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# Which is the Worst-case Nash Equilibrium?*

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**Abstract**

A *Nash equilibrium* of a *routing network* represents a stable state of the network where no *user* finds it beneficial to unilaterally deviate from its routing strategy. In this work, we investigate the structure of such equilibria within the context of a certain game that models selfish routing for a set of $n$ users each shipping its *traffic* over a network consisting of $m$ parallel *links.* In particular, we are interested in identifying the *worst-case* Nash equilibrium – the one that maximizes *social cost.* Worst-case Nash equilibria were first introduced and studied in the pioneering work of Koutsoupias and Papadimitriou [10].

More specifically, we continue the study of the *Conjecture of the Fully Mixed Nash Equilibrium,* henceforth abbreviated as FMNE *Conjecture,* which asserts that the *fully mixed Nash equilibrium,* when existing, is the worst-case Nash equilibrium. (In the *fully mixed Nash equilibrium,* the *mixed strategy* of each user assigns (strictly) positive probability to every link.) We report substantial progress towards identifying the validity, methodologies to establish, and limitations of, the FMNE Conjecture.

Our first set of results provides strong evidence for the validity of the FMNE Conjecture by establishing it in a number of interesting instances of the routing game. All these instances fall within the model of *related links,* where there is a capacity $c^j$ for each link $j$, and the individual *delay* of user $i$ (with traffic $w_i$) on link $j$ is $\frac{w_i}{c^j}$. Specifically, we prove:

- The FMNE Conjecture is valid if all mixed strategies of the players are mutually disjoint, while link *capacities* vary arbitrarily.

  In consequence, this implies the validity of the conjecture in case there are only two users and the network consists of just two links.
- The FMNE Conjecture is valid when there are just two links with identical capacities and an arbitrary (but even) number of users with identical *traffics.*

  This is shown through an elegant combinatorial analysis, involving properties of binomial coefficients, that may be of independent interest.
- The FMNE Conjecture is valid when there are just two users with identical traffics and an arbitrary number of links with arbitrary capacities.

  This is shown through elegant combinatorial arguments and an involved analytical approach.

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Our second set of results applies to the case of *unrelated links,* where no relation is assumed between the delays of a single user on different links. We obtain here simple instances of the selfish routing game for which the FMNE Conjecture is valid, and some other simple instances of it for which the FMNE Conjecture fails. Specifically, we prove:

- The FMNE Conjecture is valid if all strategies of the users are pure and $n \leq m$; that is, the social cost of any pure Nash equilibrium is no more than the social cost of the fully mixed Nash equilibrium.
- The FMNE Conjecture is valid if $n = 2$ and $m = 2$, while it fails when $n = 3$ and $m = 2$.

Most importantly, our results for the case of unrelated links provide concrete limitations on the generality of the FMNE Conjecture, since they imply that it is not valid for *every* game.

# 1 Introduction

**Motivation and Framework**. *Nash equilibrium* [14, 15] is arguably the most important solution concept in (non-cooperative) Game Theory.* It represents a stable state of the play of a strategic game in which each player holds an accurate opinion about the (expected) behavior of other players and acts *rationally.* Understanding the combinatorial structure of Nash equilibria is a necessary prerequisite to either designing efficient algorithms to compute them, or establishing corresponding hardness and thereby designing (efficient) approximation algorithms.†

In this work, we embark on a systematic study of the combinatorial structure of Nash equilibria in the context of a simple routing game that models selfish routing over a non-cooperative network such as the Internet. This game was originally introduced in a pioneering work of Koutsoupias and Papadimitriou [10]; that work defined *coordination ratio* (also known as *price of anarchy* [17]) as a worst-case measure of the impact of the selfish behavior of users on the efficiency of routing over a non-cooperative network operating at a Nash equilibrium. As a worst-case measure, the coordination ratio bounds the maximum loss of efficiency due to selfish behavior of users at the worst-case Nash equilibrium; in sharp contrast, the principal motivation of our work is to identify the *actual* worst-case Nash equilibrium of the selfish routing game.

Within the framework of the selfish routing game of Koutsoupias and Papadimitriou [10], we assume a collection of $n$ *users,* each employing a *mixed strategy,* which is a probability distribution over $m$ parallel *links,* to control the shipping of its own assigned *traffic.* For each link, a *capacity* specifies the rate at which the link processes traffic. In a Nash equilibrium, each user selfishly routes its traffic on those links that minimize its *expected latency cost,* given the network congestion caused by the other users. The *social cost* of a Nash equilibrium is the expectation, over all random choices of the users, of the maximum, over all links, *latency* through a link. The *worst-case* Nash equilibrium is one that maximizes social cost.

Our study distinguishes between *pure* Nash equilibria, where each user chooses exactly one link (with probability one), and *mixed* Nash equilibria, where the choices of each user are modeled by a probability distribution over links. Of special interest to our work is the *fully mixed Nash equilibrium* [12], where each user chooses each link with non-zero probability; henceforth, denote $\mathbf{F}$ the fully mixed Nash equilibrium. We will also introduce and study *disjointly mixed Nash equilibria,* where (loosely speaking) mixed strategies of different users do not intersect.

Allowing link capacities to vary arbitrarily gives rise to the standard model of *related links,* also known as model of *uniform links* in the scheduling literature (cf. Gonzales *et al.* [5]); the name is due to the fact that the order of the delays a user experiences on each of the links is the *same* across all users. A special case of the model of related links is the model of *identical links,* where all link capacities

*See [16] for a concise introduction to contemporary Game Theory.

†Computation of Nash equilibria has been long observed to be a very challenging, yet notoriously hard algorithmic problem; see [17] for an advocation.

are equal (cf. Graham [6]); thus, in this model, each user incurs the same delay on all links. We also consider the model of *unrelated machines,* where instead of associating a traffic and a capacity with each user and link, respectively, we assign a delay for each pair of a user and a link in an arbitrary way (cf. Horowitz and Sahni [8]); thus, in the unrelated links model, there is no relation between the delays incurred to a user on different links. Reciprocally, in the model of *identical traffics,* all user traffics are equal; they may vary arbitrarily in the model of *arbitrary traffics.* We are interested in understanding the impact of model assumptions on links and users on the patterns of the worst-case Nash equilibria for the selfish routing game we consider.

**Results and Contribution.** In this work, we embark on a systematic study of a natural conjecture due to Gairing *et al.* [4], which asserts that the fully mixed Nash equilibrium is the *worst-case* Nash equilibrium (with respect to social cost).

**Conjecture 1.1 (Fully Mixed Nash Equilibrium Conjecture [4])** *Consider the model of arbitrary traffics and related links. Then, for any traffic vector* $\mathbf{w}$ *such that the fully mixed Nash equilibrium* $\mathbf{F}$ *exists, and for any Nash equilibrium* $\mathbf{P}$, $\mathsf{SC}(\mathbf{w}, \mathbf{P}) \leq \mathsf{SC}(\mathbf{w}, \mathbf{F})$.

Henceforth, abbreviate the Fully Mixed Nash Equilibrium Conjecture as the FMNE *Conjecture.* We advocate that the FMNE Conjecture is a simultaneously intuitive and natural conjecture:

- To support intuition, observe that the fully mixed Nash equilibrium favors collisions between different users (since each user assigns its traffic with positive probability to every link); in turn, this increased probability of collisions favors a corresponding increase to the expected maximum total traffic through a link, which is precisely the social cost.
- To support significance, note that the FMNE Conjecture identifies the actual *worst-case* Nash equilibrium for the selfish routing game we consider. We consider this a significant milestone that may show the way for the settlement of more general and realistic instances of selfish routing, such as Internet protocols for source routing. Identifying the worst-case Nash equilibrium for source routing protocols will enable the managers of Internet protocols not only to avoid choosing it as an operating point for Internet, but also to evaluate the Nash equilibrium of choice by comparing it against the worst-case one.
  We stress that, in sharp contrast, the worst-case measure of coordination ratio studied before [1, 2, 9, 10, 12] only determines how far the worst-case Nash equilibrium is from optimal performance; since it does not identify the worst-case Nash equilibrium, it fails to provide a basis of comparison between different Nash equilibria on the basis of their social costs.

Our study reports substantial progress towards the settlement of the FMNE Conjecture:

- We prove the FMNE Conjecture for several interesting special cases of it (within the model of related links).
- In doing so, we provide proof techniques and tools which, while applicable to interesting special cases of it, may suffice for the general case as well.
- We reveal limitations of the FMNE Conjecture by establishing that it is not, in general, valid over the model of unrelated links; we present both positive and negative instances for the conjecture.

We start by describing our (positive) results for the model of related links.

- We first consider the case of disjointly mixed Nash equilibria, for which we provide a simple and intuitive proof of the FMNE Conjecture (Theorem 3.1). As an interesting consequence, this implies the validity of the conjecture in the case where there are only two users, and the network consists of just two links.

- We next restrict ourselves to the model of identical traffics, where we assume, in addition, that there are only two users ($n = 2$), while the number of links remains arbitrary. Through elegant combinatorial arguments that fit within an involved analysis, we verify the validity of the FMNE Conjecture in this case (Theorem 4.4).
  The proof provides a syntactic characterization of arbitrary Nash equilibria for this case; furthermore, the proof develops intuitive expressions for the social costs of the fully mixed and arbitrary (satisfying the characterization) Nash equilibria and uses them to bound their difference away from zero.
- We now turn to simultaneously considering the models of identical traffics and identical links, where we assume, in addition, that there are only two links ($m = 2$), while the number of users remains arbitrary. We also make a technical assumption, namely that the number of users ($n$) is even. Through an elegant combinatorial analysis, involving properties of binomial coefficients, we verify the validity of the FMNE Conjecture in this case (Theorem 5.1).
  The proof identifies a syntactic restriction on any *potentially* worst-case Nash equilibrium for this case; this restriction allows the derivation of an elegant combinatorial expression for the social cost of an arbitrary (potentially worst-case) Nash equilibrium. On the other hand, direct combinatorial arguments yield the social cost of the fully mixed Nash equilibrium. Then, an involved bounding technique is employed to bound the difference of the two social costs away from zero.

For the model of unrelated links, we have obtained both positive and negative results.

- We first consider the case of pure Nash equilibria, for which we provide a simple and intuitive proof of the FMNE Conjecture (Theorem 6.2).
  The proof assumes that $n \leq m$ and follows from a corresponding property we establish, which bounds the selfish cost of a user in an arbitrary pure Nash equilibrium by that in the fully mixed Nash equilibrium (Proposition 6.1). Furthermore, we are able to establish this property for any arbitrary (not necessarily pure) Nash equilibrium if we assume that $n = 2$ (Proposition 6.3). (Note that still $n \leq m$ in this case.)
- We next prove (by a simple case analysis) that the FMNE Conjecture is valid if $n = 2$ and $m = 2$ (Theorem 6.4).
- We finally provide a counterexample to prove that the FMNE Conjecture is *not* valid if $n = 3$ and $m = 2$ (Theorem 6.5). This implies that the FMNE Conjecture is not valid for *every* game.
  Since this negative result is shown for a case where $n > m$, while our previous positive results (for the model of unrelated links) have relied on the assumption that $n \leq m$, it is tempting to hypothesize that the relation between $n$ and $m$ is a crucial factor for the validity of the FMNE Conjecture in the model of unrelated links.

**Related Work, Comparison and Significance.** The selfish routing game considered in this paper was first introduced and studied in the pioneering work of Koutsoupias and Papadimitriou [10]. This game was subsequently studied in the work of Mavronicolas and Spirakis [12], where fully mixed Nash equilibria were introduced and analyzed. Both works focused mainly on proving bounds on coordination ratio. Subsequent works that provided bounds on coordination ratio include [1, 2, 9]. The work of Fotakis *et al.* [3] was the first to study the combinatorial structure and the computational complexity of Nash equilibria for the selfish routing game we consider; that work was subsequently extended by Gairing *et al.* [4]. (See details below.)

