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Sparse Topologies with Small Spectrum Size ⋆

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Abstract

One of the fundamental properties of a graph is the number of distinct eigenvalues of its adjacency or Laplace matrix. Determining this number is of theoretical interest as well as of practical impact. Sparse graphs with small spectra exhibit excellent structural properties and can act as interconnection topologies. In this paper, for any n we present graphs, for which the product of their vertex degree and the number of different eigenvalues is small. It is known that load balancing can be performed on such graphs in a small number of steps.

Key words: Laplacian of a graph; Spectra of graphs; Interconnection topologies

1 Introduction

Since their introduction, spectral methods have attracted great attention and have proved to be a valuable tool for theoretical and applied graph theory [7,3]. The set of eigenvalues of the Laplace or adjacency matrix of a graph is called its Laplace or adjacency spectrum; it is one of the most important algebraic invariants of a graph. Although in general a graph is not characterized uniquely by its spectrum, there is a strong connection between its eigenvalues and several structural properties of the graph such as the diameter, bisection width, isoperimetric number etc. See [5] for a selection of results in this area.

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An important parameter which is connected to the spectrum of a graph is its size, i.e. the number of distinct eigenvalues of the adjacency or Laplace matrix. This value is correlated to the symmetry properties of the graph: the only graph which has two distinct eigenvalues is the complete graph and its automorphism group is as rich as possible – the symmetric group. Graphs having three distinct eigenvalues are called strongly regular; they have a diameter of 2 and they possess many interesting properties [17,3]. Another well studied class of highly symmetric graphs are the distance-regular graphs [4]. The size of their spectrum is $1 + diam(G)$ which matches a lower bound for all graphs.

We are interested in constructing graphs whose product of the vertex degree and the number of different eigenvalues is small. One example of such a graph is the hypercube $Q(d)$, which is a vertex and edge symmetric graph. The d-dimensional hypercube has 2^d vertices and a diameter resp. vertex degree of d. Moreover, it has $d + 1$ distinct eigenvalues and is widely used as an interconnection topology. Other graphs such as the complete bipartite graph or the star have only 3 distinct eigenvalues, but because of their high density the product mentioned above is large and they are ill-suited to parallel or distributed network topologies.

Some graphs exist, for which the product of the vertex degree and the spectrum size is smaller than in the case of the hypercube with approximately the same cardinality. One of them is the Petersen graph, which has 10 vertices, vertex degree 3, diameter 2 and 3 different eigenvalues. Another one is the $Cage(6, 6)$, which has 62 vertices, vertex degree 6, diameter 3 and only 4 distinct eigenvalues [3]. A family of graphs with very good spectral and structural properties is the family of the Star graphs [2,1]. The Star graph of order d has $d!$ vertices, vertex degree $d-1$, diameter $\frac{3}{2}(d-1)$ and using the result of Flatto, Odlyzko and Wales it turns out that it has only $2d - 1$ distinct eigenvalues [13].

In this paper we focus our attention on scalable families of sparse graphs (maximal vertex degree $O(\log n)$ where n is the number of vertices) with small spectra. We use the term scalable to denote a family of graphs, which contains an n-vertex graph for each natural n. Our motivation behind the study of this topic stems from the area of load balancing in distributed systems. The load balancing problem is defined as follows. Let $G = (V, E)$ be an arbitrary, undirected, connected graph in which node $v \in V$ contains load $w(v)$. The goal is to determine a schedule which will move load across edges so that the load on each node will be the same at the end. In each step, load can only be moved from any node to its neighbors. Communication between non-adjacent vertices is not allowed. This problem is equivalent to load balancing in synchronous distributed processor networks and parallel machines when we associate a node with a processor, an edge with a communication link and the load with identical, independent tasks [6,20].

As described in e.g. [10], the load balancing process can be split into two phases, the flow computation phase, which computes the balancing flow and the migration phase, in which the load items are moved according to the computed flow. Algorithms for the flow computation phase have been extensively studied in the past [10,11,14–16,20]. The diffusion algorithms studied in the papers mentined above calculate an l_2-optimal flow, where the balancing flow x on the graph $G = (V, E)$ is called l_2-optimal iff $\sqrt{\sum_{e \in E} x_e^2}$ is minimal among all balancing flows. One of these algorithms is the so-called optimal scheme (OPT), which only needs $|Sp(G)|$ iteration steps to compute a balancing flow on a network. Here, $Sp(G)$ denotes the set of eigenvalues of the Laplacian of G. Additionally, in any iteration step a node has to communicate with all of its neighbors, so the total cost of the load balancing algorithm depends on the number of distinct eigenvalues of the graph and on its vertex degree. The number of steps is in fact the product of both. In this paper, we are interested in topologies whose product of the vertex degree and the number of distinct eigenvalues is small.

The graphs constructed in this paper are well-suited to load balancing applications and they can be used as interconnection topologies. Such an architecture could be realized by a bus system. In order to avoid high communication costs, the graphs defined in the next sections are used as virtual topologies in the system. These new topologies have small vertex degrees and support a fast balancing flow computation.

Related to this paper, in [8] the authors consider network topologies on which both phases of the balancing algorithm have good performance. They present several well-connected network topologies with small number of different eigenvalues and show empirically that in particular for random load distribution the migration phase has small time-requirement on such networks.

The outline of the paper is as follows. In Section 2 we present the definitions and lemmas used in this paper for the computation of the eigenvalues of the graphs constructed below. In Section 3 we propose scalable families of trees of constant degree whose spectrum consists of $O(\log^2 n)$ different eigenvalues. Since the tree topology is not very well-suited to our application, we present a scalable family of well connected graphs with at most $O(\log^3 n)$ distinct eigenvalues and a vertex degree of $O(\log n)$. As a by-product, we also present the eigenvalues of the Butterfly graph. In the last section, the previous results are improved in the case of the adjacency spectrum by using another technique and we construct graphs with $O(\log^2 n)$ distinct eigenvalues, where the vertex-degree still remains $O(\log(n))$. Parts of these results have already been presented at the Symposium on Theoretical Aspects of Computer Science in 2001 [12].

2 Definitions and Lemmas

As described in the introduction, our focus in this paper is on scalable families of graphs. A family of graphs $\tilde{G}$ is called scalable, if for each $n \in \mathbb{N}$ there is an n-vertex graph $G \in \tilde{G}$. The identity matrix will be denoted with $\mathbf{I}_n \in \mathbb{R}^{n \times n}$. The symbols $\mathbf{J}_{m,n}$, $\mathbf{0}_{m,n}$ denote $m \times n$ matrices which contain all ones and all zeros, respectively. The spectrum of a matrix $\boldsymbol{A}$ is the set of its eigenvalues: $Sp(\boldsymbol{A}) = \{\lambda \mid \exists \boldsymbol{x} \ : \ \boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}\}$

The operation "$\otimes$" denotes the *Kronecker product*, i.e., if $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ and $\boldsymbol{B} \in \mathbb{R}^{p \times q}$, then the matrix $\boldsymbol{A} \otimes \boldsymbol{B} \in \mathbb{R}^{mp \times nq}$ is obtained from $\boldsymbol{A}$ by replacing every element a_{ij} with the block $a_{ij}\boldsymbol{B}$ [7].

