less than $\phi\sqrt{q} = c\sqrt{\ln(n) \cdot q}$ (with probability $1 - O(1/(n^{\frac{c^2}{4}-1})))$.

A similar result can also be obtained w.r.t. the underloaded nodes. Summarizing, after the second phase of the randomized algorithm, the maximal load imbalance is $O(\log(n))$ and the number of tokens producing an overload is $O(n\log(n))$ (with probability $1 - o(1/n)$). $\square$

Lemma 5 handles the case when the number of tokens producing overload in the system is $O(n\log(n))$ but larger than $\omega(n)$.

LEMMA 5. *Let $G = (V, E)$ be a graph and let $qn$ be the number of tokens which perform random walks of length $a(\frac{d\ln(n)}{\lambda_2})$ on $G$. If $q$ is larger than some certain constant, then after the random walks the number of tokens producing overload in the system is less than $qn/c$ (with probability $1 - o(1)$), where $c > 4$.*

PROOF. Again, we denote with

$$P_{i,\tau} = \tilde{P}_{i,\tau}(1 + O(1/n^2))$$

the probability that on some node $i$ are more than $\tau$ tokens. For simplicity, we only consider $\tilde{P}_{i,\tau}$ in the rest of the proof.

Let $\tau = (1 + \phi/8)q$. Using the techniques from the proof of Lemma 4, we obtain

$$\begin{aligned}
\tilde{P}_{i,\tau} &= \sum_{i=(1+\phi/8)q}^{nq} \binom{nq}{i} (1/n)^i (1-1/n)^{nq-i} \\
&\leq \left(\frac{8}{8+\phi}\right)^{\frac{(8+\phi)q}{8}} e^{\frac{q\phi}{8} + \frac{\phi^2 q}{32(n-2)}}
\end{aligned}$$

If $\phi \geq 1$, then a constant $l$ exists such that for any $q \geq l$ holds

$$\left(\frac{8}{8+\phi}\right)^{\frac{(8+\phi)q}{8}} e^{\frac{q\phi}{8} + \frac{\phi^2 q}{32(n-2)}} \leq \frac{1}{e^{\phi+2}}.$$

We denote with $X$ the random variable describing the total number of nodes, which have more than $q + \frac{\phi q}{8}$ tokens. Let $X'$ be the random variable representing the number of successes among $n$ Bernoulli trials, where success occurs with probability $\frac{1}{e^{\phi+2}}$ for any trial $i$. Furthermore, let $\mu = E[X']$ be the expected value of $X'$. We know that

$$Pr\{X' - \mu \geq r\} \leq \left(\frac{\mu e}{r}\right)^r$$

for any $r$ [7, 18]. Using the result obtained above, $\mu \leq n\frac{1}{e^{\phi+2}}$ and since $Pr\{X \geq \mu + r\} \leq Pr\{X' \geq \mu + r\}$, we get

$$Pr\left\{X \geq \frac{(e^2+1)n}{e^{\phi+2}}\right\} \leq \left(\frac{en/e^{\phi+2}}{e^2 n/e^{\phi+2}}\right)^{\frac{e^2 n}{e^{\phi+2}}} = (1/e)^{n/e^{\phi}}$$

This implies that if $qn$ tokens execute a random walk of length $a\left(\frac{d}{\lambda_2}\log(n)\right)$ in $G$, then after the random walks there are at most $\frac{(e^2+1)n}{e^{\phi+2}}$ nodes having more than $q + \frac{\phi q}{8}$ tokens (with probability $1 - O(1/n^2)$), where $\phi \in \{1, \ldots, \ln(n) - \ln(\ln(n)) - 1\}$. Thus, after the random walks, the number of tokens producing an overload is less than

$$\left(n - \frac{n(e^2+1)}{e^3}\right) \cdot q/8 + \left(\frac{n(e^2+1)}{e^3} - \frac{n(e^2+1)}{e^4}\right) \cdot q/4 + \ldots.$$

Summarizing, after the third phase of the randomized algorithm, the number of tokens producing an overload is $O(n)$, and the error $||e^k||_2$ is less than $O(\sqrt{n})$. $\square$

Combining Lemma 4 together with Lemma 5, we obtain the following theorem.

THEOREM 4. *Let $G = (V, E)$ be a graph, let $d$ be the maximum vertex degree in $G$ and let $\lambda_2$ be the second smallest* *eigenvalue of the Laplacian of $G$. Furthermore, let $w^0$ be the initial load on $G$ and $K = \max_i |w_i^0 - \overline{w}_i|$ the maximal load imbalance. If $\overline{w}_i \geq 32(n+1)\ln(n)$, then the randomized algorithm reduces the error $||e^k||_2$ to $O(\sqrt{n})$ with probability $1 - o(1/n)$ in $k = O\left(\frac{d}{\lambda_2}(\log(K) + \log(n)\log(\log(n)))\right)$ iteration steps.*

This theorem can be easily generalized to dynamic networks. Dynamic networks are modelled by a sequence $(G_j)_{j\geq 0}$ of graphs, where $G_j$ is the graph which occurs in iteration step $j$ [15]. Let us assume that any graph $G_i$ is connected and let $\mathcal{A}$ be defined such that $\mathcal{A} \geq \frac{1}{B} \sum_{j=kB+1}^{(k+1)B} \frac{d^j_{\max}}{\lambda^j_2}$ for any $k \in I\!N$, where $B$ is a large constant, $d^j_{\max}$ represents the maximum vertex degree and $\lambda^j_2$ denotes the second smallest eigenvalue of the Laplacian of the graph $G_j$. Then, the randomized algorithm reduces $||e^k||_2$ to $O(\sqrt{n})$ in at most $k = O\left(\mathcal{A}(\log(K) + \log(n)\log(\log(n)))\right)$ iteration steps (with probability $1 - o(1/n)$).

## 5. CONCLUSION

Let us now summarize the results of the paper. In Section 3, we derived new spectral bounds on the expansion properties of graphs by using diffusion algorithms. We showed that these bounds are better than the classical bounds for several graph classes. In Section 4, we analyzed the load balancing problem of unit size tokens. Since known diffusion schemes do not completely balance the load for such settings, we presented a randomized strategy to reduce the load imbalance. We also showed that this algorithm can be generalized to dynamic networks.

The randomized algorithm is mainly based on FOS. However, if we could use SOS instead of FOS for this approach, then we would significantly improve the runtime of the algorithm. Another open problem is how to generalize SOS to dynamic networks.

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