The closest to our work are the one by Fotakis *et al.* [3] and the one by Gairing *et al.* [4].

- The FMNE Conjecture has been inspired by two results due to Fotakis *et al.* [3] that confirm or support the conjecture. First, Fotakis *et al.* [3, Theorem 6] establish the Fully Mixed Nash Equilibrium Conjecture for the model of identical links and assuming that $n = 2$; Theorem 4.4

in this work extends this result to the model of related links, still assuming that $n = 2$ while assuming, in addition, that traffics are identical. Second, Fotakis *et al.* [3, Theorem 7] prove that, for the model of related links and of identical traffics, the social cost of *any* Nash equilibrium is no more than 49.02 times the social cost of the fully mixed Nash Equilibrium.

- The FMNE Conjecture was explicitly stated in the work of Gairing *et al.* [4, Conjecture 1.1]. In the same paper, two results are shown that confirm or support the conjecture. First, Gairing *et al.* [4, Theorem 4.2] establish the validity of the FMNE Conjecture when restricted to pure Nash equilibria. Second, Gairing *et al.* [4, Theorem 5.1] prove that for the model of identical links, the social cost of *any* Nash equilibrium is no more than $6 + \varepsilon$ times the social cost of the fully mixed Nash equilibrium, for any constant $\varepsilon > 0$. (Note that since this result does not assume identical traffics, it is incomparable to the related result by Fotakis *et al.* [3, Theorem 7] (for the model of related links) which does.)

The ultimate settlement of the FMNE Conjecture (for the model of related links) may reveal an interesting complexity-theoretic contrast between the worst-case pure and the worst-case mixed Nash equilibria. On one hand, identifying the worst-case pure Nash equilibrium is an $\mathcal{NP}$-hard problem [3, Theorem 4]; on the other hand, if the FMNE Conjecture is valid, identification of the worst-case mixed Nash equilibrium is immediate in the cases where the fully mixed Nash equilibrium exists. (In addition, the characterization of the fully mixed Nash equilibrium shown in [12, Theorem 14] implies that such existence can be checked in polynomial time.)

**Road Map.** The rest of this paper is organized as follows. Section 2 presents our definitions and some preliminaries. The case of disjointly mixed Nash equilibria is treated in Section 3. Section 4 considers the case of identical traffics and related links with $n = 2$. The reciprocal case of identical traffics and identical links with $m = 2$ is studied in Section 5. Section 6 examines the case of unrelated links. We conclude, in Section 7, with a discussion of our results and some open problems. Due to lack of space we omitted most of the proofs. They can be found in the appendix.

## 2 Framework

Most of our definitions are patterned after those in [12, Section 2], [3, Section 2] and [4, Section 2], which, in turn, were based on those in [10, Sections 1 & 2].

**Mathematical Preliminaries and Notation.** Throughout, denote for any integer $m \geq 2$, $[m] = \{1, \ldots, m\}$. For a random variable $X$, denote $\mathcal{E}(X)$ the *expectation* of $X$.

**General.** We consider a *network* consisting of a set of $m$ parallel *links* $1, 2, \ldots, m$ from a *source* node to a *destination* node. Each of $n$ *network users* $1, 2, \ldots, n$, or *users* for short, wishes to route a particular amount of traffic along a (non-fixed) link from source to destination. (Throughout, we will be using subscripts for users and superscripts for links.) In the model of related links, denote $w_i$ the *traffic* of user $i \in [n]$, and $W = \sum_{i \in [n]} w_i$. Define the $n \times 1$ *traffic vector* $\mathbf{w}$ in the natural way. Assume throughout that $m > 1$ and $n > 1$. Assume also, without loss of generality, that $w_1 \geq w_2 \geq \ldots \geq w_n$. In the model of unrelated links, denote $C_{ij}$ the *cost* of user $i \in [n]$ on link $j \in [m]$. Define the $n \times m$ *cost matrix* $\mathbf{C}$ in the natural way.

A *pure strategy* for user $i \in [n]$ is some specific link. A *mixed strategy* for user $i \in [n]$ is a probability distribution over pure strategies; thus, a mixed strategy is a probability distribution over the set of links. The *support* of the mixed strategy for user $i \in [n]$, denoted *support*$(i)$, is the set of those pure strategies (links) to which $i$ assigns positive probability. A *pure strategy profile* is represented by an $n$-tuple $\langle \ell_1, \ell_2, \ldots, \ell_n \rangle \in [m]^n$; a *mixed strategy profile* is represented by an $n \times m$ *probability matrix* $\mathbf{P}$ of $nm$ probabilities $p_i^j$, $i \in [n]$ and $j \in [m]$, where $p_i^j$ is the probability that user $i$ chooses link $j$.

For a probability matrix $\mathbf{P}$, define *indicator variables* $I_i^j \in \{0, 1\}$, where $i \in [n]$ and $j \in [m]$, such that $I_i^j = 1$ if and only if $p_i^j > 0$. Thus, the support of the mixed strategy for user $i \in [n]$ is the set $\{j \in [m] \mid I_i^j = 1\}$. For each link $j \in [m]$, define the *view* of link $j$, denoted $view(j)$, as the set of users $i \in [n]$ that potentially assign their traffics to link $j$; so, $view(j) = \{i \in [n] \mid I_i^j = 1\}$. For each link $j \in [m]$, denote $V^j = |view(j)|$.

**Syntactic Classes of Mixed Strategies.** A mixed strategy profile $\mathbf{P}$ is *disjointly mixed* if for all links $j \in [m]$, $|\{i \in view(j) : p_i^j < 1\}| \leq 1$, that is, there is at most one non-pure user on each link. A mixed strategy profile $\mathbf{P}$ is *fully mixed* [12, Section 2.2] if for all users $i \in [n]$ and links $j \in [m]$, $I_i^j = 1$.[‡] Throughout, we will cast a pure strategy profile as a special case of a mixed strategy profile in which all (mixed) strategies are pure.

**System, Models and Cost Measures.** In the model of related links, denote $c^\ell > 0$ the *capacity* of link $\ell \in [m]$, representing the rate at which the link processes traffic, and $C = \sum_{l \in [m]} c^l$. So, the *latency* for traffic $w$ through link $\ell$ equals $w/c^\ell$. In the model of *identical capacities,* all link capacities are equal to $c$, for some constant $c > 0$; link capacities may vary arbitrarily in the model of *arbitrary capacities.* Assume throughout, without loss of generality, that $c^1 \geq c^2 \geq \ldots \geq c^m$. In the model of *identical traffics,* all user traffics are equal to 1; user traffics may vary arbitrarily in the model of *arbitrary traffics.*

For a pure strategy profile $\langle \ell_1, \ell_2, \ldots, \ell_n \rangle$, the *latency cost for user $i \in [n]$,* denoted $\lambda_i$, is the latency cost of the link it chooses, that is, $(\sum_{k:\ell_k=\ell_i} w_k)/c^{\ell_i}$. For a mixed strategy profile $\mathbf{P}$, denote $\delta^\ell$ the *actual traffic* on link $\ell \in [m]$; so, $\delta^\ell$ is a random variable. For each link $\ell \in [m]$, denote $\theta^\ell$ the *expected traffic* on link $\ell \in [m]$; thus, $\theta^\ell = \mathcal{E}(\delta^\ell) = \sum_{i=1}^n p_i^\ell w_i$. For a mixed strategy profile $\mathbf{P}$, the *expected latency cost* for user $i \in [n]$ on link $\ell \in [m]$, denoted $\lambda_i^\ell$, is the expectation, over all random choices of the remaining users, of the latency cost for user $i$ had its traffic been assigned to link $\ell$; thus,

$$\lambda_i^\ell \quad = \quad \frac{w_i + \sum_{k=1,k\neq i} p_k^\ell w_k}{c^\ell} = \frac{(1-p_i^\ell)w_i + \theta^\ell}{c^\ell}\,.$$

For each user $i \in [n]$, the *minimum expected latency cost,* denoted $\lambda_i$, is the minimum, over all links $\ell \in [m]$, of the expected latency cost for user $i$ on link $\ell$; thus, $\lambda_i = \min_{\ell \in [m]} \lambda_i^\ell$.

Associated with a traffic vector $\mathbf{w}$ and a mixed strategy profile $\mathbf{P}$ is the *social cost* [10, Section 2], denoted $\mathsf{SC}(\mathbf{w}, \mathbf{P})$, which is the expectation, over all random choices of the users, of the maximum (over all links) latency of traffic through a link; thus,

$$\mathsf{SC}(\mathbf{w}, \mathbf{P}) \quad = \quad \mathcal{E}\left(\max_{\ell \in [m]} \frac{\sum_{k:\ell_k=\ell} w_k}{c^\ell}\right) \quad = \sum_{\langle \ell_1, \ell_2, \ldots, \ell_n \rangle \in [m]^n} \left(\prod_{k=1}^n p_k^{\ell_k} \cdot \max_{\ell \in [m]} \frac{\sum_{k:\ell_k=\ell} w_k}{c^\ell}\right)\,.$$

Note that $\mathsf{SC}(\mathbf{w}, \mathbf{P})$ reduces to the maximum latency through a link in the case of pure strategies. On the other hand, the *social optimum* [10, Section 2] associated with a traffic vector $\mathbf{w}$, denoted $\mathsf{OPT}(\mathbf{w})$, is the *least possible* maximum (over all links) latency of traffic through a link. Note that while $\mathsf{SC}(\mathbf{w}, \mathbf{P})$ is defined in relation to a mixed strategy profile $\mathbf{P}$, $\mathsf{OPT}(\mathbf{w})$ refers to the *optimum* pure strategy profile.

In the model of unrelated links, the latency of user $i$ on link $l$ is its cost $C_{il}$. Thus, the expected latency cost of user $i$ on link $l$ translates to $\lambda_i^l = C_{il} + \sum_{k=1,k\neq i} p_i^l C_{kl}$, and the social cost, now depending on $\mathbf{C}$ and the strategy profile $\mathbf{P}$, is defined by $\mathsf{SC}(\mathbf{C}, \mathbf{P}) = \sum_{\langle l_1, l_2, \ldots, l_n \rangle \in [m]^n} \left(\prod_{k=1}^n p_k^{l_k} \cdot \max_{l \in [m]} \sum_{k:l_k=l} C_{kl}\right)$.

**Nash Equilibria.** We are interested in a special class of mixed strategies called Nash equilibria [15] that we describe below. Formally, the probability matrix $\mathbf{P}$ is a *Nash equilibrium* [10, Section 2] if for all users $i \in [n]$ and links $\ell \in [m]$, $\lambda_i^\ell = \lambda_i$ if $I_i^\ell = 1$, and $\lambda_i^\ell \geq \lambda_i$ if $I_i^\ell = 0$. Thus, each user assigns its traffic with positive probability only on links for which its expected latency cost is minimized; this implies that

[‡]An earlier treatment of fully mixed strategies in the context of *bimatrix games* has been found in [18], called there *completely mixed strategies.* See also [13] for a subsequent treatment in the context of *strategically zero-sum games.*

there is no incentive for a user to unilaterally deviate from its mixed strategy in order to avoid links on which its expected latency cost is higher than necessary. The *coordination ratio* [10] is the maximum value, over all traffic vectors $\mathbf{w}$ and Nash equilibria $\mathbf{P}$ of the ratio $\mathsf{SC}(\mathbf{w}, \mathbf{P}) / \mathsf{OPT}(\mathbf{w})$. In the model of unrelated links, the coordination ratio translates to the maximum value of $\mathsf{SC}(\mathbf{C}, \mathbf{P}) / \mathsf{OPT}(\mathbf{C})$.