Consider a (weighted) digraph $G = (V, E)$ with $w(e)$ being the weight of an edge e. The *adjacency matrix* of G is the matrix $A_G = (a_{ij})_{1 \leq i,j \leq |V|}$, where $a_{ij} = w(e_{ij})$ if an edge e_{ij} leads from a vertex v_i to a vertex v_j and $a_{ij} = 0$ otherwise (a_{ii} is the weight of a self-loop in v_i). The *Laplacian* of G is the matrix $\Delta_G = D - A_G$, where $D = (d_{ij})$ is a diagonal matrix with entries $d_{ii} = \sum_j a_{ij}$. The spectrum of the adjacency matrix and of the Laplacian of a graph G is denoted as $Sp_A(G)$ and $Sp_\Delta(G)$, respectively, and is called the adjacency and Laplace spectrum of G. Note that for d-regular graphs the adjacency and Laplace spectrum are equivalent. The Laplace spectrum of a d-regular graph consists of all values $\lambda_\Delta = d - \lambda_A$, where λ_A is an eigenvalue of the adjacency matrix and d is the vertex degree of the graph.

Our aim in this paper is to construct graphs, on which the optimal scheme (OPT) has a small number of iteration steps. The optimal scheme is defined as follows. Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be the m nonzero distinct eigenvalues of the Laplacian of the graph $G = (V, E)$. Now, in the t-th iteration step each vertex v_i sends a load of $\frac{1}{\lambda_{t+1}} w_i^t$ to its neighbors, where w_i^t is the load of vertex v_i after the iteration step $t - 1$. So in the t-th iteration step, the load of the vertex v takes the form

$$w^t(v) = w^{t-1}(v) - \sum_{\{v,u\} \in E} \frac{1}{\lambda_t}(w^{t-1}(v) - w^{t-1}(u))$$

After m steps, the load of the network will be totally balanced (see [11]). This implies that the diameter of the graph is less than or equal to $m - 1$ (see also [7]).

The following well known lower bound on the size of the spectrum exists:

Lemma 1 *The number of distinct eigenvalues of the Laplacian or adjacency matrix of an undirected connected graph G with n vertices and maximal degree d is* $\Omega\left(\frac{\log n}{\log d}\right)$.

PROOF. We have already seen above that for the diameter of G it holds $diam(G) \leq |Sp_{\Delta}(G)| - 1$ (e.g. [7]). Through the use of an argument known as Moore's bound,i.e., a graph with maximal degree d and diameter $diam(G)$ can have at most $1 + d + d(d-1) + \cdots + d(d-1)^{diam(G)-1}$ vertices and the lemma follows. In the case of the adjacency spectrum a similar argument holds ([7]). $\square$

In our approach to construct graphs with small spectra we will use the lemmas stated below. The reasoning behind is the following: if the adjacency (or Laplace) matrix $A \in \mathbb{R}^{n\times n}$ of a graph G is formed from a core (with the adjacency matrix $C \in \mathbb{R}^{r\times r}$) and p copies of a graph with the adjacency matrix $B \in \mathbb{R}^{m\times m}$, where the edges between the vertices of C and the vertices of each copy of B is represented by some matrix $S = R^T$, then the eigenvalues of A are obtained through by the union of the spectra of two particular block matrices of size $r + m \times r + m$ and $r \times r$, respectively. It is often convenient to view these block components as matrices of some simpler graphs. As an example, we present the case $p = 2$. We assume that there is no edge between the two different compies of B. Then, A takes the form

$$\begin{pmatrix} C & R^T & R^T \\ R & B & \mathbf{0} \\ R & \mathbf{0} & B \end{pmatrix}.$$

In the following lemma we will show that the spectrum of A corresponds to the spectrum of

$$\begin{pmatrix} C & \sqrt{2}\cdot R^T & \mathbf{0} \\ \sqrt{2}\cdot R & B & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & B \end{pmatrix}.$$

Lemma 2 *Let $n = p\cdot m + r$, $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ be a matrix of the form*

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{C} & \mathbf{J}_{1,p} \otimes \boldsymbol{S} \\ \mathbf{J}_{p,1} \otimes \boldsymbol{R} & \mathbf{I}_p \otimes \boldsymbol{B} + (\mathbf{J}_{p,p} - \mathbf{I}_p) \otimes \boldsymbol{X} \end{pmatrix}$$

where $\boldsymbol{C} \in \mathbb{R}^{r\times r}$, $\boldsymbol{S} \in \mathbb{R}^{r\times m}$, $\boldsymbol{R} \in \mathbb{R}^{m\times r}$ and $\boldsymbol{B}, \boldsymbol{X} \in \mathbb{R}^{m,m}$. Then the spectrum of $\boldsymbol{A}$ can be written as the union

$$Sp(\boldsymbol{A}) = Sp(\boldsymbol{B} - \boldsymbol{X}) \cup Sp \begin{pmatrix} \boldsymbol{C} & \sqrt{p}\cdot \boldsymbol{S} \\ \sqrt{p}\cdot \boldsymbol{R} & \boldsymbol{B} + (p-1)\boldsymbol{X} \end{pmatrix} \qquad (1)$$

PROOF. Consider the matrices

$$\boldsymbol{W} = \begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \boldsymbol{U} \otimes \mathbf{I}_m \end{pmatrix}, \boldsymbol{U} = \frac{1}{\sqrt{p}} \begin{pmatrix} \mathbf{I}_{p-1} & -\mathbf{J}_{p-1,1} \\ \mathbf{J}_{1,p-1} & 1 \end{pmatrix} \tag{2}$$

and

$$\boldsymbol{U}^{-1} = \frac{1}{\sqrt{p}} \begin{pmatrix} (p-1) \cdot \mathbf{I}_{p-1} - \mathbf{J}_{p-1,p-1} & \mathbf{J}_{p-1,1} \\ -\mathbf{J}_{1,p-1} & 1 \end{pmatrix} \tag{3}$$

Using the transformation $Sp(\boldsymbol{A}) = Sp(\boldsymbol{W}\boldsymbol{A}\boldsymbol{W}^{-1})$ we get

$$Sp(\boldsymbol{A}) = Sp \begin{pmatrix} \boldsymbol{C} & \mathbf{0} & \sqrt{p} \cdot \boldsymbol{S} \\ \mathbf{0} & \mathbf{I}_{p-1} \otimes (\boldsymbol{B} - \boldsymbol{X}) & \mathbf{0} \\ \sqrt{p} \cdot \boldsymbol{R} & \mathbf{0} & \boldsymbol{B} + (p-1)\boldsymbol{X} \end{pmatrix}$$

The lemma then follows by interchanging the second and the third row and column block. □

From now on, we will not distinguish between the notation of a matrix and an edge weighted graph. Lemma 2 implies that the spectrum of $\boldsymbol{A}$ is the union of the spectrum of two other graphs. The first is constructed from $\boldsymbol{C}$ and one copy of $\boldsymbol{B} + (p-1)\boldsymbol{X}$, where the matrix $\sqrt{p} \cdot \boldsymbol{R}$ represents the edges between $\boldsymbol{C}$ and $\boldsymbol{B} + (p-1)\boldsymbol{X}$. The second graph is $\boldsymbol{B} - (p-1)\boldsymbol{X}$.

