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A New Model for Selfish Routing*

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Abstract

In this work, we introduce and study a new model for *selfish routing* over *non-cooperative networks* that combines features from the two such best studied models, namely the *KP model* [19] and the *Wardrop model* [11, 30], in an interesting way.

We consider a set of n *users,* each using a *mixed strategy* to ship its *unsplittable traffic* over a *network* consisting of m parallel *links.* In a *Nash equilibrium,* no *user* can increase its *Individual Cost* by unilaterally deviating from its strategy. To evaluate the performance of such Nash equilibria, we introduce *Quadratic Social Cost* as a certain *sum* of Individual Costs – namely, the sum of the expectations of the squares of the incurred *link latencies.* This definition is unlike the KP model, where *Maximum Social Cost* [19] has been defined as the *maximum* of Individual Costs.

We are interested in understanding the impact of our modeling assumptions on the various algorithmic, combinatorial, structural and optimality properties of Nash equilibria in this new model. The findings of our study are outlined as follows:

- We present a comprehensive collection of elegant, combinatorial expressions for the computation of Quadratic Social Cost. These expressions readily imply some polynomial algorithms to compute Quadratic Social Cost in several special cases, and a corresponding *general,* pseudopolynomial, dynamic programming algorithm.
- Through an elegant, combinatorial analysis, we prove, as our main result, that, for the model of identical users and identical links, the *worst-case* Nash equilibrium – the one that maximizes Quadratic Social Cost – is the *fully mixed Nash equilibrium* [22], where each user assigns (strictly) positive probability to every link.
- Finally, we present a comprehensive collection of (usually *tight*) *constant* (that is, independent of m and n) lower and upper bounds on *Quadratic Coordination Ratio,* the variant of *Maximum Coordination Ratio* [19] (defined using Maximum Social Cost) that we define using Quadratic Social Cost, in a number of models.

  These constant bounds stand in very sharp contrast to corresponding $\Theta\left(\frac{\lg m}{\lg \lg m}\right)$ and $O\left(\sqrt{m}\right)$ bounds on Maximum Coordination Ratio, which were previously shown in [9, 14, 18, 19, 22], for the KP model.

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1 Introduction

1.1 Motivation and Framework

1.1.1 Nash Equilibria and Outline

Nash equilibrium [23, 24] is arguably the most robust equilibrium concept in (non-cooperative) Game Theory.* At a Nash equilibrium, no player of a strategic game can unilaterally improve its objective by switching to a different *strategy.* In a *pure* Nash equilibrium, each player chooses exactly one strategy (with probability one); in a *mixed* Nash equilibrium, the choices of each player are modeled by a probability distribution over strategies. Of special interest to our work is the *fully mixed Nash equilibrium* [22], where each user chooses each strategy with non-zero probability. Nash equilibria have some very nice properties; most notably, for finite games, there always exists a mixed Nash equilibrium [24].

In this work, we embark on a systematic study, within a new model for *selfish routing* over *non-cooperative networks* that we introduce, of some interesting algorithmic and mathematical properties of Nash equilibria for some specific *routing game* formulated in this context. Our new model for selfish routing is an interesting hybridization of the two most famous models for selfish routing that were studied in the literature before; these are the so called *KP model* [19] and *Wardrop model* [4, 30].

1.1.2 The KP Model and the Wardrop Model

The KP and the Wardrop models differ with respect to the assumptions they are making about:
- the structure of the *routing network* they consider;
- the *splittability* or *unsplittability* of the users' *traffics*;
- the definition of *Individual Cost* for a user they use for defining Nash equilibria;
- the type of Nash equilibria (pure or mixed) they consider;
- the specific definitions they employ for *Social Cost,* a performance measure for Nash equilibria, and for *Social Optimum,* an optimality measure for traffic assignments (not necessarily equilibria). The definitions for Social Cost usually relate to Individual Costs. In either model, these two definitions give rise to *Coordination Ratio,* the maximum value of the ratio of Social Cost over Social Optimum; a *worst-case* Nash equilibrium is one that maximizes its particular Social Cost.

In the KP model, a collection of n *users* is assumed; each user employs a *mixed strategy,* which is a probability distribution over m parallel *links,* to control the shipping of its own assigned *traffic.* In the KP model, traffics are *unsplittable.* For each link, a *capacity* specifies the rate at which the link processes traffic. Allowing link capacities to vary arbitrarily gives rise to the standard model of *related links.* A special case of the model of related links is the model of *identical links,* where all link capacities are identical. Reciprocally, in the model of *identical users,* all user traffics are equal; they may vary arbitrarily in the model of *arbitrary users.* In a Nash equilibrium, each user selfishly routes its traffic on those links that minimize its *Individual Cost*: its *expected latency cost* on that link, given the network congestion caused by the other users. In the KP model, the Social Cost of a Nash Equilibrium, henceforth called *Maximum Social Cost,* is the expectation, over all random choices of the users, of the maximum, over all links, *latency* through a link; the Social Optimum, henceforth called the *Maximum Social Optimum,* is the *least possible* maximum, over all links, latency through a link that could be attained had global regulation been available; correspondingly, the Coordination Ratio in the KP model will henceforth be called the *Maximum Coordination Ratio.* It follows that the Maximum Social Cost in the KP model is the *maximum* of Individual Costs.

In the Wardrop model [11, 30], there have been considered *arbitrary* networks with *latency functions* for edges. Moreover, the traffics are *splittable* into arbitrary pieces. In this model, unregulated traffic is modeled as a *network flow. Equilibrium flows* have been classified as flows with all flow paths used between a given pair of a *source* and a *destination* having the same latency. Equilibrium flows are optimal solutions to a convex program, in case the edge latency functions are convex. An equilibrium in this model can be interpreted as a Nash equilibrium in a game with infinitely many users, each carrying an infinitesimal amount of traffic from a *source* to a *destination.* Thus, the Wardrop model has been restricted to pure Nash equilibria. The *Individual Cost* of a user is defined as the sum of the edge *latencies* on a path from the user's source to its destination.

*See [25] for a concise introduction to contemporary Game Theory.

The *Social Cost* of a Nash equilibrium is the sum of all Individual Costs. The *Social Optimum* is the least possible, over all network flows, sum of Individual Costs.

1.1.3 The New Model for Selfish Routing

Our new model for selfish routing over non-cooperative networks is a hybridization of the KP model [19] and the Wardrop model [11, 30]. More specifically, we follow the KP model to consider the simple parallel links network (which, however, is also a special case for the Wardrop model). We also follow the KP model to consider unsplittable traffics and mixed Nash equilibria. The Individual Cost we adopt is also identical to that adopted in the KP model – the expected latency cost on a link. However, we follow the Wardrop model to model Social Cost as a certain *sum* of Individual Costs, which we will later describe. In some sense, our new model is the Wardrop model restricted to the simple parallel links network but modified to allow for unsplittable traffics and for mixed strategies; these two features were borrowed from the KP model. Our work is the *first* step towards accomodating unsplittable traffics within the Wardrop model.