Mavronicolas and Spirakis [12, Lemma 15] show that in the model of identical links, all links are equiprobable in a fully mixed Nash equilibrium.

**Lemma 2.1 (Mavronicolas and Spirakis [12])** *Consider the fully mixed case under the model of identical capacities. Then, there exists a unique Nash equilibrium with associated Nash probabilities $p_i^\ell = 1/m$, for any user $i \in [n]$ and link $\ell \in [m]$.*

Gairing *et al.* [4, Lemma 4.1] show that in the model of related links, the minimum expected latency cost of any user $i \in [n]$ in a Nash equilibrium $\mathbf{P}$ is bounded by its minimum expected latency cost in the fully mixed Nash equilibrium $\mathbf{F}$.

**Lemma 2.2 (Gairing *et al.* [4])** *Fix any traffic vector $\mathbf{w}$, mixed Nash equilibrium $\mathbf{P}$ and user $i$. Then, $\lambda_i(\mathbf{w}, \mathbf{P}) \leq \lambda_i(\mathbf{w}, \mathbf{F})$.*

# 3 Disjointly Mixed versus Fully Mixed Nash Equilibria

In this section, we restrict ourselves to the case of disjointly mixed Nash equilibria, and we establish the $\mathsf{FMNE}$ Conjecture for this case. We prove:

**Theorem 3.1** *Fix any traffic vector $\mathbf{w}$ such that $\mathbf{F}$ exists, and any disjointly mixed Nash equilibrium $\mathbf{P}$. Then, $\mathsf{SC}(\mathbf{w}, \mathbf{P}) \leq \mathsf{SC}(\mathbf{w}, \mathbf{F})$.*

**Corollary 3.2** *Consider the model of related links, and assume that $n = 2$ and $m = 2$. Then, the* $\mathsf{FMNE}$ *Conjecture is valid.*

# 4 Identical Traffics, Related Links and $n = 2$

In this section we restrict to 2 users with identical traffics, that is, $w_1 = w_2$. Without loss of generality we assume $w_1 = w_2 = 1$ and $c^1 \geq \cdots \geq c^m$. In the following, we denote by *support*$(1)$ and *support*$(2)$ the supports of user 1 and 2, respectively, and by $p_i^j$ and $f_i^j$ the probabilities for user $i$ to choose link $j$ in $\mathbf{P}$ and $\mathbf{F}$, respectively. Since we consider two users with identical traffics, we have $f_1^j = f_2^j$ for all $j \in [m]$, and we write $f^j = f_i^j$.

In order to prove the **FMNE** Conjecture for this type of Nash equilibria we will use the following formula for the social cost of any Nash equilibrium $\mathbf{P}$ in this setting.

**Theorem 4.1** *In case of two users with identical traffics on $m$ related links, the social cost of any Nash equilibrium $\mathbf{P}$ is*

$$\mathsf{SC}(\mathbf{w}, \mathbf{P}) = \lambda_2(\mathbf{P}) + \sum_{1 \leq i < j \leq m} p_2^i p_1^j \left( \frac{1}{c^j} - \frac{1}{c^i} \right).$$

We now show that we only have to consider Nash equilibria $\mathbf{P}$ of certain structure.

**Lemma 4.2** *For any Nash equilibrium $\mathbf{P} \neq \mathbf{F}$ of two users with identical traffics on $m$ related links the following holds:*

1. *The supports of the two users are*

$$support(1) = [r] \cup I_1 \qquad and \qquad support(2) = [r] \cup I_2,$$

*where* $I_1, I_2$ *are disjoint sets of links not containing a link* $i \in [r]$*, such that*

$$[r] \cup I_1 \cup I_2 = [r + |I_1| + |I_2|].$$

2. *All links in* $I_1$ *and all links in* $I_2$ *have the same capacity.*

In order to prove the **FMNE** Conjecture for two users with identical traffics on $m$ related links in Theorem 4.4, we show that the following lemma holds.

**Lemma 4.3** *Let* $\mathbf{G}$ *be the fully mixed Nash equilibrium of two users with identical traffics on* $m$ *related links with capacities* $c^1 \geq \ldots \geq c^m$*. Furthermore, let the last* $s \geq 1$ *links have the same capacity, and let* $\mathbf{F}$ *be the fully mixed Nash equilibrium of the instance received by increasing the capacities of the last* $s$ *links to* $c^{m-s}$*. Then* $\mathsf{SC}(\mathbf{w}, \mathbf{F}) \leq \mathsf{SC}(\mathbf{w}, \mathbf{G})$*.*

**Theorem 4.4** *Consider the model of identical traffics and related links, and assume that* $n = 2$*. Then, the* FMNE *Conjecture is valid.*

# 5 Identical Traffics, Identical Links and $m = 2$

We show:

**Theorem 5.1** *Consider the model of identical traffics and identical links, and assume that* $m = 2$ *and* $n$ *is even. Then, the* FMNE *Conjecture is valid.*

**Proof:** Since both the traffics and the link capacities are identical, we can assume without loss of generality that $w_i = 1$ for all $i \in [n]$ and $c^j = 1$ for all $j \in [m]$. Recall that in the case of identical capacities, the fully mixed Nash equilibrium $\mathbf{F}$ exists always (that is, for all traffic vectors $\mathbf{w}$). Hence, we will show that for any other Nash equilibrium $\mathbf{P}$, $\mathsf{SC}(\mathbf{w}, \mathbf{P}) \leq \mathsf{SC}(\mathbf{w}, \mathbf{F})$. Fix any Nash equilibrium $\mathbf{P}$. We can identify three sets of users in $\mathbf{P}$: $\mathcal{U}_1 = \{i : support(i) = \{1\}\}$, $\mathcal{U}_2 = \{i : support(i) = \{2\}\}$ and $\mathcal{U}_{12} = \{i : support(i) = \{1, 2\}\}$. There are $u = \min(|\mathcal{U}_1|, |\mathcal{U}_2|)$ (pure) users, which choose link 1 and link 2, respectively, with probability 1. Therefore, $\mathsf{SC}(\mathbf{w}, \mathbf{P}) = \mathsf{SC}(\mathbf{w}, \mathbf{P}') + u$, where $\mathbf{P}'$ is the Nash equilibrium derived from $\mathbf{P}$ by omitting those $2u$ users. We will show, that $\mathsf{SC}(\mathbf{w}, \mathbf{F}') \geq \mathsf{SC}(\mathbf{w}, \mathbf{P}')$ for the fully mixed Nash equilibrium $\mathbf{F}'$ of $n - 2u$ users. As $\mathsf{SC}(\mathbf{w}, \mathbf{F}) > \mathsf{SC}(\mathbf{w}, \mathbf{F}') + 2u$ (Lemma 5.2), this proves the theorem.
Without loss of generality, we can assume that $\mathbf{P}'$ is of the following form: $r$ (pure) users go on link 1 with probability 1, and $n - r$ users choose both links with positive probability. We write $\mathbf{P_r}$ for this kind of Nash equilibrium.

**Lemma 5.2** *For the fully mixed Nash equilibrium* $\mathbf{F}$*,*

$$\mathsf{SC}(\mathbf{w}, \mathbf{F}) \quad = \quad \frac{n}{2} + \frac{n}{2^n} \binom{n-1}{\frac{n}{2} - 1}.$$

**Lemma 5.3** *For the Nash equilibrium* $\mathbf{P}_r$ *with two sets of users* $\mathcal{U}_1 = \{i : support(i) = \{1\}\}$ *and* $\mathcal{U}_{12} = \{i : support(i) = \{1, 2\}\}$ *with* $|\mathcal{U}_1| = r < n$ *and* $|\mathcal{U}_{12}| = n - r$ *the Nash probabilities are*

$$p := p_i^1 = \frac{1}{2} - \frac{r}{2(n - r - 1)}, \qquad and \qquad q := p_i^2 = \frac{1}{2} + \frac{r}{2(n - r - 1)},$$

*for all users* $i \in \mathcal{U}_{12}$*. Furthermore,* $n > 2r + 1$ *holds.*

**Lemma 5.4** *The social cost of the Nash equilibrium* $\mathbf{P}_r$ *is given by*

$$\mathsf{SC}(\mathbf{w}, \mathbf{P}_r) = \frac{n}{2}\binom{n-r}{\frac{n}{2}-r} p^{\frac{n}{2}-r} q^{\frac{n}{2}} + \sum_{i=\frac{n}{2}+1}^{n} i \cdot \binom{n-r}{i-r} p^{i-r} q^{n-i} + \sum_{i=\frac{n}{2}+1}^{n-r} i \cdot \binom{n-r}{i} p^{n-r-i} q^{i}.$$

It remains to show that $\Delta := \mathsf{SC}(\mathbf{w}, \mathbf{F}) - \mathsf{SC}(\mathbf{w}, \mathbf{P}_r) \geq 0$, which is done in the appendix. ■

# 6 Unrelated Links

In this section, we consider the case of unrelated links. We first prove:

**Proposition 6.1** *Consider the model of unrelated links. Fix any cost matrix* $\mathbf{C}$ *for which* $\mathbf{F}$ *exists, and a pure Nash equilibrium* $\mathbf{P}$. *Assume that* $2 \leq n \leq m$. *Then, for any player* $i$, $\lambda_i(\mathbf{P}) < \lambda_i(\mathbf{F})$.

We continue to prove:

**Theorem 6.2** *Consider the model of unrelated links. Assume that* $2 \leq n \leq m$. *Consider any cost matrix* $\mathbf{C}$ *such that the fully mixed Nash equilibrium* $\mathbf{F}$ *exists, and any pure Nash equilibrium* $\mathbf{P}$. *Then,* $\mathsf{SC}(\mathbf{C}, \mathbf{P}) \leq \mathsf{SC}(\mathbf{C}, \mathbf{F})$.

**Proof:** Clearly, the social cost of any pure Nash equilibrium $\mathbf{P}$ is equal to the selfish cost of some user, while the social cost of a fully mixed Nash equilibrium $\mathbf{F}$ is at least the selfish cost of any user. Hence, Proposition 6.1 implies the claim. ■

We prove:

**Proposition 6.3** *Consider the model of unrelated links. Assume that* $n = 2$. *Fix any cost matrix* $\mathbf{C}$ *for which* $\mathbf{F}$ *exists, and any Nash equilibrium* $\mathbf{P}$. *Then, for any user* $i \in [2]$, $\lambda_i(\mathbf{P}) \leq \lambda_i(\mathbf{F})$.

**Theorem 6.4** *Consider the model of unrelated links. Assume that* $n = 2$ *and* $m = 2$. *Then, the* FMNE *Conjecture is valid.*

We remark that Theorem 6.4 generalizes Corollary 3.2 to the case of unrelated links. We finally prove:

**Theorem 6.5 (Counterexample to the FMNE Conjecture)** *Consider the model of unrelated links. Then, the* FMNE *Conjecture is not valid even if* $n = 3$ *and* $m = 2$.

# 7 Conclusion and Directions for Further Research

We have verified the FMNE Conjecture over several interesting restrictions of the selfish routing game we considered for the case of related links. We have also investigated the FMNE Conjecture in the case of unrelated links, for which we have identified instances of the game that validate and *falsify* the FMNE Conjecture, respectively. The most obvious problem left open by our work is to establish the FMNE Conjecture in its full generality for the case of related links. We hope that several of the combinatorial techniques introduced in this work for settling special cases of the conjecture may be handy for the general case.

The FMNE Conjecture attempts to study a possible order on the set of Nash equilibria (for the specific selfish routing game we consider) that is defined with respect to their social costs; in the terminology of partially ordered sets, the FMNE Conjecture asserts that the fully mixed Nash equilibrium is a maximal element of the defined order. We feel that this order deserves further study. For example, what are the minimal elements of the order? More generally, is there a characterization of measures on Nash equilibria such that the fully mixed Nash equilibrium is a maximal element of the order defined with respect to any specific measure? (Our study considers the social cost as one such measure of interest.)