In the following sections we consider the special case where $\boldsymbol{X}$ is either $\mathbf{I}$ or $\mathbf{0}$. We define $\boldsymbol{Q}$ as the smallest matrix with the property: $\boldsymbol{R}^T = \begin{pmatrix} \boldsymbol{Q}^T & \mathbf{0} \end{pmatrix}$. Note that $\boldsymbol{Q}^T$ has the same number of rows as the matrix $\boldsymbol{C}$. Since not all vertices of $\boldsymbol{B}$ will be connected to $\boldsymbol{C}$, the number of columns of $\boldsymbol{Q}^T$ is equal to the number of vertices of $\boldsymbol{B}$, which have an adjacent vertex in $\boldsymbol{C}$.

We denote with $\tilde{T}_{1,A}(C, B, Q, p)$ and with $\tilde{T}_{2,A}(C, B, Q, p)$ the graphs described by $\tilde{T}_{1,A}$ and $\tilde{T}_{2,A}$, respectively, where

$$\tilde{T}_{1,A} = \begin{pmatrix} C & \mathbf{J}_{1,p} \otimes R^T \\ \mathbf{J}_{p,1} \otimes R & \mathbf{I}_p \otimes B \end{pmatrix}, \quad \text{and}$$

$$\tilde{T}_{2,A} = \begin{pmatrix} C & \mathbf{J}_{1,p} \otimes R^T \\ \mathbf{J}_{p,1} \otimes R & \mathbf{I}_p \otimes B + (\mathbf{J}_{p,p} - \mathbf{I}_p) \otimes \mathbf{I} \end{pmatrix}.$$

Now we define a sequence of graphs, where each graph takes the form with the adjacency matrix $\tilde{T}_{1,A}(C, B, Q, p)$ or $\tilde{T}_{2,A}(C, B, Q, p)$. Note, that R describes Q in a unique way, so $\tilde{T}_{1,A}$ and $\tilde{T}_{2,A}$ are well-defined.

Definition 3 *Let $(G_k)_{1\leq k<\infty}$ be a sequence of graphs defined as follows. Let $(C_k)_{1\leq k<\infty}, (Q_k)_{1\leq k<\infty}$ be a sequence of matrices and $(p_k)_{1\leq k<\infty}$ a sequence of integers for any $1 \leq k < \infty$, where $G_k = \tilde{T}_{1,A}(C_k, Q_k, G_{k-1}, p_k)$. Here, Q_k represents the edges between the core C_k of G_k and the core C_{k-1} of G_{k-1}; Q_k has the same number of rows as C_k and the same number of columns as C_{k-1}. There are no edges between C_k and $G_{k-1} \setminus C_{k-1}$.*

The sequence $(G'_k)_{1\leq k<\infty}$ is defined in a similar way to $(G_k)_{1\leq k<\infty}$. For any $1 \leq k < \infty$ let $(C'_k)_{1\leq k<\infty}$ and $(Q'_k)_{1\leq k<\infty}$ be a sequence of matrices and $(p_k)_{1\leq k<\infty}$ be a sequence of integers, so that $G'_k = \tilde{T}_{2,A}(C'_k, Q'_k, G'_{k-1}, p_k)$ where Q'_k represents the edges between the core C'_k of G'_k and the core C'_{k-1} of G'_{k-1}. Again, there is no edge between C'_k and $G'_{k-1} \setminus C'_{k-1}$.

In order to show how to calculate the eigenvalues of the graphs G_k resp. G'_k defined above, we introduce the graph class $\tilde{M}_A(C_1, Q_{1,2}, C_2, Q_{2,3} \ldots, C_k)$.

Definition 4 *Let $C_1, C_2, \ldots C_k$ be a sequence of matrices, where $1 \leq k < \infty$. Furthermore, let $Q_{1,2}, Q_{2,3}, \ldots, Q_{k-1,k}$ also be a sequence of matrices.*

$\tilde{M}_A(C_1, Q_{1,2}, C_2, \ldots, Q_{k-1,n}, C_k)$ denotes the graph with a block tri-diagonal adjacency matrix of the form

$$\begin{pmatrix} C_k & Q^T_{k-1,k} & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ Q_{k-1,k} & C_{k-1} & Q^T_{k-2,k-1} & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & Q_{k-2,k-1} & C_{k-2} & \cdots & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & C_2 & Q^T_{1,2} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & Q_{1,2} & C_1 \end{pmatrix}.$$

In the following lemma we show that the eigenvalues of G_k can be reduced to the eigenvalues of some graphs $\tilde{M}_A$ which are defined above.

Lemma 5 *The spectrum of $G_k = \tilde{T}_{1,A}(C_k, Q_k, G_{k-1}, p_k)$ is the union of the spectra of n graphs $M_1, \ldots, M_k$, where $M_i = \tilde{M}_A(C_i, \sqrt{p_i} \cdot Q_i, C_{i-1}, \sqrt{p_{i-1}} \cdot Q_{i-1}, \ldots, C_1)$ for any $1 \leq i \leq k$.*

Furthermore, the spectrum of $G'_k = \tilde{T}_{2,A}(C'_k, Q'_k, G'_{k-1}, p)$ is the union of the spectra of some graphs $M_{i,j}$, where $1 \leq i \leq k$ and $1 \leq j \leq k-i$ for $i \neq k$ and $j = 1$ for $i = k$. The graph $M_{i,j} = \tilde{M}_A(C_{i,j,0}, \sqrt{p} \cdot Q'_i, C_{i,j,1}, \ldots, C_{i,j,i-1})$, where $C_{i,j,l} = C'_{i-l} + (i - k + p(j-1) + l(p-1)) \cdot \mathbf{I}$.

PROOF. The first statement of the lemma can be proved using induction. When Lemma 2 is applied to G_k, it turns out that the eigenvalues of G_k are the union of the spectra of G_{k-1} and of the graph $\tilde{T}_{1,A}(C'_{k,k-1}, Q_{k-1}, G_{k-2}, p_{k-1})$, where $C'_{k,k-1} = \tilde{M}_A(C_k, \sqrt{p_k} \cdot Q_k, C_{k-1})$. For example if we apply twice Lemma 2 to G_k, where $p_{k-2} = 2$, we get

$$\begin{pmatrix} C_k & \sqrt{p_k}Q_k^T & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \sqrt{p_k}Q_k & C_{k-1} & \sqrt{p_{k-1}}Q_{k-1}^T & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \sqrt{p_{k-1}}Q_{k-1} & C_{k-2} & R_{k-2}^T & R_{k-2}^T \\ \mathbf{0} & \mathbf{0} & R_{k-2} & G_{k-3} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & R_{k-2} & \mathbf{0} & G_{k-3} \end{pmatrix}.$$

Now we assume that if we apply Lemma 2 i times, the eigenvalues will result from the union of the spectra of $G_{k-1} \dots G_{k-i}$ and of the graph

$$\tilde{T}_{1,A}(C^*_{k,k-1,\dots,k-i}, Q^*_{k-i}, G_{k-i-1}, p_{k-i}),$$

with

$$C^*_{k,k-1,\dots,k-i} = \tilde{M}_A(C_k, \sqrt{p_k} \cdot Q_k, C_{k-1}, \dots, \sqrt{p_{k-i+1}} \cdot Q_{k-i+1}C_{k-i})$$

and $Q^{*T}_{k-i} = \begin{pmatrix} 0 \\ Q^T_{k-i} \end{pmatrix}$. When Lemma 2 is applied again, it turns out that the spectrum of G_k can be obtained from the spectra of $G_{k-1} \dots G_{k-i-1}$ and from the spectrum of the graph

$$\tilde{T}_{1,A}(C^*_{k,k-1,\dots,k-i-1}, Q^*_{k-i-1}, G_{k-i-2}, p_{k-i-1}),$$

where

$$C^*_{k,k-1,\dots,k-i-1} = \tilde{M}_A(C_k, \sqrt{p_k} \cdot Q_k, C_{k-1}, \dots, C_{k-i-1})$$

and $Q^{*T}_{k-i-1} = \begin{pmatrix} 0 \\ Q^T_{k-i-1} \end{pmatrix}$. After k steps we obtain the first statement of the lemma.