For any link, consider the *square* of the traffic through the link divided by the link capacity; taking the expectation of this and adding up over all links yields the Social Cost for our model. Call it *Quadratic Social Cost.* In correspondence with Quadratic Social Cost, we also define and study in our new model *Quadratic Optimum* and *Quadratic Coordination Ratio.* Naturally, the former is the *least possible* sum of the squares of total traffic through a link divided by the link capacity; the latter is the maximum value of the ratio of Quadratic Social Cost over Quadratic Social Optimum. Since Nash equilibria are defined with respect to Individual Costs (but are independent of Social Cost), the Nash equilibria in our new model coincide with those in the KP-model since the two adopt the same Individual Costs.

Note that the commutativity between expectation and sum in the definition of Quadratic Social Cost has been unavailable (between expectation and maximum) in the definition of Maximum Social Cost for the KP model. So, this commutativity allows hopes for some more tractable analysis of several problems regarding some interesting algorithmic, combinatorial, structural and optimality properties of Nash equilibria in the new model.

1.2 Contribution and Significance

We partition our results into three major groups. Results related to combinatorial expressions for Quadratic Social Cost are described in Section 1. Section 1 describes results on identifying the worst-case Nash equilibrium (with respect to Quadratic Social Cost). Finally, results on the Quadratic Coordination Ratio are outlined in Section 1.

1.2.1 Combinatorial Expressions for Quadratic Social Cost

We present a comprehensive collection of combinatorial expressions for Quadratic Social Cost.

- We start with the general model of arbitrary users and arbitrary links. We obtain an elegant, recursive combinatorial formula for Quadratic Social Cost in this most general case (Theorem 3.1). This formula forms the basis for an elegant, dynamic programming algorithm to compute Quadratic Social Cost in the most general setting. The algorithm is pseudopolynomial; it runs in time $O(mnW)$, where $W = \sum_{i \in [n]} w_i$.
- We continue with the model of arbitrary users and identical links. In this setting, we obtain a very simple, combinatorial expression for the Quadratic Social Cost of the fully mixed Nash equilibrium (Theorem 3.2). The expression immediately implies a $\Theta(n)$-time algorithm to compute the Quadratic Social Cost of the fully mixed Nash equilibrium in this setting.

The combinatorial expression significantly simplifies when users also become identical; this simplification results in dropping the time complexity of the algorithm computing the Quadratic Social Cost of the fully mixed Nash equilibrium to $\Theta(1)$.

- We finally turn to the model of identical users and related links. We discover a surprisingly simple, combinatorial expression for the Quadratic Social Cost of the fully mixed Nash equilibrium (Theorem 3.5). The expression immediately implies an algorithm to compute the Quadratic Social Cost of the fully mixed Nash equilibrium in time $\Theta(m)$.

Moreover, the traffic vector $\mathbf{c}$ is involved in this combinatorial expression via the sum $C = \sum_{j \in [m]} c^j$, which appears in the denominator of the fraction. This dependence excludes any possible Braess-like paradoxes [5], where increasing the capacity of a link might result in an increase to the Quadratic Social Cost of the fully mixed Nash equilibrium. This dependence also implies that the Quadratic Social Cost of the fully mixed Nash equilibrium is insensitive to capacity reallocations among the links.

1.2.2 The Worst-case Nash Equilibrium

A natural problem that arises in the context of Quadratic Social Cost is to identify the *worst-case* Nash equilibrium – the one that maximizes, for each specific choice of user traffics and link capacities, the Quadratic Social Cost. We address this problem in the particular setting of the model of identical users and identical links. We use an elegant, combinatorial analysis to prove, as our main result, that, in this particular setting, the worst-case Nash equilibrium is the fully mixed Nash equilibrium.

A natural conjecture due to Gairing *et al.* [17], known as the *Fully Mixed Nash Equilibrium Conjecture,* asserts that the worst-case Nash equilibrium with respect to Social Cost is the fully mixed Nash equilibrium. This conjecture remains yet unproven.

1.2.3 Bounds on Quadratic Coordination Ratio

We next turn to study the Quadratic Coordination Ratio. We obtain a collection of lower and upper bounds on Quadratic Coordination Ratio. The bounds we prove are always constant and usually tight. (Tightness is proved by exhibiting instances attaining the bounds.) To establish our upper bounds, we prove some new structural properties of Nash equilibria for the specific settings we consider; we use these properties to formalize the maximization of Quadratic Coordination Ratio as a non-linear maximization problem, which we solve. We believe that the technique of expressing the maximization of (Quadratic) Coordination Ratio as a non-linear maximization problem, and subsequently solving it, may be of further applicability and interest. Also, some proofs for upper bounds on Quadratic Coordination Ratio employ some combinatorial expressions for Quadratic Social Cost that we derived, and some simple lower bounds on Quadratic Social Optimum.

- We start with the model of arbitrary users and identical links. For this model, we restrict ourselves to pure Nash equilibria. We prove that the Quadratic Coordination Ratio is precisely $\frac{9}{8}$.
- We next turn to the model of identical users and related links. We still restrict ourselves to pure Nash equilibria. For this model, we discover that the Quadratic Coordination Ratio increases slightly to $\frac{4}{3}$.

The different tight bounds of $\frac{9}{8}$ and $\frac{4}{3}$ on Quadratic Coordination Ratio we have described so far imply a *separation* between the model of arbitrary users and identical links and the model of identical users and related links, when both are restricted to pure Nash equilibria, with respect to Quadratic Coordination Ratio.

- We next turn to the model of arbitrary users and identical links. Here, we restrict ourselves to the fully mixed Nash equilibrium. For this setting, we prove an upper bound of $2 - \frac{1}{m}$ on Quadratic Coordination Ratio.
- We next prove that for the model of identical users and identical links, and still restricting ourselves to the fully mixed Nash equilibrium, the Quadratic Coordination Ratio drops slightly to $1 + \min\{\frac{m-1}{,}n, \frac{n-1}{m}\}$, which is always strictly less than 2. In fact, this upper bound reduces to $1 + \varepsilon$, for any constant $\varepsilon > 0$, when either n or m is unboundedly large. By Theorem 4.8, an identical fact follows for the Quadratic Coordination Ratio that takes the maximum over *all* Nash equilibria in the model of identical users and identical links.

1.3 Related Work and Comparison

The KP model was first introduced in the work of Koutsoupias and Papadimitriou [19]; it was further studied in [9, 10, 12, 14, 16, 17, 18, 21, 22]. Fully mixed Nash equilibria were introduced and analyzed in [22]. Bounds on Maximum Coordination Ratio were proved in [9, 14, 18, 22]. The works by Fotakis *et al.* [16], by Gairing *et al.* [17], and by Lücking *et al.* [21] dwelved into the combinatorial structure and the computational complexity of Nash equilibria for the KP model. In particular, the Fully Mixed Nash Equilibrium Conjecture was motivated by some results in [16], explicitly formulated in [17] and further studied in [21].