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## Appendix

**Proof of Theorem 3.1:**
Consider the $k$ mixed users $i_1, \ldots, i_k$ in the disjointly mixed Nash equilibrium $\mathbf{P}$, and denote their supports by $support(i_1), \ldots, support(i_k)$. (By definition of disjointly mixed strategies, $support(i) \cap support(i') = \emptyset$ for each pair of distinct mixed users $i, i' \in [m]$.)

For any link $l \in [m]$, denote $W^l$ the total traffic assigned by pure users to link $l$. Denote $l_0$ the link to which only pure users are assigned that maximizes $\frac{W^l}{c^l}$ over all links to which only pure users are assigned. Since $\mathbf{P}$ is a Nash equilibrium, the expected latency cost $\lambda_i^l = \frac{W^l + w_i}{c^l}$ is constant over all links $l \in support(i)$, for any mixed user $i$. Since $\mathbf{P}$ is disjointly mixed, this implies that

$$
\begin{aligned}
\mathsf{SC}(\mathbf{w}, \mathbf{P}) &= \sum_{l_1 \in support(i_1), \ldots, l_k \in support(i_k)} \max \left\{ \frac{W^{l_1} + w_{i_1}}{c^{l_1}}, \ldots, \frac{W^{l_k} + w_{i_k}}{c^{l_k}}, \frac{W^{l_0}}{c^{l_0}} \right\} p_{i_1}^{l_1} \cdots p_{i_k}^{l_k} \\
&= \max \left\{ \frac{W^{\widehat{l_1}} + w_{i_1}}{c^{\widehat{l_1}}}, \ldots, \frac{W^{\widehat{l_k}} + w_{i_k}}{c^{\widehat{l_k}}}, \frac{W^{l_0}}{c^{l_0}} \right\} \sum_{l_1 \in support(i_1), \ldots, l_k \in support(i_k)} p_{i_1}^{l_1} \cdots p_{i_k}^{l_k} \\
&= \max \left\{ \frac{W^{\widehat{l_1}} + w_{i_1}}{c^{\widehat{l_1}}}, \ldots, \frac{W^{\widehat{l_k}} + w_{i_k}}{c^{\widehat{l_k}}}, \frac{W^{l_0}}{c^{l_0}} \right\},
\end{aligned}
$$

where links $\widehat{l_1}, \ldots, \widehat{l_k}$ are arbitrarily chosen from $support(i_1), \ldots, support(i_k)$, respectively.

Note that since $\mathbf{P}$ is a Nash equilibrium, for any mixed user $i$, $\lambda_i(\mathbf{w}, \mathbf{P}) = \frac{W^{\widehat{l_i}} + w_i}{c^{\widehat{l_i}}}$, and for any pure user $i'$ assigned to link $i_0$, $\lambda_{i'}(\mathbf{w}, \mathbf{P}) = \frac{W^{l_0}}{c^{l_0}}$. Thus, either $\mathsf{SC}(\mathbf{w}, \mathbf{P}) = \lambda_i(\mathbf{w}, \mathbf{P})$ or $\mathsf{SC}(\mathbf{w}, \mathbf{P}) = \lambda_{i'}(\mathbf{w}, \mathbf{P})$. Lemma 2.2 implies that both $\lambda_i(\mathbf{w}, \mathbf{P}) \leq \mathsf{SC}(\mathbf{w}, \mathbf{F})$ and $\lambda_{i'}(\mathbf{w}, \mathbf{P}) \leq \mathsf{SC}(\mathbf{w}, \mathbf{F})$. Hence, in all cases, $\mathsf{SC}(\mathbf{w}, \mathbf{P}) \leq \mathsf{SC}(\mathbf{w}, \mathbf{F})$, as needed.

**Proof of Lemma 4.2:**
The probabilities of the fully mixed Nash equilibrium $\mathbf{F}$ have to fulfill the Nash equilibrium conditions

$$\frac{1 + f^i}{c^i} = \frac{1 + f^j}{c^j} \quad \text{for all } i, j \in [m].$$

This is the case for

$$f^i = \frac{(m+1)c^i}{C} - 1 \quad \text{for all } i \in [m]. \tag{1}$$

Since $\sum_{i \in [n]} f^i = 1$, the fully mixed Nash equilibrium $\mathbf{F}$ exists, that is, all probabilities are at least 0 if and only if

$$C < (m+1)c^m. \tag{2}$$

We can exclude the following cases:

1. $\underline{support(1) = support(2) = [m]}$
   This implies $\mathbf{P} = \mathbf{F}$, and we are done.

2. $\underline{support(1) = support(2) \subsetneq [m]}$
   Let $s \geq 1$ be the number of links which are used in $\mathbf{F}$ but not in $\mathbf{P}$. In a Nash equilibrium all

users only choose links with minimum expected latency cost. Thus, these $s$ links have to be the $s$ slowest links in $[m]$. On the one hand we have

$$\lambda_1(\mathbf{P}) \leq \frac{1}{c^m} < \frac{1+f^m}{c^m} = \lambda_1(\mathbf{F}). \tag{3}$$

On the other hand $C < (m+1)c^m$ implies

$$(m+1)(c^m + c^{m-1} + \cdots + c^{m+1-s}) + (m+1)C \quad > \quad sC + (m+1)C.$$

Thus

$$\frac{m-s+1}{C - c^m - c^{m-1} - \cdots - c^{m+1-s}} \quad > \quad \frac{m+1}{C}.$$

The probabilities in $\mathbf{P}$ can be computed in the same way as for the fully mixed Nash equilibrium, ignoring the links in $[m] \setminus [m-s]$. We get

$$p_1^j = p_2^j = \frac{(m-s+1)c^j}{C - c^m - c^{m-1} - \cdots - c^{m+1-s}} - 1 \quad > \quad \frac{(m+1)c^j}{C} - 1 = f^j,$$

yielding

$$\lambda_1(\mathbf{P}) = \frac{1+p_2^j}{c^j} > \frac{1+f^j}{c^j} = \lambda_1(\mathbf{F}) \overset{(3)}{>} \lambda_1(\mathbf{P}),$$

a contradiction.

3. <u>$support(2) \subsetneq support(1)$</u>
   Let $k \in support(1) \setminus support(2)$. Then the minimum expected latency costs of user 1 in $\mathbf{F}$ and $\mathbf{P}$ are

$$\lambda_1(\mathbf{F}) = \frac{1+f^k}{c^k} \qquad \text{and} \qquad \lambda_1(\mathbf{P}) = \frac{1}{c^k} = \lambda_1(\mathbf{F}) - \frac{f^k}{c^k}.$$

   Thus, for all $j \in support(2)$, we have

$$\lambda_1(\mathbf{P}) \quad = \quad \frac{1+p_2^j}{c^j} = \frac{1}{c^k} = \lambda_1(\mathbf{F}) - \frac{f^k}{c^k} = \frac{1+f^j}{c^j} - \frac{f^k}{c^k},$$

   which implies $p_2^j = f^j - \frac{c^j}{c^k} f^k < f^j$ for all $j \in support(2)$. We get

$$1 \quad = \quad \sum_{j \in support(2)} p_2^j < \sum_{j \in support(2)} f^j = 1 - \sum_{j \in [m] \setminus support(2)} f^j < 1,$$

   a contradiction.

4. <u>$\exists j, k \in support(1) \setminus support(2) : c^j \neq c^k$</u>
   By definition of the minimum expected latency cost, we have

$$\frac{1}{c^j} = \lambda_1(\mathbf{P}) = \frac{1}{c^k},$$

   a contradiction.

5. $\underline{\exists j \in support(1) \setminus support(2) \land k \in support(1) \cap support(2) : c^j > c^k}$
   By definition of the minimum expected latency cost, we have
$$\frac{1}{c^j} = \lambda_1(\mathbf{P}) = \frac{1+p_2^k}{c^k} > \frac{1}{c^k} > \frac{1}{c^j},$$
   a contradiction.
6. $\underline{\exists j \in support(1) \setminus support(2) \land k \in [m] \setminus (support(1) \cup support(2)) : c^j < c^k}$
   By definition of the minimum expected latency cost, we have
$$\frac{1}{c^j} = \lambda_1(\mathbf{P}) \leq \frac{1}{c^k},$$
   a contradiction.

Exchanging user 1 and user 2, we get symmetric cases which can be proven in the same way. This completes the proof of the lemma.

**Proof of Theorem 4.1:**
The social cost $\mathsf{SC}(\mathbf{w},\mathbf{P})$ and the minimum expected latency cost $\lambda_2(\mathbf{P})$ of a Nash equilibrium $\mathbf{P}$ are defined by

$$\begin{aligned}
\mathsf{SC}(\mathbf{w},\mathbf{P}) &= \sum_{1\leq i,j\leq m} p_2^i p_1^j \begin{cases} \frac{1}{c^j} & i<j \\ \frac{1}{c^i} & j<i \\ \frac{2}{c^i} & i=j \end{cases} \\
&= \sum_{1\leq i<j\leq m} \frac{p_2^i p_1^j}{c^j} + \sum_{1\leq j<i\leq m} \frac{p_2^i p_1^j}{c^i} + \sum_{1\leq i\leq m} \frac{2p_2^i p_1^i}{c^i}, \\
\lambda_2(\mathbf{P}) &= \sum_{1\leq i,j\leq m} p_2^i p_1^j \begin{cases} \frac{1}{c^i} & i\neq j \\ \frac{2}{c^i} & i=j \end{cases} \\
&= \sum_{1\leq i<j\leq m} \frac{p_2^i p_1^j}{c^i} + \sum_{1\leq j<i\leq m} \frac{p_2^i p_1^j}{c^i} + \sum_{1\leq i\leq m} \frac{2p_2^i p_1^i}{c^i}.
\end{aligned}$$

This directly implies the claim.

**Proof of Theorem 4.4:**
Since the traffics of user 1 and user 2 are equal, we can assume without loss of generality that $c^i \geq c^j$ for all $i \in I_1, j \in I_2$.
Denote $m_1 = |I_1|$ and $m_2 = |I_2|$. Due to Lemma 4.2 we write

$$c^i = \begin{cases} \gamma^i & \text{if } i \in [r], \\ \beta & \text{if } i \in I_1 = [r+m_1] \setminus [r], \\ \alpha & \text{if } i \in I_2 = [r+m_1+m_2] \setminus [r+m_1]. \end{cases}$$

Note that due to $c^1 \geq \ldots \geq c^m$ we have $\gamma^1 \geq \ldots \geq \gamma^r \geq \beta \geq \alpha$. We denote $\Gamma = \sum_{1\leq i\leq r} \gamma^i$. Moreover, we define

$$A = \frac{(m+1)\alpha - C}{\alpha C} \qquad \text{and} \qquad B = \frac{(m+1)\beta - C}{\beta C}.$$

We first consider the case $r + m_1 + m_2 = m$. The probabilities and the social cost of the Nash equilibria $\mathbf{F}$ and $\mathbf{P}$ are as follows:

Fully mixed Nash equilibrium $\mathbf{F}$
As seen in Equation (1), the probabilities in $\mathbf{F}$ are

$$f^i = \frac{(m+1)c^i}{C} - 1 \quad \text{for all } i \in [m].$$

Note that

$$f^{r+j} = \beta B \quad \text{for all } j \in [m_1], \text{ and} \tag{4}$$

$$f^{r+m_1+j} = \alpha A \quad \text{for all } j \in [m_2]. \tag{5}$$

Due to Theorem 4.1 the social cost of $\mathbf{F}$ is

$$\begin{aligned}
\mathsf{SC}(\mathbf{w}, \mathbf{F}) &= \frac{1+f^1}{\gamma^1} + \sum_{1 \le i < j \le r} \left(\frac{1}{\gamma^j} - \frac{1}{\gamma^i}\right) f^i f^j \\
&\quad + \sum_{1 \le i \le r} f^i \cdot \sum_{1 \le j \le m_1} f^{r+j} \left(\frac{1}{\beta} - \frac{1}{\gamma^i}\right) \\
&\quad + \sum_{1 \le i \le r} f^i \cdot \sum_{1 \le j \le m_2} f^{r+m_1+j} \left(\frac{1}{\alpha} - \frac{1}{\gamma^i}\right) \\
&\quad + \sum_{1 \le i \le m_1} f^{r+i} \cdot \sum_{1 \le j \le m_2} f^{r+m_1+j} \left(\frac{1}{\alpha} - \frac{1}{\beta}\right) \\
&= \frac{1+f^1}{\gamma^1} + \sum_{1 \le i < j \le r} \left(\frac{1}{\gamma^j} - \frac{1}{\gamma^i}\right) f^i f^j \\
&\quad + \sum_{1 \le i \le r} f^i \cdot \left(m_1 f^{r+1} \left(\frac{1}{\beta} - \frac{1}{\gamma^i}\right) + m_2 f^{r+m_1+1} \left(\frac{1}{\alpha} - \frac{1}{\gamma^i}\right)\right) \\
&\quad + m_1 f^{r+1} \cdot m_2 f^{r+m_1+1} \left(\frac{1}{\alpha} - \frac{1}{\beta}\right).
\end{aligned}$$

Equation (2) implies that

$$C < (m+1)\alpha. \tag{6}$$

Nash equilibrium $\mathbf{P}$:
The probabilities of **user 1** can be computed with help of the Nash equilibrium condition $\lambda_2(\mathbf{P}) = \frac{1+p_1^i}{\gamma^i} = \frac{1}{\alpha}$ for all $i \in [r]$. We have

$$\begin{array}{lll}
p_1^i & = \frac{\gamma^i}{\alpha} - 1 & \text{for all } i \in [r], \\
p_1^{r+i} & = \frac{1}{m_1}\left(1 - \sum_{1 \le j \le r} p_1^j\right) = \frac{1}{\alpha m_1}\left((r+1)\alpha - \Gamma\right) & \text{for all } i \in [m_1], \text{ and} \\
p_1^{r+m_1+i} & = 0 & \text{for all } i \in [m_2].
\end{array}$$

In the same way we can compute the probabilities of **user 2**, getting

$$\begin{array}{lll}
p_2^i & = \frac{\gamma^i}{\beta} - 1 & \text{for all } i \in [r], \\
p_2^{r+i} & = 0 & \text{for all } i \in [m_1], \text{ and} \\
p_2^{r+m_1+i} & = \frac{1}{\beta m_2}\left((r+1)\beta - \Gamma\right) & \text{for all } i \in [m_2].
\end{array}$$

Due to Theorem 4.1 the social cost of $\mathbf{P}$ is

$$\begin{aligned}\mathsf{SC}(\mathbf{w},\mathbf{P}) &= \frac{1+p_1^1}{\gamma^1} + \sum_{1\le i<j\le r}\left(\frac{1}{\gamma^j}-\frac{1}{\gamma^i}\right)p_2^i p_1^j \\ &\quad + \sum_{1\le i\le r} p_2^i \cdot \sum_{1\le j\le m_1} p_1^{r+j}\left(\frac{1}{\beta}-\frac{1}{\gamma^i}\right).\end{aligned}$$

Note that the following properties hold for all $i \in [r]$:

$$f^i - p_1^i = \frac{(m+1)\gamma^i}{C} - 1 - \frac{\gamma^i}{\alpha} + 1 = \gamma^i \cdot \frac{(m+1)\alpha - C}{\alpha C} = \gamma^i A, \tag{7}$$

$$f^i - p_2^i = \frac{(m+1)\gamma^i}{C} - 1 - \frac{\gamma^i}{\beta} + 1 = \gamma^i \cdot \frac{(m+1)\beta - C}{\beta C} = \gamma^i B. \tag{8}$$

We have

$$\Delta = \mathsf{SC}(\mathbf{w},\mathbf{F}) - \mathsf{SC}(\mathbf{w},\mathbf{P}) \ge \Delta_1 + \Delta_2 + \Delta_3,$$

where

$$\Delta_1 = \frac{f^1 - p_1^1}{\gamma^1}, \tag{9}$$

$$\Delta_2 = \sum_{1\le i<j\le r}\left(f^i f^j - p_2^i p_1^j\right)\left(\frac{1}{\gamma^j}-\frac{1}{\gamma^i}\right), \quad \text{and} \tag{10}$$

$$\begin{aligned}\Delta_3 &= \sum_{1\le i\le r} f^i \left(m_1 f^{r+1}\left(\frac{1}{\beta}-\frac{1}{\gamma^i}\right) + m_2 f^{r+m_1+1}\left(\frac{1}{\alpha}-\frac{1}{\gamma^i}\right)\right) \\ &\quad - \sum_{1\le i\le r} p_2^i \cdot \sum_{1\le j\le m_1} p_1^{r+j}\left(\frac{1}{\beta}-\frac{1}{\gamma^i}\right).\end{aligned} \tag{11}$$

We separately simplify $\Delta_1$, $\Delta_2$ and $\Delta_3$, combine the simplifications and show that $\Delta_1+\Delta_2+\Delta_3 \ge 0$.

1. Simplification of $\Delta_1$:
   From Equations (7) and (9) we immediately get $\Delta_1 = A$.

2. Simplification of $\Delta_2$:
   By definition of $f^i$, we have

$$\begin{aligned} f^i f^j &\overset{(7),(8)}{=} (p_2^i + \gamma^i B)(p_1^j + \gamma^j A) \\ &= p_2^i p_1^j + p_2^i \gamma^j A + p_1^j \gamma^i B + \gamma^i \gamma^j AB. \end{aligned} \tag{12}$$

This implies

$$
\begin{aligned}
\Delta_2 &\overset{(10),(12)}{=} \sum_{1\le i<j\le r} (p_2^i\gamma^j A + p_1^j\gamma^i B + \gamma^i\gamma^j AB)\left(\frac{1}{\gamma^j}-\frac{1}{\gamma^i}\right)\\
&= A\cdot\sum_{1\le i<j\le r}\left(1-\frac{\gamma^j}{\gamma^i}\right)p_2^i + B\cdot\sum_{1\le i<j\le r}\left(\frac{\gamma^i}{\gamma^j}-1\right)p_1^j\\
&\quad +AB\cdot\sum_{1\le i<j\le r}(\gamma^i-\gamma^j)\\
&\overset{B\ge A}{\ge} A\left(\sum_{1\le i<j\le r}\left(1-\frac{\gamma^j}{\gamma^i}\right)p_2^i + \left(\frac{\gamma^i}{\gamma^j}-1\right)p_1^j\right)\\
&= A\left(\sum_{1\le i<j\le r}\left(1-\frac{\gamma^j}{\gamma^i}\right)\left(\frac{\gamma^i-\beta}{\beta}\right) + \left(\frac{\gamma^i}{\gamma^j}-1\right)\left(\frac{\gamma^j-\alpha}{\alpha}\right)\right)\\
&\overset{\beta\ge\alpha}{\ge} A\cdot\sum_{1\le i\le r}\left(\frac{\gamma^i-\beta}{\beta}\right)\left(\sum_{1\le j\le r}\left|1-\frac{\gamma^j}{\gamma_i}\right|\right)\\
&= \frac{A}{\beta}\cdot\sum_{1\le i\le r}\left(\frac{\gamma^i-\beta}{\gamma^i}\right)\sum_{1\le j\le r}|\gamma^i-\gamma^j|.
\end{aligned}
$$

3. Simplification of $\Delta_3$:
   We have

$$
\begin{aligned}
\Delta_3 &\overset{(11)}{=} \sum_{1\le i\le r} m_1(f^if^{r+1}-p_2^ip_1^{r+1})\left(\frac{1}{\beta}-\frac{1}{\gamma^i}\right)\\
&\quad+\sum_{1\le i\le r} m_2 f^i f^{r+m_1+1}\left(\frac{1}{\alpha}-\frac{1}{\gamma^i}\right)\\
&= \Delta_{31}+\Delta_{32}.
\end{aligned}
$$

4. Simplification of $\Delta_{31}$:
   We have

$$
\begin{aligned}
\Delta_{31} &\overset{(8)}{=} m_1\cdot\sum_{1\le i\le r}((p_2^i+\gamma^iB)f^{r+1}-p_2^ip_1^{r+1})\left(\frac{1}{\beta}-\frac{1}{\gamma^i}\right)\\
&= m_1\cdot\sum_{1\le i\le r}(p_2^i(f^{r+1}-p_1^{r+1})+\gamma^iBf^{r+1})\left(\frac{1}{\beta}-\frac{1}{\gamma^i}\right)\\
&\overset{(4)}{=} m_1\cdot\sum_{1\le i\le r}\left(\left(\frac{\gamma^i}{\beta}-1\right)\left(\beta B-\frac{(r+1)\alpha-\Gamma}{m_1\alpha}\right)\left(\frac{1}{\beta}-\frac{1}{\gamma^i}\right)+\beta\gamma^iB^2\left(\frac{1}{\beta}-\frac{1}{\gamma^i}\right)\right)\\
&= m_1\cdot\sum_{1\le i\le r}\left(\left(B(\gamma^i-\beta)-\frac{\gamma^i-\beta}{\beta}\cdot\frac{(r+1)\alpha-\Gamma}{m_1\alpha}\right)\left(\frac{1}{\beta}-\frac{1}{\gamma^i}\right)+B^2(\gamma^i-\beta)\right)\\
&= \left(m_1\beta B-\frac{(r+1)\alpha-\Gamma}{\alpha}\right)\frac{1}{\beta^2}\sum_{1\le i\le r}\frac{(\gamma^i-\beta)^2}{\gamma^i}+m_1B^2\sum_{1\le i\le r}(\gamma^i-\beta).
\end{aligned}
$$

Since

$$
\begin{aligned}
& m_1\beta B - \frac{(r+1)\alpha - \Gamma}{\alpha} - A(m_1\beta - C) \\
&= m_1\beta B - \frac{(m+1)\alpha - C + m_1(\beta-\alpha)}{\alpha} - m_1\beta A + AC \\
&= m_1\beta B - AC - m_1\frac{(\beta-\alpha)}{\alpha} - m_1\beta A + AC \\
&= m_1(\beta B - \frac{\beta-\alpha}{\alpha} - \beta A) \\
&= \frac{m_1}{\alpha C}((m+1)\alpha\beta - \alpha C - (m+1)\alpha\beta + \beta C - \beta C + \alpha C) = 0,
\end{aligned}
$$

we get

$$
\begin{aligned}
\Delta_{31} &= A(m_1\beta - C)\frac{1}{\beta^2}\sum_{1\le i\le r}\frac{(\gamma^i-\beta)^2}{\gamma^i} + m_1B^2\sum_{1\le i\le r}(\gamma^i-\beta) \\
&\ge A(m_1\beta - C)\frac{1}{\beta^2}\sum_{1\le i\le r}\frac{(\gamma^i-\beta)^2}{\gamma^i}.
\end{aligned}
$$