The second statement is also proved using induction. We assume that after Lemma 2 is applied i times, the eigenvalues of G'_k will result from the union of the spectra of the graphs which are defined as follows. We add self loops with edge weights $(p-1)(l-1)-1$ for $1 \le l \le i$ to the graphs $G'_{k-1} \dots, G'_{k-l}, \dots, G'_{n-i}$. The last graph has the adjacency matrix of the form $\tilde{T}_{2,A}({C'}^*_{k,k-1,\dots,k-i}, {Q'}^*_{k-i}, B'_{k-i-1}, p)$, where B'_{k-i-1} is obtained from G'_{k-i-1} by adding self-loops of weight $i(p-1)$ to the vertices of the graph, ${C'}^*_{k,k-1,\dots,k-i} = \tilde{M}_A({C'}^*_k, \sqrt{p} \cdot Q'_k, {C'}^*_{k-1}, \dots, {C'}^*_{k-i})$ with ${C'}^*_l = {C'}_l + l(p-1)\mathbf{I}$ and ${Q'}^{*T} =$

$\begin{pmatrix} 0 \\ Q'^T \end{pmatrix}$ for any $1 \leq l \leq i$. When Lemma 2 is applied again, we obtain our assumption for the case $i + 1$. After k such steps the spectrum of G'_k will be equal to the union of the spectra of $G'_{k-1}, \ldots, G'_1$ and of $\tilde{M}_A(C_{k,1,0}, \sqrt{p} \cdot Q'_k, \ldots, C_{k,1,k-1})$, where $C_{k,1,l} = C'_{k-l} + l(p-1) \cdot \mathbf{I}$ for any $0 \leq l \leq k-1$. When we use the technique described above for the graphs $G'_{k-1}, \ldots, G'_1$, we obtain the second statement of the lemma. $\square$

3 Spectra of well-suited interconnection topologies

In this section we introduce families of graphs whose product of different eigenvalues and vertex degree is small, and we consider the Butterfly graph which is widely used as an interconnection network. First, we turn our attention to scalable families of graphs. We construct the tree-like graphs $T_{(d,n)}$ with $O\left(\left(\frac{\log n}{\log d}\right)^2\right)$ distinct eigenvalues of both adjacency and Laplace matrix. Since trees are not well suited as interconnection topologies we define a new graph class with much better network properties. In order to compute their eigenvalues, we use the technique presented in the previous section. In the last subsection we present the eigenvalues of the Butterfly.

3.1 Spectra of scalable families of graphs

Now we turn our attention to scalable families of graphs. In the sequel, n will always denote the number of vertices. Let us define the graph class $T_{(d,n)}$ as follows.

Definition 6 *The tree $T_{(d,n)}$ is a rooted tree defined recursively. $T_{(d,1)}$ contains only the root vertex. For $n < d + 1$, $T_{(d,n)}$ is a star with one root and $n - 1$ leaves connected to it. For $n > d$, $T_{(d,n)} = (V, E)$ is constructed as follows. Let $s = \left\lfloor \frac{n-1}{d} \right\rfloor$ and $q = n - 1 - ds$. Let $(V_1, E_1), ..., (V_d, E_d)$ be d disjoint copies of $T_{(d,s)}$ with respective roots $r_1,...,r_d$. Then $V = \{r\} \cup \{v_1, ..., v_q\} \cup \bigcup_{i=1}^{d} V_i$ and $E = \{(r, v_i) \mid i = 1...q\} \cup \{(r, r_i) \mid i = 1..d\} \cup \bigcup_{i=1}^{d} E_i$.*

In an informal context, the construction of a tree $T_{(d,n)}$ involves setting one vertex r as a root, then dividing the remaining $n - 1$ vertices evenly and constructing a number of copies of $T_{(d,s)}$. The roots of these copies are connected to r. Remaining vertices are added as vertices with degree 1 connected to r.

Remark 7 *The graph $T_{(d,n)}$ has n vertices and maximal degree of at most $2d+1$.*

A bound on the number of different eigenvalues of $T_{(d,n)}$ is given in the next theorem.

Theorem 8 *The graph $T_{(d,n)}$ has at most $O\left(\left(\frac{\log n}{\log d}\right)^2\right)$ different eigenvalues.*

PROOF. Consider a graph $T_{(d,n)}$. Now let us define the sequence of integers $(n_j)_{0\leq j<\infty}$, where

$$
\begin{aligned}
n_0 &= n \\
n_k &= \left\lfloor \frac{n_{k-1}-1}{d} \right\rfloor
\end{aligned}
$$

Let l be defined such that $n_{l-1} > d+1$ and $n_l \leq d+1$. Let q_i be the number of vertices of degree 1 which are attached to the root of the subtree on level i, i.e. $q_i = n_{l-i} - 1 - dn_{l-i+1}$ for $i = 1, ..., l$ and $q_0 = n_l - 1$. The adjacency matrix of $T_{(d,n)}$ takes the form of Lemma 5, where $T_{(d,n)} = \tilde{T}_{1,A}(C_n, T_{(d,s)}, Q_n, d)$ with $C_k = \begin{pmatrix} 0 & \mathbf{J}_{1,q_l} \\ \mathbf{J}_{q_l,1} & \mathbf{0} \end{pmatrix}$ and Q_n^T has a column containing one 1 and the rest are zero entries. The other columns contain only zeros.

Using Lemma 5 we see that the spectrum of $T_{(d,n)}$ is the union of spectra of graphs $M_{(k)}$ for $k = 0, ..., l$, where $M_{(k)}$ is a weighted caterpillar – a graph which consists of a path of length k, where the edges of the backbone has weight $\sqrt{d}$ and each vertex v on the path has a set $\tilde{V}_v$ of vertices of degree 1 connected to it. For each such v we can use Lemma 2 with the adjacency matrix of $M_{(k)} \setminus \{v\}$ as $\boldsymbol{C}$, and we eventually obtain a graph with at most $2(l+1)$ vertices and a number of single-vertex graphs. Therefore, $Sp(T_{(d,n)})$ is the union of the spectra of the $M_{(i)}$'s, $i = 0, \ldots, l$, where each of them has at most $2(l+1)$ vertices, and of several graphs consisting of only a single vertex with eigenvalue 0. As $l = O(\log_d n) = O\left(\frac{\log n}{\log d}\right)$ the result follows.

In the case of Laplace spectrum, adding self loops in the same way as in the proof of Lemma 1 reduces the problem to the adjacency spectrum of a similar graph. $\square$

The choice of the parameter d results in the construction of graphs with different properties. Depending on the application, one may ask either for a minimal number of eigenvalues or for the minimal value of $deg \cdot |Sp|$, where deg is the maximal degree of the graph. In order to get a small number of eigenvalues, some reasonable large value of d should be chosen (e.g., $d = O(\log n)$), and in order to get a small value of the product, some small value should be chosen (e.g., $d = 2$).