The Wardrop model was defined in [30] and further studied in [3, 4, 11]. Recent studies of selfish routing within the Wardrop model include [27, 28, 29].

Fotakis *et al.* [16, Theorem 8] proved that computing the Maximum Social Cost of an arbitrary Nash equilibrium is a $\#\mathcal{P}$-complete problem. This hardness result stands in very sharp contrast to our general, pseudopolynomial algorithm to compute Quadratic Social Cost (Theorem 3.1).

For the KP model, there are known bounds on Maximum Coordination Ratio of $\Theta\left(\frac{\lg m}{\lg \lg m}\right)$ for the model of arbitrary users and identical links [9, 18, 19, 22], of $\Theta\left(\frac{\lg m}{\lg \lg m}\right)$ for the model of arbitrary users and arbitrary links [9], and of $O(\sqrt{m})$ for the model of arbitrary users and arbitrary links and for pure Nash equilibria [14]. These super-constant bounds stand in very sharp contrast to some of the constant bounds (independent of m and n) on Quadratic Coordination Ratio we prove in this work. However, for the Wardrop model, there have been shown constant bounds on Coordination Ratio [27, 28, 29].

Other works that have studied Coordination Ratio include [13] for a network creation game and [2] for a network design game. For a survey of recent work on selfish routing in non-cooperative networks, see [15]. Work in the scheduling literature that has considered quadratic cost functions for *makespan* includes [1, 6, 8, 20]; work in the networking literature that has considered quadratic cost functions for network delay includes [7].

1.4 Road Map

The rest of this paper is organized as follows. Section 2 presents our definitions and some preliminaries. The Quadratic Social Cost of Nash equilibria is studied in Section 3. Section 4 proves that the fully mixed Nash equilibrium maximizes Quadratic Social Cost in the model of identical users and identical links. Our bounds on Quadratic Coordination Ratio are presented in Section 5. We conclude, in Section 6, with a discussion of our results and some open problems.

2 Framework

2.1 Mathematical Preliminaries and Notation

Throughout, denote for any integer $m \geq 2$, $[m] = \{1, \ldots, m\}$. For a random variable X, denote $\mathcal{E}(X)$ the *expectation* of X. We continue to prove a simple combinatorial inequality.

Lemma 2.1 *For any integers k, a and b, such that $0 < k \leq a \leq b$,* $\frac{1}{a^k}\binom{a}{k} \leq \frac{1}{b^k}\binom{b}{k}$.

The proof can be found in the appendix.

Finally, we prove a combinatorial lemma that will be useful in a later proof.

Lemma 2.2 *Fix any real number a, where $0 < a < 1$, and positive integer r, and set $A = \frac{r}{a}$. Then,* $A + \frac{r-1}{r}A^2 = \sum_{1 \leq k \leq r} \binom{r}{k} k^2 \left(\frac{1}{a}\right)^k \left(1 - \frac{1}{a}\right)^{r-k}$.

The proof can be found in the appendix.

2.2 General

We consider a *network* consisting of a set of m parallel *links* $1, 2, \ldots, m$ from a *source* node to a *destination* node. Each of n *network users* $1, 2, \ldots, n$, or *users* for short, wishes to route a particular amount of traffic along a (non-fixed) link from source to destination. (Throughout, we will be using subscripts for users and superscripts for links.)

Denote w_i the *traffic* of user $i \in [n]$. Define the $n \times 1$ *traffic vector* $\mathbf{w}$ in the natural way. Assume throughout that $m > 1$ and $n > 1$. Assume also, without loss of generality, that $w_1 \geq w_2 \geq \ldots \geq w_n$. Denote $W = \sum_{i \in [n]} w_i$.

A *pure strategy* for user $i \in [n]$ is some specific link. A *mixed strategy* for user $i \in [n]$ is a probability distribution over pure strategies; thus, a mixed strategy is a probability distribution over the set of links. The *support* of the mixed strategy for user $i \in [n]$, denoted $\mathsf{sup}(i)$, is the set of those pure strategies (links) to which i assigns positive probability. A *pure strategy profile* is represented by an n-tuple $\langle \ell_1, \ell_2, \ldots, \ell_n \rangle \in [m]^n$; a *mixed strategy profile* is represented by an $n \times m$ *probability matrix* $\mathbf{P}$ of nm probabilities p_i^j, $i \in [n]$ and $j \in [m]$, where p_i^j is the probability that user i chooses link j.

For a probability matrix $\mathbf{P}$, define *indicator variables* $I_i^j \in \{0, 1\}$, where $i \in [n]$ and $j \in [m]$, such that $I_i^j = 1$ if and only if $p_i^j > 0$. Thus, the support of the mixed strategy for user $i \in [n]$ is the set $\{j \in [m] \mid I_i^j = 1\}$. For each link $j \in [m]$, define the *view* of link j, denoted $view(j)$, as the set of users $i \in [n]$ that potentially assign their traffics to link j; so, $view(j) = \{i \in [n] \mid I_i^j = 1\}$. For each link $j \in [m]$, denote $V^j = |view(j)|$.

A mixed strategy profile $\mathbf{P}$ is *fully mixed* [22, Section 2.2] if for all users $i \in [n]$ and links $j \in [m]$, $I_i^j = 1$. Throughout, we will cast a pure strategy profile as a special case of a mixed strategy profile in which all (mixed) strategies are pure.

2.3 System, Models and Cost Measures

Denote $c^\ell > 0$ the *capacity* of link $\ell \in [m]$, representing the rate at which the link processes traffic. So, the *latency* for traffic w through link ℓ equals w/c^{ℓ}. In the model of *uniform capacities,* all link capacities are equal to c, for some constant $c > 0$; link capacities may vary arbitrarily in the model of *arbitrary capacities.* Assume throughout, without loss of generality, that $c^1 \geq c^2 \geq \ldots \geq c^m$. Denote $C = \sum_{j \in [m]} c^j$. In the model of *identical traffics,* all user traffics are equal to 1; user traffics may vary arbitrarily in the model of *arbitrary traffics.*