5. Simplification of $\Delta_{32}$:
We have

$$
\begin{aligned}
\Delta_{32} &= m_2\sum_{1\le i\le r} f^i f^{r+m_1+1}\left(\frac{1}{\alpha} - \frac{1}{\gamma^i}\right) \\
&\overset{(7)}{=} m_2\sum_{1\le i\le r}(p_1^i + \gamma^i A) f^{r+m_1+1}\left(\frac{1}{\alpha} - \frac{1}{\gamma^i}\right) \\
&\overset{(5)}{=} m_2\sum_{1\le i\le r}(p_1^i + \gamma^i A)\alpha A\left(\frac{1}{\alpha} - \frac{1}{\gamma^i}\right) \\
&= m_2\alpha A\left(\sum_{1\le i\le r}\frac{\gamma^i-\alpha}{\alpha}\left(\frac{1}{\alpha} - \frac{1}{\gamma^i}\right) + \sum_{1\le i\le r} A\left(\frac{\gamma^i-\alpha}{\alpha}\right)\right) \\
&= m_2\frac{A}{\alpha}\sum_{1\le i\le r}\frac{(\gamma^i-\alpha)^2}{\gamma^i} + m_2A^2\sum_{1\le i\le r}(\gamma^i-\alpha).
\end{aligned}
$$

Since the function $f(x) = \frac{1}{x}(\gamma^i - x)^2$ is monotonically decreasing in $x$ for all $0 \le x \le \gamma^i$, this implies

$$
\begin{aligned}
\Delta_{32} &\ge m_2A\sum_{1\le i\le r}\frac{(\gamma^i-\beta)^2}{\beta\gamma^i} + m_2A^2\sum_{1\le i\le r}(\gamma^i-\beta) \\
&= m_2\frac{A}{\beta}\sum_{1\le i\le r}\frac{(\gamma^i-\beta)^2}{\gamma^i} + m_2A^2(\Gamma - r\beta) \\
&\ge m_2\frac{A}{\beta}\sum_{1\le i\le r}\frac{(\gamma^i-\beta)^2}{\gamma^i}.
\end{aligned}
$$

Denote $S(i) = \sum_{1 \le j \le r} |\gamma^i - \gamma^j|$. Combining the results for $\Delta_1$, $\Delta_2$, $\Delta_{31}$ and $\Delta_{32}$, we get

$$\begin{aligned}
\Delta &\ge A + \frac{A}{\beta} \sum_{1 \le i \le r} \frac{\gamma^i - \beta}{\gamma^i} S(i) + A(m_1\beta - C)\frac{1}{\beta^2} \sum_{1 \le i \le r} \frac{(\gamma^i - \beta)^2}{\gamma^i} + m_2 \frac{A}{\beta} \sum_{1 \le i \le r} \frac{(\gamma^i - \beta)^2}{\gamma^i} \\
&= A + \frac{A}{\beta} \sum_{1 \le i \le r} \frac{\gamma^i - \beta}{\gamma^i} S(i) + A((m_1 + m_2)\beta - C)\frac{1}{\beta^2} \sum_{1 \le i \le r} \frac{(\gamma^i - \beta)^2}{\gamma^i} \\
&\ge A + \frac{A}{\beta} \sum_{1 \le i \le r} \frac{\gamma^i - \beta}{\gamma^i} S(i) + A((m_1\beta + m_2\alpha - C)\frac{1}{\beta^2} \sum_{1 \le i \le r} \frac{(\gamma^i - \beta)^2}{\gamma^i}.
\end{aligned}$$

Since

$$\begin{aligned}
\Gamma + S(i) &= \Gamma + \sum_{1 \le j \le r} |\gamma^i - \gamma^j| \\
&= \Gamma + \sum_{1 \le j < i} (\gamma^j - \gamma^i) + \sum_{i < j \le r} (\gamma^i - \gamma^j) \\
&= \sum_{1 \le j \le i} \gamma^j + (r - i)\gamma^i + \sum_{1 \le j < i} (\gamma^j - \gamma^i) \\
&\ge i\gamma^i + (r - i)\gamma^i \\
&= r\gamma^i,
\end{aligned}$$

this implies

$$\begin{aligned}
&\sum_{1 \le i \le r} \frac{\gamma^i - \beta}{\gamma^i} S(i) - \frac{C - m_1\beta - m_2\alpha}{\beta} \sum_{1 \le i \le r} \frac{(\gamma^i - \beta)^2}{\gamma^i} \\
&= \sum_{1 \le i \le r} \frac{\gamma^i - \beta}{\gamma^i} S(i) - \frac{\Gamma}{\beta} \sum_{1 \le i \le r} \left( (\gamma^i - \beta) - \frac{\beta(\gamma^i - \beta)}{\gamma^i} \right) \\
&= \sum_{1 \le i \le r} \frac{\gamma^i - \beta}{\gamma^i} (S(i) + \Gamma) - \frac{\Gamma}{\beta} \sum_{1 \le i \le r} (\gamma^i - \beta) \\
&= \sum_{1 \le i \le r} \frac{\gamma^i - \beta}{\gamma^i} (S(i) + \Gamma) - \frac{\Gamma}{\beta} (\Gamma - r\beta) \\
&\ge \sum_{1 \le i \le r} \frac{\gamma^i - \beta}{\gamma^i} r\gamma^i - \frac{\Gamma}{\beta} (\Gamma - r\beta) \\
&= (\Gamma - r\beta) r - \frac{\Gamma}{\beta} (\Gamma - r\beta) \\
&= -\frac{1}{\beta} (\Gamma - r\beta)^2 \\
&\ge -\beta.
\end{aligned}$$

We get

$$\Delta \ \ge \ A + \frac{A}{\beta}(-\beta) = 0.$$

This completes the proof for the case $r + m_1 + m_2 = m$.

Now assume $r+m_1+m_2<m$. We increase the capacities of the links not used in $\mathbf{P}$ to $c^{r+m_1+m_2}$ and consider the corresponding fully mixed Nash equilibrium $\mathbf{F}'$ and the Nash equilibrium $\mathbf{P}'$ with

$$\begin{aligned} support'(1) &= support(1) \quad \text{and} \\ support'(2) &= support(2)\cup([m]\setminus[r+m_1+m_2]). \end{aligned}$$

It is easy to see that $\mathbf{P}'$ always exists, and that $\mathbf{P}$ and $\mathbf{P}'$ have the same social cost. Now we can apply the restricted case $r+m_1+m_2'=m$ and get $\mathsf{SC}(\mathbf{w},\mathbf{F}')\geq\mathsf{SC}(\mathbf{w},\mathbf{P}')=\mathsf{SC}(\mathbf{w},\mathbf{P})$.

Since Lemma 4.3 shows that we can increase the capacities of all links in $[m]\setminus[r+m_1+m_2]$ to $c^{r+m_1+m_2}$ without increasing the social cost of the fully mixed Nash equilibrium, we have $\mathsf{SC}(\mathbf{w},\mathbf{F})\geq\mathsf{SC}(\mathbf{w},\mathbf{F}')\geq\mathsf{SC}(\mathbf{w},\mathbf{P})$.

**Proof of Lemma 4.3:**
We denote by $\delta=c^{m-s}-c^m$ and $g^j$ the probability of a user to choose link $j$ in $\mathbf{G}$. Set $r=m-s$. We have

$$f^j \quad = \quad \begin{cases} \frac{(m+1)c^j}{C+s\delta}-1 & \forall i\in[r] \\ \frac{(m+1)c^r}{C+s\delta}-1 & \forall i\in[m]\setminus[r], \end{cases}$$

and

$$g^j \quad = \quad \frac{(m+1)c^j}{C}-1 \quad \forall i\in[m].$$

In the same way as in Theorem 4.4 we show that $\Delta'=\mathsf{SC}(\mathbf{w},\mathbf{G})-\mathsf{SC}(\mathbf{w},\mathbf{F})\geq\Delta_1'+\Delta_2'+\Delta_3'\geq 0$ where

$$\begin{aligned} \Delta_1' &= \frac{1+g^1}{c^1}-\frac{1+f^1}{c^1}, \\ \Delta_2' &= \sum_{1\leq i<j\leq r}\left(\frac{1}{c^j}-\frac{1}{c^i}\right)\left(g^ig^j-f^if^j\right), \\ \Delta_3' &= \sum_{1\leq i\leq r<j\leq m}\left(\left(\frac{1}{c^{r+1}}-\frac{1}{c^i}\right)g^ig^{r+1}-\left(\frac{1}{c^r}-\frac{1}{c^i}\right)f^if^{r+1}\right). \end{aligned}$$

Using $A'=\frac{(m+1)\delta}{C(C+s\delta)}$ we can simplify these terms similar to the proof of Theorem 4.4 yielding

$$\begin{aligned} \Delta_1' &= A', \\ \Delta_2' &\geq A'\sum_{1\leq i\leq r}f^i\frac{\sum_{1\leq j\leq r}|c^i-c^j|}{c^i}, \\ \Delta_3' &\geq \sum_{1\leq i\leq r}f^i\sum_{r<j\leq m}\left(g^{r+1}\left(\frac{1}{c^{r+1}}-\frac{1}{c^i}\right)-g^{r+1}\left(\frac{1}{c^r}-\frac{1}{c^i}\right)\right) \\ &\geq (m-r)\sum_{1\leq i\leq r}f^i\left(\frac{1}{c^{r+1}}-\frac{1}{c^i}\right)\left(g^{r+1}-f^{r+1}\right) \\ &\geq (m-r)\sum_{1\leq i\leq r}f^i\left(\frac{1}{c^{r+1}}-\frac{1}{c^i}\right)\left(-\frac{A'}{s}(C-sc^{r+1})\right). \end{aligned}$$

Combining $\Delta_2'$ and $\Delta_3'$ as in the proof of Theorem 4.4 implies $\Delta_2'+\Delta_3'\geq -A'$, proving the claim.

**Proof of Lemma 5.2:**
By Lemma 2.1, all of the $2^n$ assignments of users to links are equiprobable; so each occurs with probability $\frac{1}{2^n}$. The maximum latency attains the following values:

- $\frac{n}{2}$, attained when $\frac{n}{2}$ users are assigned to both links 1 and 2; this occurs in $\binom{n}{\frac{n}{2}}$ ways;
- $i$, where $\frac{n}{2} < i \leq n$, attained when $i$ users are assigned to one link and the remaining $n-i < i$ users are assigned to the other link; this occurs in $2\binom{n}{i}$ ways (where, the factor 2 takes care of exchanging the links where the maximum latency is attained).

Thus,

$$\begin{aligned}
&\mathsf{SC}\left(\mathbf{F}\right)\\
= &\quad \tfrac{1}{2^n}\left(\tfrac{n}{2}\tbinom{n}{\frac{n}{2}} + 2\textstyle\sum_{i=\frac{n}{2}+1}^{n} i\tbinom{n}{i}\right)\\
= &\quad \tfrac{1}{2^n}\left(\tfrac{n}{2}\,\tfrac{n}{\frac{n}{2}}\tbinom{n-1}{\frac{n}{2}-1} + 2\textstyle\sum_{i=\frac{n}{2}+1}^{n} n\tbinom{n-1}{i-1}\right) \quad \text{(since } k\tbinom{n}{k} = n\tbinom{n-1}{k-1}\text{)}\\
= &\quad \tfrac{n}{2^n}\left(\tbinom{n-1}{\frac{n}{2}-1} + 2\textstyle\sum_{i=\frac{n}{2}+1}^{n}\tbinom{n-1}{i-1}\right)\\
= &\quad \tfrac{n}{2^n}\left(\tbinom{n-1}{\frac{n}{2}-1} + 2\textstyle\sum_{i=\frac{n}{2}}^{n-1}\tbinom{n-1}{i}\right)\\
= &\quad \tfrac{n}{2^n}\left(\tbinom{n-1}{\frac{n}{2}-1} + \textstyle\sum_{i=0}^{n-1}\tbinom{n-1}{i}\right) \quad \text{(since } \tbinom{n}{k} = \tbinom{n}{n-k}\text{)}\\
= &\quad \tfrac{n}{2^n}\left(\tbinom{n-1}{\frac{n}{2}-1} + 2^{n-1}\right)\\
= &\quad \tfrac{n}{2} + \tfrac{n}{2^n}\tbinom{n-1}{\frac{n}{2}-1}
\end{aligned}$$

as needed.