However, trees are extremely ill-suited to the application of load balancing because of their poor connectivity properties. To overcome this weakpoint, we provide another construction of a graph $H_{(n)}$ with $O\left((\log n)^3\right)$ distinct eigenvalues which is much more suitable as a topology for load balancing.

Definition 9 *Every graph $H_{(n)}$ has a set of distinguished vertices (core), which is denoted by $C(H_{(n)})$. The graph $H_{(n)}$ is defined recursively as follows. $H_{(1)} = C(H_{(1)}) = K_1$, $H_{(2)} = C(H_{(2)}) = K_2$. For $H_{(n)} = (V, E)$, $n > 2$, let $H_{(n)}$ be of the form $\tilde{T}_{2,A}(C_n, H_{(\lfloor \frac{n-1}{2} \rfloor)}, Q_n, 2)$, where*

$$C_n = \begin{cases} K_2 \text{ if } n \text{ is even,} \\ K_1 \text{ if } n \text{ is odd,} \end{cases}$$

and $Q_n = \mathbf{J}$.

In an informal context, in order to construct $H_{(n)}$, first construct two copies of $H_{(\lfloor \frac{n-1}{2} \rfloor)}$ and connect the corresponding vertices. The remaining 1 or 2 vertices of $H_{(n)}$ form its core and are connected mutually and to all vertices of the cores of both $H_{(\lfloor \frac{n-1}{2} \rfloor)}$s (see Figure 1). It follows that $H_{(n)}$ has a vertex degree of at most $\log n + 5$.

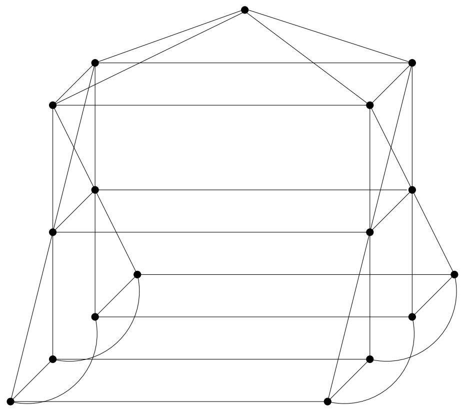

Fig. 1. The graph $H_{(17)}$.

Theorem 10 *The graph $H_{(n)}$ has at most $O(\log^3 n)$ distinct eigenvalues.*

PROOF. Similar to the proof of Theorem 8, let $(n_j)_{0 \le j < \infty}$ be a sequence of integers, such that

$$n_0 = n$$
$$n_k = \left\lfloor \frac{n_{k-1} - 1}{2} \right\rfloor$$

where l is defined by $n_{l-1} > 2$ and $n_l \leq 2$. Let c_i be the size of the core of the i-th subgraph, i.e. $c_i = n_{l-i} - 2n_{l-i+1}$ for $i = 1,..,l$ and $c_0 = n_l$. The adjacency matrix of $H_{(n)}$ takes the form described in Lemma 5, where $H_{(n)} = \tilde{T}_{2,A}(C_n, H_{(n_1)}, Q_n, 2)$ with $C_n = \mathbf{J}_{c_n,c_n} - \mathbf{I}_{c_n}$ and $Q_n^T = \mathbf{J}_{c_n,c_{n_1}}$. Using Lemma 5 we see that the spectrum of $H_{(n)}$ is the union of spectra of some matrices $\boldsymbol{M}_{i,j}$, here $i = 1, ..., l$ and $j = 1, \ldots, l-i$. As every core has the size of at most two, $\boldsymbol{M}_{i,j}$ is a square matrix with dimension of at most $2(l+1)$. This results in at most $O(l^2)$ matrices of dimension $O(l)$ which altogether amounts to at most $O(l^3)$ distinct eigenvalues.

Again, a similar technique can be used to compute the number of different eigenvalues of the Laplacian. $\square$

3.2 The spectrum of the Butterfly graph

The spectral properties of some interconnection networks like grids, tori, hypercubes [7] or DeBruijn graphs [9] have widely been investigated. Now we present the adjacency and Laplace spectrum of the Butterfly with and without wrap-around edges. Note that Schmidt has independently computed the eigenvalues of the Butterfly [18].

The Butterfly graph without wrap-around edges ($BF_{(k)}$) consists of $k+1$ columns, each column containing 2^k vertices labeled with unique binary strings of length k. An edge connects two vertices in $BF_{(k)}$ if and only if they are in consecutive i-th and $(i+1)$-st columns and their labels are either equal or differ only in the i-th bit.

Theorem 11 *The adjacency spectrum $Sp_A(BF_{(k)})$ of the Butterfly graph $BF_{(k)}$ is*

$$Sp_A(BF_{(k)}) = \left\{ 4\cos\left(\frac{\pi i}{j+1}\right) \mid 1 \leq i \leq j \leq k+1 \right\}$$

PROOF. The Butterfly graph $BF_{(k)}$ can be recursively constructed by taking two copies of $BF_{(k-1)}$, adding one column of 2^k vertices and connecting these vertices to both $BF_{(k-1)}$'s. Thus the Butterfly graph $BF_{(k)}$ has the structure $\tilde{T}_{1,A}(C_k, BF_{k-1}, Q_k, 2)$ of Lemma 5, where $\boldsymbol{C}_k = \mathbf{0}$ and $\boldsymbol{Q}_k^T = \begin{pmatrix} \mathbf{I}_{2^{k-1}} \\ \mathbf{I}_{2^{k-1}} \end{pmatrix}$. Using Lemma 5, it holds that $Sp_A(BF_{(k)}) = \cup_{i=0}^{k} \{Sp(\boldsymbol{M}_i)\}$. The matrix $\boldsymbol{M}_i$ can be viewed as an adjacency matrix of a weighted binary tree with all edges having weight $\sqrt{2}$. After permuting the vertices, Lemma 5 can be used again with $C_k = (0)$, $p_k = 2$ and $Q_k = (\sqrt{2})$. Thus $Sp_A(BF_{(k)}) = \cup_{i=0}^{k} \{Sp(\boldsymbol{M}_i)\} = \cup_{i=0}^{k} \cup_{j=0}^{i} \left\{Sp(\boldsymbol{M}'_j)\right\}$. The matrix $\boldsymbol{M}'_j$ can be viewed as an adjacency matrix of a weighted path with $j+1$ vertices where all edges have a weight of 2. Since the adjacency spectrum of a path with l vertices is (e.g. [7])

$$Sp_A\left(P_{(l)}\right) = \left\{2\cos\left(\frac{\pi i}{l+1}\right) \mid i = 1, ..., l\right\},$$

it holds that $Sp(\boldsymbol{M}'_j) = 2Sp(P_{j+1}) = 4\left\{\cos\left(\frac{\pi l}{j+2}\right) \mid l = 1, ..., j+1\right\}$. □

The Laplace spectrum of the Butterfly can also be computed using similar techniques.