For a pure strategy profile $\langle \ell_1, \ell_2, \ldots, \ell_n \rangle$, the *latency cost for user $i \in [n]$*, denoted λ_i, is $(\sum_{k:\ell_k=\ell_i} w_k)/c^{\ell_i}$; that is, the latency cost for user i is the latency of the link it chooses. For a mixed strategy profile $\mathbf{P}$, denote δ^{ℓ} the *actual traffic* on link $\ell \in [m]$; so, δ^{ℓ} is a random variable. For each link $\ell \in [m]$, denote θ^{ℓ} the *expected traffic* on link $\ell \in [m]$; thus, $\theta^{\ell} = \mathcal{E}(\delta^{\ell}) = \sum_{i=1}^{n} p_i^{\ell} w_i$. Given $\mathbf{P}$, define the $m \times 1$ *expected traffic vector* $\mathbf{\Theta}$ induced by $\mathbf{P}$ in the natural way. Given $\mathbf{P}$, denote Λ^{ℓ} the *expected latency* on link $\ell \in [m]$; clearly, $\Lambda^{\ell} = \frac{\theta^{\ell}}{c^{\ell}}$. Define the $m \times 1$ *expected latency vector* $\mathbf{\Lambda}$ in the natural way. For a mixed strategy profile $\mathbf{P}$, the *expected latency cost* for user $i \in [n]$ on link $\ell \in [m]$, denoted λ_i^{ℓ}, is the expectation, over all random choices of the remaining users, of the latency cost for user i had its traffic been assigned to link ℓ; thus,

$$\lambda_i^{\ell} \quad = \quad \frac{w_i + \sum_{k=1, k \neq i} p_k^{\ell} w_k}{c^{\ell}} = \frac{(1 - p_i^{\ell}) w_i + \theta^{\ell}}{c^{\ell}} .$$

For each user $i \in [n]$, the *minimum expected latency cost,* denoted λ_i, is the minimum, over all links $\ell \in [m]$, of the expected latency cost for user i on link ℓ; thus, $\lambda_i = \min_{\ell \in [m]} \lambda_i^{\ell}$. For a probability matrix $\mathbf{P}$, define the $n \times 1$ *minimum expected latency cost vector* λ induced by $\mathbf{P}$ in the natural way.
Associated with a traffic vector $\mathbf{w}$, a capacity vector $\mathbf{c}$ and a mixed strategy profile $\mathbf{P}$ is the *quadratic social cost*, denoted $\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P})$, which is the expectation of the sum of squares of the incurred link latencies; thus,

$$\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P}) \quad = \quad \mathcal{E}\left(\sum_{\ell \in [m]} \frac{\left(\sum_{k:\ell_k=\ell} w_k\right)^2}{c^{\ell}} \right) \quad = \sum_{\langle \ell_1, \ell_2, \ldots, \ell_n \rangle \in [m]^n} \left(\prod_{k=1}^{n} p_k^{\ell_k} \cdot \sum_{\ell \in [m]} \frac{\left(\sum_{k:\ell_k=\ell} w_k\right)^2}{c^{\ell}} \right) .$$

Since the expectation of a sum is equal to the sum of expectations, we can write

$$\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P}) \quad = \quad \sum_{\ell \in [m]} \sum_{A \subset [n]} \left(\prod_{i \in A} p_i^{\ell} \right) \left(\prod_{i \notin A} (1 - p_i^{\ell}) \right) \frac{\left(\sum_{k:\ell_k=\ell} w_k\right)^2}{c^{\ell}} . \tag{1}$$

Note that $\mathsf{QSC}\,(\mathbf{w}, \mathbf{P})$ reduces to

$$\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P}) \quad = \quad \sum_{\ell \in [m]} \sum_{A \subset [n]} \left(\prod_{i \in A} p_i^{\ell} \right) \left(\prod_{i \notin A} (1 - p_i^{\ell}) \right) \frac{|A|^2}{c^{\ell}} .$$

if all traffics are equal, that is, $w_i = 1$ for all $i \in [n]$. The *quadratic social optimum* associated with a traffic vector $\mathbf{w}$ and a capacity vector $\mathbf{c}$, denoted $\mathsf{QOPT}(\mathbf{w}, \mathbf{c})$, is the *least possible* sum of squares of the incurred link latencies; thus,

$$\mathsf{QOPT}(\mathbf{w}, \mathbf{c}) \quad = \quad \min_{\langle \ell_1, \ell_2, \ldots, \ell_n \rangle \in [m]^n} \sum_{\ell \in [m]} \frac{\left(\sum_{k:\ell_k=\ell} w_k\right)^2}{c^{\ell}} .$$

Note that while $\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P})$ is defined in relation to a mixed strategy profile $\mathbf{P}$, $\mathsf{QOPT}(\mathbf{w}, \mathbf{c})$ refers to the *optimum* pure strategy profile.

The *Social Cost*, denoted $\mathsf{MSC}(\mathbf{w}, \mathbf{c}, \mathbf{P})$, which is used in the original KP-Model, is defined as the expectation of the maximum of the incurred link latencies. Correspondingly, the *optimum social cost*, denoted $\mathsf{MOPT}(\mathbf{w}, \mathbf{c})$, is the minimum, over all assignments, maximum incurred link latency.

2.4 Nash Equilibria

We are interested in a special class of mixed strategies called Nash equilibria [24] that we describe below. Formally, the probability matrix $\mathbf{P}$ is a *Nash equilibrium* [19, Section 2] if for all users $i \in [n]$ and links $\ell \in [m]$, $\lambda_i^\ell = \lambda_i$ if $I_i^\ell = 1$, and $\lambda_i^\ell > \lambda_i$ if $I_i^\ell = 0$. Thus, each user assigns its traffic with positive probability only on links (possibly more than one of them) for which its expected latency cost is minimized; this implies that there is no incentive for a user to unilaterally deviate from its mixed strategy in order to avoid links on which its expected latency cost is higher than necessary.

Mavronicolas and Spirakis [22, Lemma 15] show that in the model of arbitrary users and identical links, all links are equiprobable in a fully mixed Nash equilibrium.

Lemma 2.3 (Mavronicolas and Spirakis [22]) *Consider the model of arbitrary users and identical links. Then, there exists a unique fully mixed Nash equilibrium with associated Nash probabilities $p_i^\ell = 1/m$, for any user $i \in [n]$ and link $\ell \in [m]$.*

2.5 Coordination Ratio and Quadratic Coordination Ratio

The *Quadratic Coordination Ratio* is the maximum value, over all traffic vectors $\mathbf{w}$, capacity vectors $\mathbf{c}$, and Nash equilibria $\mathbf{P}$ of the ratio $\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P}) / \mathsf{QOPT}(\mathbf{w}, \mathbf{c})$. In a corresponding way, we define *Coordination Ratio* as the maximum value, over all traffic vectors $\mathbf{w}$, capacity vectors $\mathbf{c}$ and Nash equilibria $\mathbf{P}$ of the ratio $\mathsf{MSC}(\mathbf{w}, \mathbf{c}, \mathbf{P}) / \mathsf{MOPT}(\mathbf{w}, \mathbf{c})$.