**Proof of Lemma 5.3:**
Take any user $i \in \mathcal{U}_{12}$. Then, by definition of selfish cost, $\lambda_i^1 = r + 1 + (n-r-1)p_i^1$ and $\lambda_i^2 = 1 + (n-r-1)p_i^2 = 1 + (n-r-1)(1-p_i^1)$. Thus, the Nash equilibrium condition for $\mathbf{P}_r$, that is, $\lambda_i^1 = \lambda_i^2$ for all $i \in \mathcal{U}_{12}$ and $\lambda_i^1 \leq \lambda_i^2$ for all $i \in \mathcal{U}_1$, is equivalent to the above setting of $p_i^1$ and $p_i^2$. Furthermore, $p_i^1 > 0$ implies $n > 2r + 1$.

**Proof of Lemma 5.4:**
In any possible assignment according to $\mathbf{P_r}$, $r$ users are assigned to link 1, while $(n-r)$ users choose link 1 with probability $p$ or link 2 with probability $q$. Let $i$ be the maximum number of users per link for a particular assignment. Then, $i \in \{n/2, \ldots, n\}$. There are $\alpha_1 = \binom{n-r}{n/2-r}$ assignments which map $\frac{n}{2}$ users to each of the two links. Each assignment occurs with probability $p^{n/2-r}q^{n/2}$. For $\frac{n}{2}+1 \leq i \leq n$, $\alpha_2 = \binom{n-r}{i-r}$ assignments map $i$ users to link 1. Each assignment occurs with probability $p^{i-r}q^{n-i}$. And for $\frac{n}{2}+1 \leq i \leq n-r$, $\alpha_3 = \binom{n-r}{i}$ assignments map $i$ users to link 2, each occuring with probability $p^{n-i-r}q^i$. Thus,

$$\begin{aligned}
\mathsf{SC}\left(\mathbf{w}, \mathbf{P}_r\right) &= \frac{n}{2}\cdot\alpha_1\cdot p^{n/2-r}q^{n/2} + \sum_{i=\frac{n}{2}+1}^{n} i\cdot\alpha_2\cdot p^{i-r}q^{n-i} + \sum_{i=\frac{n}{2}+1}^{n-r} i\ \alpha_3\ p^{n-r-i}q^i\\
&= \frac{n}{2}\binom{n-r}{\frac{n}{2}-r}p^{\frac{n}{2}-r}q^{\frac{n}{2}} + \sum_{i=\frac{n}{2}+1}^{n} i\cdot\binom{n-r}{i-r}p^{i-r}q^{n-i} + \sum_{i=\frac{n}{2}+1}^{n-r} i\cdot\binom{n-r}{i}p^{n-r-i}q^i\,.
\end{aligned}$$

**Proof of Theorem 5.1:**
For $\mathsf{SC}\left(\mathbf{w}, \mathbf{P}_r\right)$ we can write

$$\mathsf{SC}\left(\mathbf{w}, \mathbf{P}_r\right) = \frac{n}{2}\binom{n-r}{\frac{n}{2}-r}p^{\frac{n}{2}-r}q^{\frac{n}{2}} + (A - B)$$

with

$$
\begin{aligned}
A &= \sum_{i=\frac{n}{2}+1}^{n} (i-r)\binom{n-r}{i-r} p^{i-r-1}q^{n+1-i} + (n-r)\binom{n-r-1}{i-1} p^{n-r-i}q^{i} \\
B &= \sum_{i=\frac{n}{2}+1}^{n} (i-r)\binom{n-r}{i-r} p^{i-r-1}q^{n+1-i} - i\binom{n-r}{i-r} p^{i-r}q^{n-i}
\end{aligned}
$$

It holds

$$
\begin{aligned}
A &= q(n-r)\left\{\sum_{i=\frac{n}{2}+1}^{n} \binom{n-r-1}{i-r-1} p^{i-r-1}q^{n-i} + \sum_{i=\frac{n}{2}+1}^{n} \binom{n-r-1}{n-r-i} p^{n-r-i}q^{i-1}\right\} \\
&= q(n-r)\left\{\sum_{i=\frac{n}{2}+1}^{n} \binom{n-r-1}{i-r-1} p^{i-r-1}q^{n-i} + \sum_{j=r+1}^{n/2} \binom{n-r-1}{j-r-1} p^{j-r-1}q^{n-j}\right\} \\
&\quad \text{with } j = n-i+1. \\
\Rightarrow A &= q(n-r)\sum_{i=0}^{n-r-1} \binom{n-r-1}{i} p^{i}q^{n-r-1-i} \\
&= q(n-r)(p+q)^{n-r-1} = q(n-r)
\end{aligned}
$$

And

$$
\begin{aligned}
B &= \sum_{i=\frac{n}{2}+1}^{n} \binom{n-r}{i-r} p^{i-r-1}q^{n-i}\left\{(i-r)q - ip\right\} \\
&= \sum_{i=n/2+1-r}^{n-r} \binom{n-r}{i} p^{i-1}q^{n-r-i}\left\{(q-p)i - pr\right\} \\
&= (q-p)(n-r)\sum_{i=\frac{n}{2}+1-r}^{n-r} \binom{n-r-1}{i-1} p^{i-1}q^{n-r-i} - pr\sum_{i=\frac{n}{2}+1-r}^{n-r} \binom{n-r}{i} p^{i-1}q^{n-r-i} \\
&= \frac{r(n+1)}{2(n-r-1)}\sum_{i=n/2+1-r}^{n-r} \binom{n-r-1}{i-1} p^{i-1}q^{n-r-i} - r\sum_{i=\frac{n}{2}+1-r}^{n-r-1} \binom{n-r-1}{i} p^{i}q^{n-r-i}, \\
&\quad \text{since } \binom{n-r}{i} = \binom{n-r-1}{i} + \binom{n-r-1}{i-1} \\
&\quad \text{and } (q-p)(n-r) - pr = \frac{r(n+1)}{2(n-r-1)}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
B &= \frac{r(n+1)}{2(n-r-1)}\sum_{i=\frac{n}{2}+1-r}^{n-r} \binom{n-r-1}{i-1} p^{i-1}q^{n-r-i} - qr\sum_{i=\frac{n}{2}+2-r}^{n-r} \binom{n-r-1}{i-1} p^{i-1}q^{n-r-i} \\
&= \frac{r(n+1)}{2(n-r-1)}\binom{n-r-1}{n/2-r} p^{n/2-r}q^{n/2-1} + \frac{r}{n-r-1}\sum_{i=\frac{n}{2}+2-r}^{n-r} \binom{n-r-1}{i-1} p^{i-1}q^{n-1-i}, \\
&\quad \text{since } \frac{r(n+1)}{2(n-r-1)} - qr = \frac{r}{n-r-1}.
\end{aligned}
$$

It follows:

$$
\begin{aligned}
\Delta &= C + D + E \text{ with} \\
C &= \frac{n}{2} - q(n-r) \\
D &= \frac{n}{2^n}\binom{n-1}{n/2-1} - \frac{n}{2}\binom{n-r}{n/2-r} p^{n/2-r} q^{n/2} \\
&\quad + \frac{r(n+1)}{2(n-r-1)}\binom{n-r-1}{n/2-r} p^{n/2-r} q^{n/2-1} \\
E &= \frac{r}{n-r-1} \sum_{i=\frac{n}{2}+2-r}^{n-r} \binom{n-r-1}{i-1} p^{i-1} q^{n-r-i}
\end{aligned}
$$

It is

$$C = -\frac{r}{2(n-r-1)}$$

and

$$
\begin{aligned}
&- \frac{nq}{2}\binom{n-r}{n/2-r} + \frac{r(n+1)}{2(n-r-1)}\binom{n-r-1}{n/2-r} \\
&= -\frac{n(n-2r-1)}{2(n-r-1)}\binom{n-r-1}{n/2-r} = -np\binom{n-r-1}{n/2-r}.
\end{aligned}
$$

Therefore,

$$\Delta \geq \frac{n}{2^n}\binom{n-1}{n/2-1} - n\binom{n-r-1}{n/2-r} p^{n/2+1-r} q^{n/2-1} - \frac{r}{2(n-r-1)}. \tag{13}$$

One can check that both terms

$$n\binom{n-r-1}{n/2-r} p^{n/2+1-r} q^{n/2-1}$$

and

$$\frac{r}{2(n-r-1)}$$

are increasing functions of the argument $r$. Hence, the minimum of the right hand side in (13) is attained when $r = n/2 - 1$. Thus,

$$\Delta \geq \frac{n}{2^n}\binom{n-1}{n/2-1} - \frac{1}{2}\left(\frac{n-1}{n}\right)^{n/2-1} - \frac{n-2}{2n}. \tag{14}$$

For $n \leq 6$, one can check directly that the right hand side of (14) is nonnegative. For $n \geq 8$, observe that

$$\frac{n}{2^n}\binom{n-1}{n/2-1} > 1$$

while

$$\frac{1}{2}\left(\frac{n-1}{n}\right)^{n/2-1} + \frac{n-2}{2n} < 1.$$

This completes our proof.

**Proof of Proposition 6.1:**
By case analysis on the pattern of supports in $\mathbf{P}$.

1. Assume first that $\mathbf{P}$ assigns each user to a different link; that is, for each user $k$ there exists a unique link $l$ such that $I_k^j = 1$. (This is possible since $n \leq m$.) By definition of selfish cost, this implies that $\lambda_i(\mathbf{P}) = C_{ij}$, where $j$ is the link to which player $i$ is assigned by $\mathbf{P}$. However, $\lambda_i(\mathbf{F}) = C_{ij} + \sum_{k\in[n],k\neq i} f_k^j C_{kj}$. By definition of fully mixed Nash equilibrium, $f_k^j > 0$ for all players $k \in [n]$. It follows that $\lambda_i(\mathbf{P}) < \lambda_i(\mathbf{F})$.

2. Assume now that not all players are assigned by $\mathbf{P}$ to different links. Since $n \leq m$, this implies that there is some empty link $j \in [m]$; that is, no player is assigned to link $j$ by $\mathbf{P}$. By definition of selfish cost, $\lambda_i(\mathbf{P}) = C_{ij}$, while $\lambda_i(\mathbf{F}) = C_{ij} + \sum_{k\in[n],k\neq i} f_k^j C_{kj}$. By definition of fully mixed Nash equilibrium, $f_k^j > 0$ for all players $k \in [n]$. It follows that $\lambda_i(\mathbf{P}) < \lambda_i(\mathbf{F})$.

Thus, in all cases, $\lambda_i(\mathbf{P}) < \lambda_i(\mathbf{F})$, as needed.

**Proof of Proposition 6.3:**
We proceed by case analysis on the relation between *support*(1) and *support*(2).

1. Assume first that *support*(1) = *support*(2). There are two subcases to consider.

   (a) If *support*(1) = *support*(2) = $[m]$, then $\mathbf{P} = \mathbf{F}$ and the claim holds vacuously.

   (b) If *support*(1) = *support*(2) $\neq [m]$, then there exists some link $k \in [m]$ such that both $k \notin$ *support*(1) and $k \notin$ *support*(2). Note that $\lambda_1(\mathbf{P}) \leq \lambda_1^k(\mathbf{P}) = C_{1k}$, while $\lambda_1(\mathbf{F}) = \lambda_1^k(\mathbf{F}) = C_{1k} + f_2^k C_{2k}$. Since $f_2^k > 0$ and $C_{2k} \geq 0$, it follows that $\lambda_1(\mathbf{P}) \leq \lambda_1(\mathbf{F})$, as needed. Corresponding arguments for user 2 establish that $\lambda_2(\mathbf{P}) \leq \lambda_2(\mathbf{F})$, as well.