Theorem 12 *The Laplace spectrum $Sp_\Delta(BF_{(k)})$ of the Butterfly graph $BF_{(k)}$ is*

$$\begin{aligned} Sp_\Delta(BF_{(k)}) = &\left\{4 - 4\cos\left(\frac{\pi i}{j+1}\right) \mid 0 \le i \le j \le k\right\} \cup \\ &\cup \left\{4 - 4\cos\left(\frac{\pi(2i-1)}{2j+1}\right) \mid 1 \le i \le j \le k\right\} \end{aligned}$$

PROOF. Let $BF'_{(k)}$ be a graph which is obtained from $BF_{(k)}$ by adding a self-loop to every vertex of degree 2. Then the Laplacian spectrum of the Butterfly takes the form $Sp_\Delta\left(BF_{(k)}\right) = \left\{4 - \lambda \mid \lambda \in Sp_A\left(BF'_{(k)}\right)\right\}$. Using similar arguments to Theorem 11 we see that $Sp_A(BF'_{(k)})$ is the union of the adjacency spectra of three sets of weighted graphs which all have edges of weight 2:

- paths with 1 to $k-1$ vertices
- paths with 1 to k vertices which all have a self-loop added to one end-vertex
- a path with $k+1$ vertices which has a self-loop added to both end-vertices

The adjacency spectrum of a path with l vertices is (e.g. [7])

$$Sp_A\left(P_{(l)}\right) = \left\{2\cos\left(\frac{\pi i}{l+1}\right) \mid i = 1, ..., l\right\}.$$

The adjacency spectrum of a "double-looped" path with l vertices is

$$Sp_A\left(P''_{(l)}\right) = \left\{2 - \lambda \mid \lambda \in Sp_\Delta(P_{(l)})\right\} = \left\{2\cos\left(\frac{\pi i}{l}\right) \mid i = 0, ..., l-1\right\}.$$

The characteristic polynomial of the “one-looped” path is given by the reccurence

$$P'_0(\lambda) = 1$$
$$P'_1(\lambda) = \lambda - 1$$
$$P'_l(\lambda) = \lambda P'_{l-1}(\lambda) - P'_{l-2}(\lambda)$$

Using a technique from [19] the roots of the characteristic polynomial are

$$Sp_A\left(P'_{(l)}\right) = \left\{ 2\cos\left(\frac{\pi(2i-1)}{2l+1}\right) \mid i = 1..l \right\}.$$

□

For the next, we consider the Butterfly graph with wrap-around edges $wBF_{(k)}$. The structure of the wrapped Butterfly is very similar to the Butterfly without wrap-around edges. The graph $wBF_{(k)}$ has k columns with each column consisting of 2^k vertices. An edge connects two vertices if and only if they are in consecutive levels (i and $i+1$) and their labels are either equal or differ only in the i-th bit, **or** one of the adjacent vertices is in the first level, the other one is in the last level and their labels are equal or differ in precisely the last bit.

Theorem 13 *The adjacency spectrum $Sp_A(wBF_{(k)})$ of the wrapped Butterfly $wBF_{(k)}$ is*

$$Sp_A(wBF_{(k)}) = \left\{ 4\cos\left(\frac{\pi i}{j+1}\right) \mid 0 \le i \le j \le k \right\} \cup$$
$$\cup \left\{ 4\cos\left(\frac{\pi(2i)}{k}\right) \mid 1 \le i \le k \right\}$$

PROOF. The adjacency matrix of the wrapped Butterfly has the form:

$$A_{wBF_{(k)}} = \begin{pmatrix} F_{2^{k-1}} & Q_{2^{k-1}} \\ Q_{2^{k-1}} & F_{2^{k-1}} \end{pmatrix},$$

where $F_{2^{k-1}}$ represents the adjacency matrix of the first 2^{k-1} rows and $Q_{2^{k-1}}$ represents the edges between the first 2^{k-1} and the last 2^{k-1} rows. The eigenvalues of $A_{wBF_{(k)}}$ consist of the eigenvalues of the matrices $F_{2^{k-1}} + Q_{2^{k-1}}$ and $F_{2^{k-1}} - Q_{2^{k-1}}$. The matrix $F_{2^{k-1}} + Q_{2^{k-1}}$ corresponds to a graph constructed from a wrapped Butterfly by adding an extra vertex to each row and connecting this extra vertex to two adjacent nodes in the corresponding row with an edge of weight 2. The matrix $F_{2^{k-1}} - Q_{2^{k-1}}$ represents a wrapped Butterfly of dimension k and 2^{k-1} separate vertices. As we can observe, both matrices have the form

$$\begin{pmatrix} F_{2^{k-2}} & Q_{2^{k-2}} \\ Q_{2^{k-2}} & F_{2^{k-2}} \end{pmatrix} \text{ and } \begin{pmatrix} F'_{2^{k-2}} & Q'_{2^{k-2}} \\ Q'_{2^{k-2}} & F'_{2^{k-2}} \end{pmatrix}$$

When the transformation described above is applied $k-1$ times, it turns out that the eigenvalues of the wrapped Butterfly are the union of the eigenvalues of

- paths with 1 to k vertices, where all edges have a weight of 2
- a cycle with k vertices, where all edge-weights equal 2

Since the eigenvalues of paths and cycles are known [7], we easily obtain the theorem. □

Since the wrapped Butterfly is a regular graph of degree 4, the following corollary holds

Corollary 14 *The Laplace spectrum $Sp_\Delta(wBF_{(k)})$ of the wrapped Butterfly $wBF_{(k)}$ is*

$$\begin{aligned} Sp_\Delta(wBF_{(k)}) = & \left\{ 4 - 4\cos\left(\frac{\pi i}{j+1}\right) \mid 0 \le i \le j \le k \right\} \cup \\ & \cup \left\{ 4 - 4\cos\left(\frac{\pi(2i)}{k}\right) \mid 1 \le i \le k \right\} \end{aligned}$$

4 Graphs with small adjacency spectrum

In this section we improve the results of the previous section w.r.t. the product of different eigenvalues of the adjacency matrix and the vertex degree. However, we can not achieve the same improvements for the Laplace matrix. Using another technique we construct a scalable family of graphs $G_{(n)}$ with $O\left((\log n)^2\right)$ distinct eigenvalues of the adjacency matrix and a vertex degree of $O(\log n)$.

The graph $G_{(n)}$ is constructed by taking a hypercube, subdividing each edge and then replicating the vertices from the original hypercube a number of times (see Figure 2).

Definition 15 *Let d ($d > 0$) satisfy the inequality $2^{d-1}(d+2) \le n < 2^d(d+3)$.*

Let $\{\Delta_i\}_{i=0}^{d}$ be a sequence which is defined as follows:

$$\Delta_i = \begin{cases} 0 & \textit{for } i > \left\lfloor \frac{d}{2} \right\rfloor, \\ \Big(n - 2^{d-1}(d+2)\Big) \bmod 2^d & \textit{for } i = \left\lfloor \frac{d}{2} \right\rfloor, \\ \Delta_{i+1} \bmod \binom{d}{i+1} & \textit{for } 0 \le i < \left\lfloor \frac{d}{2} \right\rfloor, \end{cases}$$

where $a \bmod b = a - b \cdot \left\lfloor \frac{a}{b} \right\rfloor$.

Let $Q(d)$ be a hypercube. We refer to the vertices as binary strings from $\{0,1\}^d$. The k-th level is defined as $\tilde{L}_k = \Big\{x \in \{0,1\}^d \mid \#_1(x) = k\Big\}$.