3 The Quadratic Social Cost of Nash Equilibria

In this section, we study the Quadratic Social Cost of arbitrary (mixed) Nash equilibria. We start by proving:

Theorem 3.1 (Quadratic Social Cost of Arbitrary Nash Equilibrium) *Fix any traffic vector $\mathbf{w}$, any capacity vector $\mathbf{c}$, and any arbitrary Nash equilibrium $\mathbf{P}$. Then, $\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P})$ can be computed in time $O(nmW)$.*

The proof can be found in the appendix.We next establish that the Quadratic Social Cost takes a particularly nice form for the case of the fully mixed Nash equilibrium. We prove:

Theorem 3.2 (Quadratic Social Cost of Fully Mixed Nash Equilibrium) *Consider the model of arbitrary users and identical links. Then, for any traffic vector $\mathbf{w}$,*

$$\mathsf{QSC}(\mathbf{w}, \mathbf{F}) \quad = \quad W_1 + \frac{2}{m} W_2 \,,$$

where $W_1 = \sum_{i \in [n]} w_i^2$ and $W_2 = \sum_{i,k \in [n], i \neq k} w_i w_k$.

The proof can be found in the appendix. The next Lemma is used in the proof of Proposition 3.4.

Lemma 3.3 *Let $a, n \in \mathbb{N}$, a even, let $p_i \in [0,1]$ for all $1 \leq i \leq n$. Denote $P = (p_1, \ldots, p_n)$ and $p = \sum_{1 \leq i \leq n} p_i$. Set*

$$H(P) = \sum_{A \subset [n]} |A|^a \left\{ \prod_{i \in A} p_i \right\} \left\{ \prod_{j \notin A} (1 - p_j) \right\}.$$

Define $\tilde{P}$ by $\tilde{p}_i = \frac{1}{n} \cdot p$ for all $1 \leq i \leq n$. Then $H(P) \leq H(\tilde{P})$.

The proof can be found in the appendix.

Proposition 3.4 *Consider the model of identical users and identical links. Then, for any arbitrary Nash equilibrium $\mathbf{P}$,*

$$\mathsf{QSC}(\mathbf{P}) \quad \leq \quad \sum_{j \in [m]} \left(\theta^j + \frac{r_j - 1}{r_j} (\theta^j)^2 \right),$$

where $\theta^j = \sum_{i \in [n]} p_i^j$ and $r_j = |view(j)|$.

The proof can be found in the appendix.We finally prove:

Theorem 3.5 *Consider the model of identical users and related links. Then,*

$$\mathsf{QSC}(\mathbf{c}, \mathbf{F}) = \frac{n(n+m-1)}{C}.$$

The proof can be found in the appendix.

Corollary 3.6 (Quadratic Social Cost of Fully Mixed Nash Equilibrium) *Consider the model of identical users and identical links. Then,*

$$\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{F}) = \frac{n(n+m-1)}{m}.$$

4 The Worst-case Nash Equilibrium

In this section, we establish that, for the model of identical users and identical links, the worst-case Nash equilibrium is the fully mixed Nash equilibrium. We start our proof with a technical lemma which holds in the more general model of arbitrary users, and then return to the model of identical users and identical links.

Lemma 4.1 *Consider n arbitrary users on m identical links, and let $j, k \in [m]$.*

1. *If $view(j) = view(k) \neq \emptyset$, then $|view(j)| = 1$ or $\theta^j = \theta^k$ and $p_i^j = p_i^k$ for all $i \in [n]$.*
2. *If $view(j) \subsetneq view(k)$, then $\theta^j > \theta^k$.*

The proof can be found in the appendix.

Theorem 4.2 *Consider the model of identical users and identical links. Then, for any arbitrary Nash equilibrium $\mathbf{P}$, $\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P}) \leq \mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{F})$.*

The proof can be found in the appendix.We give here only a sketch of the full proof:
Let $\mathbf{P}$ be an arbitrary Nash equilibrium. Let $\alpha = \frac{n}{m}$ be the ratio of the number of users and the number of links. Proposition 3.4 implies

$$\begin{aligned} \mathsf{QSC}(\mathbf{P}) &\leq \sum_{j \in [m]} \left(\theta^j + \frac{r_j - 1}{r_j} (\theta^j)^2 \right) \\ &= n + \sum_{j \in [m]} \frac{r_j - 1}{r_j} (\theta^j)^2. \end{aligned}$$

Corollary 3.6 implies

$$\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{F}) = n + \frac{n(n-1)}{m} = n + \sum_{j \in [m]} \frac{n-1}{n} \alpha^2.$$

In general some of the θ^j are larger than α and some are smaller. $\sum_{j \in [m]} (\theta^j)^2$ is minimized among all distributions $\{\theta^j\}_{j=1,\ldots,m}$ with $\sum_{j \in [m]} \theta^j = n$, if $\theta^j = \alpha$ for all j. So the influence of the r_j has to be taken into account.
Lemma 4.1 implies, that, if $r_j = n$, then $\theta^j \leq \theta^k$ for all $k \in [m]$. So $r_j = n$ implies $\theta^j \leq \alpha$.
A careful analysis (given in the full proof) shows, that this property together with some upper bound which holds for the θ^j is sufficient to prove the fully mixed conjecture for this model.

5 Bounds on Quadratic Coordination Ratio

In this section, we present our bounds on Quadratic Coordination Ratio. We start by proving:

Theorem 5.1 (Quadratic Coordination Ratio for Pure Nash Equilibria) *Consider the model of arbitrary users and identical links, restricted to pure Nash equilibria. Then,*

$$\max_{\mathbf{w},\mathbf{P}} \frac{\mathsf{QSC}(\mathbf{w},\mathbf{P})}{\mathsf{QOPT}(\mathbf{w})} = \frac{9}{8}.$$

The proof can be found in the appendix.We continue to prove:

Theorem 5.2 (Quadratic Coordination Ratio for Pure Nash Equilibria) *Consider the model of identical users and related links, restricted to pure Nash equilibria. Then,*

$$\max_{\mathbf{w},\mathbf{c},\mathbf{P}} \frac{\mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{P})}{\mathsf{QOPT}(\mathbf{w},\mathbf{c})} = \frac{4}{3}.$$

The proof can be found in the appendix.We next prove:

Theorem 5.3 *Consider the model of arbitrary users and identical links. Then,*

$$\max_{\mathbf{w},\mathbf{c}} \frac{\mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{F})}{\mathsf{QOPT}(\mathbf{w},\mathbf{c})} \leq 2 - \frac{1}{m}.$$

The proof can be found in the appendix.We next prove:

Theorem 5.4 *Consider the model of identical users and identical links. Then, for any traffic vector* $\mathbf{w}$ *and capacity vector* $\mathbf{c}$*, and any mixed Nash equilibrium* $\mathbf{P}$*,*

$$\frac{\mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{P})}{\mathsf{QOPT}(\mathbf{w},\mathbf{c})} \leq 1 + \min\left\{\frac{m-1}{n}, \frac{n-1}{m}\right\} \leq 2 - \frac{1}{m}.$$

6 Conclusion and Directions for Further Research

We have presented a new model for selfish routing over non-cooperative networks as an interesting hybrid variant of the two best studied models for selfish routing, namely the KP model [19] and the Wardrop model [4, 30]. Within this model, we focused on certain interesting algorithmic, combinatorial, structural and optimality properties of Nash equilibria for a certain routing game formulated in this context. Our findings furnish a significant milestone to the theory of Nash equilibria in this new model for selfish routing.