2. Assume now that *support*(1) $\neq$ *support*(2). Then, either *support*(1)$\setminus$*support*(2) $\neq \emptyset$ or *support*(2)$\setminus$ *support*(1) $\neq \emptyset$.

   (a) Consider first the subcase where *support*(1) $\setminus$ *support*(2) $\neq \emptyset$. Then, there exists some link $k \in [m]$ such that $k \in$ *support*(1) while $k \notin$ *support*(2). Note that $\lambda_1(\mathbf{P}) = \lambda_1^k(\mathbf{P}) = C_{1k}$ while $\lambda_1(\mathbf{F}) = \lambda_1^k(\mathbf{F}) = C_{1k} + f_2^k C_{2k}$. Since $f_2^k > 0$ while $C_{2k} \geq 0$, it follows that $\lambda_1(\mathbf{P}) \leq \lambda_1(\mathbf{F})$.
   We proceed to consider user 2. To show that $\lambda_2(\mathbf{P}) \leq \lambda_2(\mathbf{F})$, assume, by way of contradiction, that $\lambda_2(\mathbf{P}) > \lambda_2(\mathbf{F})$. Fix any link $j \in$ *support*(2). Then, $\lambda_2(\mathbf{P}) = \lambda_2^j(\mathbf{P}) = C_{2j} + p_1^j C_{1j}$ and $\lambda_2(\mathbf{F}) = \lambda_2^j(\mathbf{F}) = C_{2j} + f_1^j C_{1j}$. Since $\lambda_2(\mathbf{P}) > \lambda_2(\mathbf{F})$, it follows that $p_1^j > f_1^j$ for all $j \in$ *support*(2). Since, however, $\sum_{l=1}^m p_1^l = 1 = \sum_{l=1}^m f_1^l$, this implies that there exists some link $k \notin$ *support*(2) such that $p_1^k < f_1^k$. Thus, $\lambda_2(\mathbf{P}) \leq \lambda_2^k(\mathbf{P}) = C_{2k} + p_1^k C_{1k} \leq C_{2k} + f_1^k C_{1k} = \lambda_2^k(\mathbf{F}) = \lambda_2(\mathbf{F})$, a contradiction. It follows that $\lambda_2(\mathbf{P}) \leq \lambda_2(\mathbf{F})$.

   (b) Corresponding arguments suffice to establish that both $\lambda_1(\mathbf{P}) \leq \lambda_1(\mathbf{F})$ and $\lambda_2(\mathbf{P}) \leq \lambda_2(\mathbf{F})$ in the subcase where *support*(2) $\setminus$ *support*(1) $\neq \emptyset$ as well.

**Proof of Theorem 6.4:**
Assume that the fully mixed Nash equilibrium $\mathbf{F}$ exists. We will show that for any other Nash equilibrium $\mathbf{P}$, $\mathsf{SC}(\mathbf{C}, \mathbf{P}) \leq \mathsf{SC}(\mathbf{C}, \mathbf{F})$. We proceed by case analysis on the supports of $\mathbf{P}$.

1. Assume first that $\mathbf{P}$ is pure. Since $m = n$, Theorem 6.2 implies that $\mathsf{SC}(\mathbf{C}, \mathbf{P}) \leq \mathsf{SC}(\mathbf{C}, \mathbf{F})$, as needed.

2. Assume now that $\mathbf{P}$ is not pure. Since $\mathbf{P} \neq \mathbf{F}$, it follows that in $\mathbf{P}$ one user is pure and the other user is mixed. Assume, without loss of generality, that 1 is a pure user assigned to link 1, while user 2 is mixed.

We calculate the selfish cost of user 2 in $\mathbf{P}$. Clearly, $\lambda_2^1(\mathbf{P}) = C_{21} + C_{11}$ and $\lambda_2^2(\mathbf{P}) = C_{22}$. Since $\mathbf{P}$ is a Nash equilibrium, $\lambda_2^1(\mathbf{P}) = \lambda_2^2(\mathbf{P})$ or $C_{21} + C_{11} = C_{22}$. Clearly, the social cost of $\mathbf{P}$ is

$$\begin{aligned} & \mathsf{SC}(\mathbf{C}, \mathbf{P}) \\ = \ & p_2^1 (C_{11} + C_{21}) + p_2^2 \max\{C_{11}, C_{22}\} \\ = \ & p_2^1 C_{22} + p_2^2 C_{22} \qquad (\text{since } C_{22} = C_{11} + C_{21} \geq C_{11}) \\ = \ & C_{22}\,. \end{aligned}$$

So,

$$\begin{aligned} & \mathsf{SC}(\mathbf{C}, \mathbf{F}) \\ \geq \ & \lambda_2(\mathbf{F}) \\ = \ & \lambda_2^2(\mathbf{F}) \qquad (\text{since } \mathbf{F} \text{ is fully mixed}) \\ \geq \ & C_{22} \qquad (\text{since } \lambda_2^2(\mathbf{F}) = C_{22} + f_{12} C_{12}) \\ = \ & \mathsf{SC}(\mathbf{C}, \mathbf{P})\,. \end{aligned}$$

as needed.

Thus, in all cases, $\mathsf{SC}(\mathbf{C}, \mathbf{P}) \leq \mathsf{SC}(\mathbf{C}, \mathbf{F})$, as needed.

**Proof of Theorem 6.5:**
Consider an instance of the game with $n = 3$ and $m = 2$.

- Set $C_{11} = C_{21} = 2$, $C_{31} = 0$ and $C_{12} = C_{22} = \frac{3}{2}$ and $C_{32} = 3$.

Consider first the pure allocation $\mathbf{P}$ assigning players 1 and 2 to link 1, and player 3 to link 2. We verify that $\mathbf{P}$ is a Nash equilibrium:

- $\lambda_1^1(\mathbf{P}) = C_{11} + C_{21} = 2 + 2 = 4$, while $\lambda_1^2(\mathbf{P}) = C_{12} + C_{32} = \frac{3}{2} + 3 = \frac{9}{2}$; thus, $\lambda_1^1(\mathbf{P}) < \lambda_1^2(\mathbf{P})$.
- $\lambda_2^1(\mathbf{P}) = C_{21} + C_{11} = 2 + 2 = 4$, while $\lambda_2^2(\mathbf{P}) = C_{22} + C_{32} = \frac{3}{2} + 3 = \frac{9}{2}$; thus, $\lambda_2^1(\mathbf{P}) < \lambda_2^2(\mathbf{P})$.
- $\lambda_3^2(\mathbf{P}) = C_{32} = 3$, while $\lambda_3^1(\mathbf{P}) = C_{31} + C_{11} + C_{21} = 0 + 2 + 2 = 4$; thus, $\lambda_3^2(\mathbf{P}) < \lambda_3^1(\mathbf{P})$.

It follows that $\mathbf{P}$ is a Nash equilibrium. The social cost of $\mathbf{P}$ is $\mathsf{SC}(\mathbf{C}, \mathbf{P}) = \max\{C_{11} + C_{21}, C_{32}\} = \max\{2 + 2, 3\} = 4$.

Consider now the fully mixed allocation $\mathbf{F}$ with $f_1^1 = f_2^1 = \frac{6}{7}$, $f_3^1 = \frac{1}{3}$, $f_1^2 = f_2^2 = \frac{1}{7}$ and $f_3^2 = \frac{2}{3}$. We verify that $\mathbf{F}$ is a (fully mixed) Nash equilibrium:

- $\lambda_1^1(\mathbf{F}) = C_{11} + f_2^1 C_{21} + f_3^1 C_{31} = 2 + \frac{6}{7} \cdot 2 + \frac{3}{2} \cdot 0 = \frac{26}{7}$, and $\lambda_1^2(\mathbf{F}) = C_{12} + f_2^2 C_{22} + f_3^2 C_{32} = \frac{3}{2} + \frac{1}{7} \cdot \frac{3}{2} + \frac{2}{3} \cdot 3 = \frac{26}{7}$; thus, $\lambda_1^1(\mathbf{F}) = \lambda_1^2(\mathbf{F})$.
- $\lambda_2^1(\mathbf{F}) = C_{21} + f_1^1 C_{11} + f_3^1 C_{31} = 2 + \frac{6}{7} \cdot 2 + \frac{3}{2} \cdot 0 = \frac{26}{7}$, and $\lambda_2^2(\mathbf{F}) = C_{22} + f_1^2 C_{12} + f_3^2 C_{32} = \frac{3}{2} + \frac{1}{7} \cdot \frac{3}{2} + \frac{2}{3} \cdot 3 = \frac{26}{7}$; thus, $\lambda_2^1(\mathbf{F}) = \lambda_2^2(\mathbf{F})$.
- $\lambda_3^1(\mathbf{F}) = C_{31} + f_1^1 C_{11} + f_2^1 C_{21} = 0 + \frac{6}{7} \cdot 2 + \frac{6}{7} \cdot 2 = \frac{24}{7}$, and $\lambda_3^2(\mathbf{F}) = C_{32} + f_1^2 C_{12} + f_2^2 C_{22} = 3 + \frac{1}{7} \cdot \frac{3}{2} + \frac{1}{7} \cdot \frac{3}{2} = \frac{24}{7}$; thus, $\lambda_3^1(\mathbf{F}) = \lambda_3^2(\mathbf{F})$.

It follows that $\mathbf{F}$ is a Nash equilibrium. The social cost of $\mathbf{F}$ is

$$
\begin{aligned}
\mathsf{SC}(\mathbf{C},\mathbf{F}) &= f_1^1 f_2^1 f_3^1 (C_{11}+C_{21}+C_{31}) + f_1^1 f_2^1 f_3^2 \max\{C_{11}+C_{21}, C_{32}\} \\
&\quad + f_1^1 f_2^2 f_3^1 \max\{C_{11}+C_{31}, C_{22}\} + f_1^1 f_2^2 f_3^2 \max\{C_{11}, C_{22}+C_{32}\} \\
&\quad + f_1^2 f_2^1 f_3^1 \max\{C_{12}, C_{21}+C_{31}\} + f_1^2 f_2^1 f_3^2 \max\{C_{12}+C_{32}, C_{21}\} \\
&\quad + f_1^2 f_2^2 f_3^1 \max\{C_{12}+C_{22}, C_{31}\} + f_1^2 f_2^2 f_3^2 (C_{12}+C_{22}+C_{32}) \\
&= \frac{6}{7}\cdot\frac{6}{7}\cdot\frac{1}{3}(2+2+0) + \frac{6}{7}\cdot\frac{6}{7}\cdot\frac{2}{3}\max\{2+2,3\} + \frac{6}{7}\cdot\frac{1}{7}\cdot\frac{1}{3}\max\left\{2+0,\frac{3}{2}\right\} \\
&\quad + \frac{6}{7}\cdot\frac{1}{7}\cdot\frac{2}{3}\max\left\{2,\frac{3}{2}+3\right\} + \frac{1}{7}\cdot\frac{6}{7}\cdot\frac{1}{3}\max\left\{\frac{3}{2},2+0\right\} + \frac{1}{7}\cdot\frac{6}{7}\cdot\frac{2}{3}\max\left\{\frac{3}{2}+3,2\right\} \\
&\quad + \frac{1}{7}\cdot\frac{1}{7}\cdot\frac{1}{3}\max\left\{\frac{3}{2}+\frac{3}{2},0\right\} + \frac{1}{7}\cdot\frac{1}{7}\cdot\frac{2}{3}\left(\frac{3}{2}+\frac{3}{2}+3\right) \\
&= \frac{1}{7\cdot 7\cdot 3}\cdot(144+288+12+54+12+54+3+12) \\
&= \frac{193}{49}\,.
\end{aligned}
$$

Thus, $\mathsf{SC}(\mathbf{C},\mathbf{F}) = \frac{193}{49} < 4 = \mathsf{SC}(\mathbf{C},\mathbf{P})$, which implies that the $\mathsf{FMNE}$ Conjecture is not valid.