The graph $G_{(n)}$ is defined as follows. Consider the graph $S(Q(d))$ with $2^{d-1}(d+2)$ vertices which are obtained from the hypercube $Q(d)$ by the subdivision of each edge. For each node x from the original graph $Q(d)$ add a set $\tilde{V}_x$ of isolated vertices of cardinality $|\tilde{V}_x| = \left\lfloor \frac{\Delta_k}{\binom{d}{k}} \right\rfloor + \left\lfloor \frac{n - 2^{d-1}(d+2)}{2^d} \right\rfloor$ where $x \in \tilde{L}_k$. For each $y \in \tilde{V}_x$ add edges (y, v) for all edges (x, v) from $S(Q(d))$.

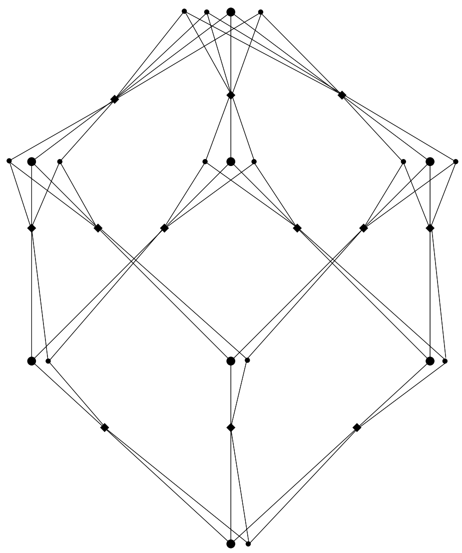

Fig. 2. The graph $G_{(33)}$.

Since the diameter of a hypercube $Q(d)$ with $n = 2^d$ vertices is $d = \log(n)$, it follows from the construction of the graph that:

Remark 16 *The graph $G_{(n)}$ ($n > 2$) has n vertices and $diam(G) \le 2 \log n$.*

The maximal degree of $G_{(n)}$ also follows from the definition.

Remark 17 *The maximal degree of $G_{(n)}$ ($n > 2$) is at most $3 \log n + o(\log n)$.*

To compute the number of distinct eigenvalues of $G_{(n)}$ we need two further lemmas.

Lemma 18 *[7] Let $\boldsymbol{M}$ be a non-singular square matrix, then*

$$\begin{vmatrix} \boldsymbol{M} & \boldsymbol{N} \\ \boldsymbol{P} & \boldsymbol{Q} \end{vmatrix} = |\boldsymbol{M}| \cdot |\boldsymbol{Q} - \boldsymbol{P}\boldsymbol{M}^{-1}\boldsymbol{N}|$$

Lemma 19 *Let G be a graph which is obtained from the hypercube $Q(d)$ using the following method. For every vertex x add a self-loop with some weight w_k^2, where $x \in \tilde{L}_k$. Each edge which connects the vertices on level k with vertices on level $k+1$ has weight $w_k \cdot w_{k+1}$. Then the graph G has at most $O(d^2)$ distinct eigenvalues.*

PROOF. Let $Q_1(l)$ and $Q_2(l)$ denote two hypercubes of dimension l which are obtained from the graph G by considering only the levels $\frac{d-l}{2}, \ldots, \frac{d+l}{2} - 1$, where the edge weights and self-loops are added as in the graph G, and which are generated by one vertex of level $\tilde{L}_{\frac{d-l}{2}}$ each. First, we show that if the entry of the eigenvector $x[v] \neq 0$ for the vertex $v \in \tilde{L}_0$, then in each level the entries of the eigenvector for the vertices of one level have the same value. Formally, if $v, w \in \tilde{L}_k$ then $x[v] = x[w]$ for any $1 \leq k \leq d$ and x eigenvector of G. By permuting the vertices of a level we obtain the same graph. Therefore we can apply any permutation to an eigenvector on the vertices of a level and the vector obtained will also be an eigenvector of the graph. We say, that we apply a permutation π to a vector x on the vertices of a level, when we set $x[v_{\pi_1}], \ldots, x[v_{\pi_{|\tilde{L}_k|}}]$ for the vertices $v_1, \ldots, v_{|\tilde{L}_k|}$ as the entries of a vector. This vector will also be an eigenvector of the graph, if x is an eigenvector of G. Now the first level $\tilde{L}_0$ consists of only one vertex. Let the entry of an eigenvector x for $v \in \tilde{L}_0$ be 1. Then we apply the following permutations to x on the level $\tilde{L}_1$:

$$\begin{pmatrix} 1 \; 2 \ldots & |\tilde{L}_1| \\ 2 \; 3 \ldots & 1 \end{pmatrix} \begin{pmatrix} 1 \; 2 \ldots & |\tilde{L}_1| \\ 3 \; 4 \ldots & 2 \end{pmatrix} \cdots \begin{pmatrix} 1 & 2 \ldots & |\tilde{L}_1| \\ |\tilde{L}_1| & 1 \ldots & |\tilde{L}_1| - 1 \end{pmatrix}.$$

Now if we take the sum of the eigenvectors which are obtained in this way, then we get the same entry of the eigenvector for each vertex of the level. Let us now assume that the vertices on a level k have the same entry. For the next we use the permutations described above for the $k+1^{th}$ level by replacing L_1 with L_{k+1}. Then, the entries of the eigenvector on level $k+1$ for each vertex will have the same value.

For the next, we consider the case where $x[v] = 0$ for $v \in \tilde{L}_0$. It follows, that the sum of the entries of the eigenvector for the vertices on level $\tilde{L}_1$ have to be 0. If some vertices v_1 and v_2 on this level exist, which have a nonzero entry in the eigenvector, then we can apply some permutations to this eigenvector on the vertices of this level and obtain $x[v_1] = 1$ and $x[v_2] = -1$ (the other entries will be zero). Now we can apply similar permutations to the levels below and we then conclude that only the vertices contained in the hypercubes $Q_1(d-2)$ and $Q_2(d-2)$ of dimension $d-2$ generated by the vertices v_1 and v_2 have nonzero entries. Note that for any vertex v which is adjacent to v_1 or v_2 it holds that $x[v] = 0$. Furthermore the entries of these eigenvectors have the same value on a level and in one subcube. Analog results can be obtained for the case where $x[v] = 0$ for any $v \in \tilde{L}_i$, where $0 \le i \le k$ with an arbitrary k, $0 \le k < [\frac{d}{2}]$. As a result, G can have at most $O(d^2)$ different eigenvalues. □

Theorem 20 *The adjacency matrix of $G_{(n)}$ $(n > 2)$ has $O(\log^2 n)$ different eigenvalues.*

PROOF. Let A be the adjacency matrix of $G_{(n)}$. Consider an arbitrary vertex x from the original hypercube $Q(d)$ together with all vertices from $\tilde{V}_x$. Let C be the adjacency matrix of a graph obtained from $G_{(n)}$ by removing the vertices $\{x\} \cup \tilde{V}_x$. Clearly A takes the form of Lemma 2, i.e., $A = \begin{pmatrix} C & \mathbf{J}_{1,p} \otimes R \\ \mathbf{J}_{p,1} \otimes R^T & \mathbf{I}_p \end{pmatrix}$, where R represents the edges between $\{x\} \cup \tilde{V}_x$ and the remaining vertices of $G_{(n)}$. After Lemma 2 is iteratively applied to the sets $\{x\} \cup \tilde{V}_x$ for each $x \in Q(d)$, we get a matrix A' which has the same spectrum as A (except possibly the eigenvalue 0). The matrix A' can be viewed as the adjacency matrix of a weighted graph G' which is defined as follows. Consider the subdivided hypercube $S(Q(d))$. Each edge incident with a vertex x from $Q(d)$ has a weight $\sqrt{1 + |\tilde{V}_x|}$.