We conclude this article with a glimpse of questions which naturally pose themselves within the hybrid model for selfish routing we have defined, and which remain tantalizingly open.

1. What is the time complexity of computing Quadratic Social Cost of an arbitrary Nash equilibrium in the general model of arbitrary users and arbitrary links? Theorem 3.1 provides a corresponding pseudopolynomial algorithm; does there exist a polynomial one? Also, no hardness results are known for the problem of computing the Quadratic Social Cost of an arbitrary Nash equilibrium.

2. Theorem 4.2 proves that the fully mixed Nash equilibrium maximizes Quadratic Social Cost in the model of identical users and identical links. Does a corresponding fact hold for the most general model of arbitrary users and arbitrary links? In close analogy to the yet unproven Fully Mixed Nash Equilibrium Conjecture [17] (namely, that the fully mixed Nash equilibrium maximizes, when existing, Social Cost), we conjecture it does:

   Conjecture 6.1 (Quadratic Fully Mixed Nash Equilibrium Conjecture) *Consider the model of arbitrary users and related links. Then, the fully mixed Nash equilibrium maximizes Quadratic Social Cost when existing.*

3. What is the value of Quadratic Coordination Ratio for the model of arbitrary users and identical links, and for the model of identical users and related links, respectively, when all Nash equilibria are considered?

   Theorems 5.1, 5.2 and 5.3 show constant upper bounds when restricted to pure Nash equilibria or to the fully mixed Nash equilibrium. Does the quadratic coordination ratio remain constant when all Nash equilibria are considered? Of course, this would follow from Conjecture 6.1

4. What is the value of Quadratic Coordination Ratio for the model of arbitrary users and related links, when restricted to pure Nash equilibria?

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Appendix

Proof of Lemma 2.1:

Note that for all integers $i \leq k$, $\frac{a-i}{a} \leq \frac{b-i}{b}$. Hence,

$$\begin{aligned}
\frac{1}{a^k}\binom{a}{k} &= \frac{a(a-1)\cdots(a-k+1)}{a^k}\cdot\frac{1}{k!} \\
&\leq \frac{b(b-1)\cdots(b-k+1)}{b^k}\cdot\frac{1}{k!} \\
&= \frac{1}{b^k}\binom{b}{k}
\end{aligned}$$

as needed.

Proof of Lemma 2.2:

We have

$$\begin{aligned}
G &= \sum_{1\leq k\leq r}\binom{r}{k}k^2\left(\frac{1}{a}\right)^k\left(1-\frac{1}{a}\right)^{r-k} \\
&= \frac{1}{a^r}\sum_{1\leq k\leq r} r\cdot k\cdot\binom{r-1}{k-1}(a-1)^{r-k} \\
&= \frac{r}{a^r}\left(\sum_{1\leq k\leq r}\binom{r-1}{k-1}(a-1)^{r-k}+\sum_{2\leq k\leq r}(k-1)\binom{r-1}{k-1}(a-1)^{r-k}\right) \\
&= \frac{r}{a^r}\left(\sum_{0\leq k\leq r-1}\binom{r-1}{k}(a-1)^{r-1-k}+(r-1)\sum_{2\leq k\leq r}\binom{r-2}{k-2}(a-1)^{r-k}\right) \\
&= \frac{r}{a^r}\left(a^{r-1}+(r-1)\sum_{0\leq k\leq r-2}\binom{r-2}{k}(a-1)^{r-2-k}\right) \\
&= \frac{r}{a}+\frac{r(r-1)}{a^2} \\
&= A+\frac{r-1}{r}A^2,
\end{aligned}$$

proving the claim.

Proof of Theorem 3.1:

Clearly,

$$\begin{aligned}
&\mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{P}) \\
=\ & \textstyle\sum_{j\in[m]}\mathcal{E}\left(\frac{(\delta^j(\mathbf{P}))^2}{c^j}\right) && \text{(by definition of Quadratic Social Cost)} \\
=\ & \textstyle\sum_{j\in[m]}\frac{\sum_{A\subseteq[n]}\prod_{i\in A}p_i^j\prod_{i\notin A}(1-p_i^j)\left(\sum_{i\in A}w_i\right)^2}{c^j} && \text{(by definition of expected quadratic latency)}\,.
\end{aligned}$$

Define now the function

$$\varepsilon_{\mathbf{P}}(j,k,r) = \sum_{A\subseteq[k]}\prod_{i\in A}p_i^j\prod_{i\notin A}(1-p_i^j)\left(r+\sum_{i\in A}w_i\right)^2.$$

This definition implies that

$$\mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{P}) = \sum_{j\in[m]}\frac{\varepsilon_{\mathbf{P}}(j,n,0)}{c^j}.$$

Note, however, that

$$
\begin{aligned}
& \varepsilon_{\mathbf{P}}(j,k,r) \\
= & \sum_{A\subseteq[k],k\in A}\prod_{i\in A}p_i^j\prod_{i\notin A}(1-p_i^j)\left(r+\sum_{i\in A}w_i\right)^2+\sum_{A\subseteq[k],k\notin A}\prod_{i\in A}p_i^j\prod_{i\notin A}(1-p_i^j)\left(r+\sum_{i\in A}w_i\right)^2 \\
= & \sum_{A\subseteq[k],k\in A}p_k^j\prod_{i\in A,i\neq k}p_i^j\prod_{i\notin A}(1-p_i^j)\left(r+w_k+\sum_{i\in A,i\neq k}w_i\right)^2 \\
& +\sum_{A\subseteq[k],k\notin A}\prod_{i\in A}p_i^j\,(1-p_k^j)\prod_{i\notin A,i\neq k}(1-p_i^j)\left(r+\sum_{i\in A}w_i\right)^2 \\
= & p_k^j\sum_{A\subseteq[k-1]}\prod_{i\in A}p_i^j\prod_{i\notin A}(1-p_i^j)\left(r+w_k+\sum_{i\in A,i\neq k}w_i\right)^2 \\
& +(1-p_k^j)\sum_{A\subseteq[k-1]}\prod_{i\in A}p_i^j\prod_{i\notin A}(1-p_i^j)\left(r+\sum_{i\in A}w_i\right)^2 \\
= & p_k^j\varepsilon(j,k-1,r+w_k)+(1-p_k^j)\varepsilon(j,k-1,r)\,.
\end{aligned}
$$

It follows that

$$
\mathsf{QSC}\left(\mathbf{w},\mathbf{c},\mathbf{P}\right) = \sum_{j\in[m]}\frac{p_n^j\varepsilon(j,n-1,w_n)+(1-p_n^j)\varepsilon(j,n-1,0)}{c^j}\,.
$$

Hence, $\mathsf{QSC}\left(\mathbf{w},\mathbf{c},\mathbf{P}\right)$ can be computed using this dynamic programming formula in time $O(nmW)$, as needed.
Proof of Theorem 3.2:
Clearly,

$$
\begin{aligned}
& \mathsf{QSC}\left(\mathbf{F}\right) & \\
= & \sum_{j\in[m]}\mathcal{E}\left((\delta^j(\mathbf{F}))^2\right) & \text{(by def. of Quadratic Social Cost)} \\
= & m\,\mathcal{E}\left((\delta^1(\mathbf{F}))^2\right) & \text{(since all links are identical)} \\
= & m\sum_{k\in[n]}\sum_{A\subseteq[n],|A|=k}\left(\prod_{i\in A}p_i^1\prod_{i\notin A}\left(1-p_i^1\right)\left(\sum_{i\in A}w_i\right)^2\right) & \text{(by def. of expected quadratic latency)} \\
= & m\sum_{k\in[n]}\sum_{A\subseteq[n],|A|=k}\left(\left(\tfrac{1}{m}\right)^k\left(1-\tfrac{1}{m}\right)^{n-k}\left(\sum_{i\in A}w_i\right)^2\right) & \text{(by Lemma 2.3)} \\
= & m\sum_{k\in[n]}\left(\tfrac{1}{m}\right)^k\left(1-\tfrac{1}{m}\right)^{n-k}\sum_{A\subseteq[n],|A|=k}\left(\sum_{i\in A}w_i\right)^2 & \\
= & \tfrac{1}{m^{n-1}}\sum_{k\in[n]}(m-1)^{n-k}\sum_{A\subseteq[n],|A|=k}\left(\sum_{i\in A}w_i\right)^2\,. &
\end{aligned}
$$

Consider now the sum $\sum_{A\subseteq[n],|A|=k}\left(\sum_{i\in A}w_i\right)^2$, where $k\geq 0$, and note that:

- For $k=0$, $\sum_{A\subseteq[n],|A|=k}\left(\sum_{i\in A}w_i\right)^2=0$.
- Fix now $k=1$. Then, for each $i\in[n]$, the term w_i^2 appears $1=\binom{n-1}{k-1}$ time in this sum, and no other term appears. Thus, for $k=1$,

$$
\begin{aligned}
\sum_{A\subseteq[n],|A|=k}\left(\sum_{i\in A}w_i\right)^2 &= \sum_{i\in[n]}\binom{n-1}{k-1}w_i^2 \\
&= \binom{n-1}{k-1}\sum_{i\in[n]}w_i^2 \\
&= \binom{n-1}{k-1}W_1\,.
\end{aligned}
$$

- Fix now any $k \geq 2$. Then, for each $i \in [n]$, the term w_i^2 appears $\binom{n-1}{k-1}$ times in this sum; for each pair $i, k \in [n]$ with $i \neq k$, the term $w_i w_k$ appears $2\binom{n-2}{k-2}$ times in the same sum. Hence,

$$\begin{aligned}
\sum_{A \subseteq [n], |A|=k} \left(\sum_{i \in A} w_i \right)^2 &= \sum_{i \in [n]} \binom{n-1}{k-1} w_i^2 + \sum_{i,j \in [n], i \neq j} 2 \binom{n-2}{k-2} w_i w_j \\
&= \binom{n-1}{k-1} \sum_{i \in [n]} w_i^2 + 2 \binom{n-2}{k-2} \sum_{i.j \in [n], i \neq j} w_i w_j \\
&= \binom{n-1}{k-1} W_1 + 2 \binom{n-2}{k-2} W_2
\end{aligned}$$

Hence, it follows that

$$\begin{aligned}
\mathsf{QSC}(\mathbf{F}) &= \frac{1}{m^{n-1}} \sum_{k \in [n]} (m-1)^{n-k} \cdot \begin{cases} \binom{n-1}{k-1} W_1, & \text{for } k = 1 \\ \left(\binom{n-1}{k-1} W_1 + 2 \binom{n-2}{k-2} W_2 \right) & \text{for } k \geq 2 \end{cases} \\
&= \frac{1}{m^{n-1}} \left(W_1 \sum_{1 \leq k \leq n} (m-1)^{n-k} \binom{n-1}{k-1} + 2W_2 \sum_{2 \leq k \leq n} (m-1)^{n-k} \binom{n-2}{k-2} \right) \\
&= \frac{1}{m^{n-1}} \left(W_1 \sum_{0 \leq k \leq n-1} (m-1)^{n-k-1} \binom{n-1}{k} + 2W_2 \sum_{0 \leq k \leq n-2} (m-1)^{n-k-2} \binom{n-2}{k} \right) \\
&= \frac{1}{m^{n-1}} \left(W_1 \, m^{n-1} + 2 \, W_2 \, m^{n-2} \right) \\
&= W_1 + \frac{2}{m} W_2 \, ,
\end{aligned}$$

as needed.

Proof of Lemma 3.3:
Define q_i, $1 \leq i \leq n$, by

$$q_i = \begin{cases} p_i & \text{for } i \geq 3, \\ \frac{1}{2}(p_1 + p_2) & \text{for } i = 1, 2. \end{cases}$$

We first show $H(P) \leq H(Q)$. We have

$$H(P) = \sum_{B \subset [3,n]} \left\{ \prod_{i \in B} p_i \right\} \left\{ \prod_{j \notin B \cup \{1,2\}} (1 - p_j) \right\} \cdot F(B, P)$$

with

$$F(B, P) = |B|^2 (1 - p_1)(1 - p_2) + (|B| + 2)^2 p_1 p_2 + (|B| + 1)^2 (p_1(1 - p_2) + p_2(1 - p_1)).$$

Here, $F(B, P)$ represents the sets $A = B$, $A = B \cup \{1\}$, $A = B \cup \{2\}$ and $A = B \cup \{1, 2\}$ in the formula for $H(P)$. Denote $b = |B| + 1$. It follows that

$$\begin{aligned}
F(B, P) &= (b-1)^2 (1 - (p_1 + p_2) + p_1 p_2) + (b+1)^2 p_1 p_2 + b^2 (p_1 + p_2 - 2p_1 p_2) \\
&= \alpha(p_1 + p_2) + \beta p_1 p_2 + (b-1)^2
\end{aligned}$$

with

$$\begin{aligned}
\alpha &= b^2 - (b-1)^2 = 2b - 1 > 0, \\
\beta &= (b-1)^2 + (b+1)^2 - 2b^2 = 1 > 0
\end{aligned}$$

It also holds that

$$H(P) = \sum_{B\subset[3,n]} \left\{\prod_{i\in B} p_i\right\} \left\{\prod_{j\notin B\cup\{1,2\}} (1-p_j)\right\} \cdot F(B,Q)$$

with

$$F(B,Q) \quad = \quad \alpha(p_1+p_2) + \beta\left(\frac{p_1+p_2}{2}\right)^2 + (b-1)^2$$

It follows that

$$\begin{aligned} F(B,Q) - F(B,P) &= \beta\left(\frac{p_1+p_2}{2}\right) - \beta p_1 p_2 \\ &= \frac{1}{4}\beta\cdot(p_1^2 + 2p_1p_2 + p_2^2 - 4p_1p_2) \\ &= \frac{1}{4}\beta\cdot(p_1-p_2)^2 \geq 0. \end{aligned}$$

This implies $H(Q) \geq H(P)$. Proceeding like this, we get $H(\tilde{P}) \geq H(P)$.