Clearly, $A' = \begin{pmatrix} \mathbf{0} & R^\top D \\ DR & \mathbf{0} \end{pmatrix}$, where $R \in \mathbb{R}^{2^d \times d2^{d-1}}$ is the vertex-edge incidency matrix of $Q(d)$ and $D \in \mathbb{R}^{2^d \times 2^d}$ is a diagonal matrix with diagonal entries $d_i = \sqrt{1 + |\tilde{V}_i|}$. Now using Lemma 18 we get $|\lambda\mathbf{I} - A'| = \begin{vmatrix} \lambda\mathbf{I} & -R^\top D \\ -DR & \lambda\mathbf{I} \end{vmatrix} = |\lambda\mathbf{I}| \cdot |\lambda\mathbf{I} - DR\frac{1}{\lambda}R^\top D| = \lambda^{d2^{d-1}} \cdot |\frac{1}{\lambda}(\lambda^2\mathbf{I} - DRR^\top D)| = \lambda^{d2^{d-1}-2^d} \cdot |\lambda^2\mathbf{I} - DRR^\top D|$. Using the fact that $RR^\top = Q + d\mathbf{I}$, where Q is the adjacency matrix of hypercube $Q(d)$, we can conclude that $G_{(n)}$ has at most $2m+1$ distinct eigenvalues, where m is the number of distinct eigenvalues of a graph G'' which is defined as follows. Consider a hypercube $Q(d)$. For each $k = 0, ..., d$, let $w_k = \sqrt{1 + |\tilde{V}_k|}$. Add to each vertex x from $Q(d)$ a self-loop with weight w_k^2, where $x \in \tilde{L}_k$. Each edge which connects vertices in levels k and $k+1$ has weight $w_k \cdot w_{k+1}$. Since G'' takes the form of a weighted hypercube described in Lemma 19, it follows that G'' has only $O(\log^2 n)$ distinct eigenvalues. □

Unfortunately, a similar theorem could not be obtained for the Laplace spectrum of these graphs, because the diagonal elements of the Laplacian do not equal zero. Therefore, we are not able to determine how fast the optimal diffusion load balancing scheme OPT converges on these topologies. Note that these graphs are more suitable as interconnection topologies than the graph classes defined in the previous subsection. However, the lower bound presented at the beginning of this section is still not tight for $G_{(n)}$. To achieve this, we can use a simple trick in order to reduce the number of different eigenvalues of the adjacency matrix of $G_{(n)}$ to $O(\log(n))$.

If we assign the value $1/\sqrt{|\tilde{V}_x| + 1}$ to each edge (y, v) of $G_{(n)}$, where $y \in \{x\} \cup \tilde{V}_x$ and x is a vertex of the original hypercube $Q(d)$, then the lower bound w.r.t. the size of the spectrum will be nearly tight. The resulting graph is a weighted graph. However, the polynomial-based diffusion schemes also work for weighted graphs.

Similar transformations can also be applied to other graphs. For example, the Star graph can be used instead of the hypercube to perform the described transformations and then, a scalable graph class, for which the lower bound w.r.t. the size of its adjacency spectrum is achieved, can immediately be constructed. Again, we can not obtain similar results w.r.t. the size of the Laplace spectrum.

5 Conclusion

We have investigated families of sparse graphs which have a small number of distinct eigenvalues of their adjacency or Laplace matrices. We introduced scalable families of such graphs with degree of at most $O(\log n)$ and a polylogarithmic number of distinct eigenvalues (of the adjacency and Laplace matrix) w.r.t. the number of vertices. In the last section, we presented another technique for the construction of scalable graphs with vertex degree $O(\log n)$ and $O(\log n)$ distinct eigenvalues of the adjacency matrix.

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References

[1] Akers, S.B., Harel, D., Krishnamurthy, B.: *The Star Graph: An Attractive Alternative to the n-Cube*, Proc. of the International Conference on Parallel Processing, 1987, pp.393–400

[2] Akers, S.B., Krishnamurthy, B.: *A Group-Theoretic Model for Symmetric Interconnection Networks*, IEEE Transactions on Computers **38**, 1989, pp. 555–565

[3] Biggs, N. L.:*Algebraic Graph Theory*, (2nd ed.), Cambridge University Press, Cambridge, 1993

[4] Brouwe, A. E., Cohen, A. M., Neumaier A.: *Distance-Regular Graphs*, Springer Verlag 1989

[5] Chung, F.R.K.: *Spectral Graph Theory*, American Mathematical Society, 1994

[6] Cybenko, G: *Load balancing for distributed memory multiprocessors* J. of Parallel and Distributed Computing **7**, 1989, pp. 279–301

[7] Cvetković, D. M., Doob, M., Sachs, H.: *Spectra of graphs, Theory and Application*, Academic Press, 1980

[8] Decker, T., Monien, B., Preis, R.: *Towards Optimal Load Balancing Topologies*, Proceedings of the 6th EuroPar Conference, LNCS **1900**, 2000, pp. 277-287

[9] Delorme, C., Tillich, J. P.: *The spectrum of DeBruijn and Kautz Graphs*, European Journal of Combinatorics **19**, 1998, pp. 307–319

[10] Diekmann R., Frommer, A., Monien, B.: *Efficient schemes for nearest neighbor load balancing*, Parallel Computing **25**, 1999, pp. 789–812

[11] Elsässer, R., Frommer, A., Monien, B., Preis, R.: *Optimal and Alternating-Direction Load Balancing*, EuroPar'99, LNCS **1685**, 1999, pp. 280–290

[12] Elsässer, R., Královič, R., Monien, B.: *Scalable Sparse Topologies with Small Spectrum*, STACS'01, LNCS **2010**, 2001, pp. 218-229

[13] Flatto, L., Odlyzko, A.M., Wales, D.B.: *Random Shuffles and Group Representations*, The Annals of Probability **13**, 1985 pp. 154–178

[14] Ghosh, B., Muthukrishnan, S., Schultz, M.H.: *First and Second order diffusive methods for rapid, coarse, distributed load balancing*, SPAA, 1996, pp. 72–81

[15] Hu, Y.F., Blake, R.J.: *An improved diffusion algorithm for dynamic load balancing*, Parallel Computing **25**, 1999, pp. 417–444

[16] Hu, Y.F., Blake, R.J., Emerson, D.R.: *An optimal migration algorithm for dynamic load balancing*, Concurrency: Prac. and Exp. **10**, 1998, pp. 467–483

[17] Hubaut, X. L.:*Strongly Regular Graphs*, Discrete Math. **13**, 1975, pp. 357–381

[18] Schmidt, G.:Über die Spektren wichtiger Graphklassen, Bachelor Thesis, 2001

[19] Tillich, J.-P.: *The spectrum of the double-rooted tree*, personal communication

[20] Xu, C., Lau, F.C.M.: *Load Balancing in Parallel Computers*, Kluwer, 1997