Proof of Proposition 3.4:

We have

$$\begin{aligned} \mathsf{QSC}(\mathbf{P}) &= \sum_{1\leq j\leq m}\sum_{B\subset[n]} |B|^2 \left(\prod_{i\in B} p_i^j\right)\left(\prod_{i\notin B}(1-p_i^j)\right) \\ &= \sum_{1\leq j\leq m}\sum_{B\subset view(j)} |B|^2 \left(\prod_{i\in B} p_i^j\right)\left(\prod_{i\notin B}(1-p_i^j)\right) \\ &\leq \sum_{1\leq j\leq m}\sum_{B\subset view(j),} |B|^2 \left(\frac{1}{a_j}\right)^{|B|}\left(1-\frac{1}{a_j}\right)^{n-|B|} \end{aligned}$$

with $a_j = \frac{r_j}{\theta^j}$ due to Lemma 3.3. Lemma 2.2 implies

$$\begin{aligned} \mathsf{QSC}(\mathbf{P}) &\leq \sum_{1\leq j\leq m}\sum_{1\leq k\leq r_j} \binom{r_j}{k} k^2 \left(\frac{1}{a_j}\right)^k \left(1-\frac{1}{a_j}\right)^{r_j-k} \\ &= \sum_{1\leq j\leq m}\left(\theta^j + \frac{r_j-1}{r_j}(\theta^j)^2\right). \end{aligned}$$

Proof of Theorem 3.5:

Mavronicolas and Spirakis ([22]) showed, that for any user $i\in[n]$ and link $j\in[m]$,

$$\begin{aligned} f_i^j &= \left(1-\tfrac{mc^j}{C}\right)\cdot\left(1-\tfrac{W}{(n-1)w_i}\right) + \tfrac{c^j}{C} \\ &= \left(1-\tfrac{mc^j}{C}\right)\cdot\left(1-\tfrac{n}{n-1}\right) + \tfrac{c^j}{C} \qquad \text{(since users are identical)} \\ &= \left(1-\tfrac{mc^j}{C}\right)\cdot\left(-\tfrac{1}{n-1}\right) + \tfrac{c^j}{C} \\ &= \left(\tfrac{m}{(n-1)C} + \tfrac{1}{C}\right)c^j - \tfrac{1}{n-1} \\ &= \tfrac{m+n-1}{(n-1)C}\,c^j - \tfrac{1}{n-1}\,. \end{aligned}$$

Using (1) for $\mathsf{QSC}(\mathbf{c}, \mathbf{F})$ and the fact that all users are identical, that is, $w_i = 1$ for all $i \in [n]$, we get

$$
\begin{aligned}
& \mathsf{QSC}(\mathbf{c}, \mathbf{F}) \\
= & \textstyle\sum_{j\in[m]} \mathcal{E}\left(\frac{\tau_{\mathbf{F}}(j)}{c^j}\right) \\
& \text{(by definition of Quadratic Social Cost)} \\
= & \textstyle\sum_{j\in[m]} \frac{\mathcal{E}(\tau_{\mathbf{F}})}{c^j} \\
= & \textstyle\sum_{j\in[m]} \left(\frac{1}{c^j} \sum_{k\in[n]} \sum_{A\subseteq[n],|A|=k} \left(\prod_{i\in A} f_i^j \cdot \prod_{i\notin A}\left(1-f_i^j\right) k^2\right)\right) \\
& \text{(by definition of expected quadratic traffic)} \\
= & \textstyle\sum_{j\in[m]} \left(\frac{1}{c^j} \sum_{k\in[n]} \sum_{A\subseteq[n],|A|=k} \left(f_1^j\right)^k \cdot \left(1-f_1^j\right)^{n-k} k^2\right) \\
& \text{(since users are identical)} \\
= & \textstyle\sum_{j\in[m]} \left(\frac{1}{c^j} \sum_{k\in[n]} \binom{n}{k} \left(f_1^j\right)^k \left(1-f_1^j\right)^{n-k} k^2\right) \\
= & \textstyle\sum_{j\in[m]} \frac{1}{c^j} \left(n f_1^j + \frac{n-1}{n}\left(f_1^j\right)^2\right) \\
& \text{(by Lemma 2.2)} \\
= & \textstyle\sum_{j\in[m]} \frac{1}{c^j} \left(n\left(\frac{m+n-1}{(n-1)C} c^j - \frac{1}{n-1}\right) + \frac{n-1}{n}\left(n\left(\frac{m+n-1}{(n-1)C} c^j - \frac{1}{n-1}\right)\right)^2\right) \\
= & \textstyle\sum_{j\in[m]} \frac{n(m+n-1)}{(n-1)C} - \sum_{j\in[m]} \frac{n}{n-1}\frac{1}{c^j} + \sum_{j\in[m]} \left(n(n-1)\frac{1}{c^j}\left(\frac{m+n-1}{(n-1)C} c^j - \frac{1}{n-1}\right)^2\right) \\
= & \textstyle\frac{n(m+n-1)}{(n-1)C} \sum_{j\in[m]} 1 - \frac{n}{n-1} \sum_{j\in[m]} \frac{1}{c^j} + \sum_{j\in[m]} \left(n(n-1)\frac{1}{c^j}\frac{(m+n-1)^2}{(n-1)^2C^2}(c^j)^2\right) \\
& \textstyle - \sum_{j\in[m]} \left(n(n-1)\frac{1}{c^j} \cdot 2\frac{m+n-1}{(n-1)C} c^j \cdot \frac{1}{n-1}\right) + \sum_{j\in[m]} \left(n(n-1)\frac{1}{c^j}\left(\frac{1}{n-1}\right)^2\right) \\
= & \textstyle\frac{mn(m+n-1)}{(n-1)C} - \frac{n}{n-1} \sum_{j\in[m]} \frac{1}{c^j} + \frac{n(m+n-1)^2}{(n-1)C^2} \sum_{j\in[m]} c^j \\
& \textstyle - \frac{2n(m+n-1)}{(n-1)C} \sum_{j\in[m]} 1 + \frac{n}{n-1} \sum_{j\in[m]} \frac{1}{c^j} \\
= & \frac{mn(m+n-1)}{(n-1)C} + \frac{n(m+n-1)^2}{(n-1)C^2} C - \frac{2mn(m+n-1)}{(n-1)C} \\
= & \frac{n(m+n-1)}{(n-1)C} \cdot (m+m+n-1-2m) \\
= & \frac{n(m+n-1)}{C},
\end{aligned}
$$

as needed.

Proof of Lemma 4.1:

1. From the Nash equilibrium condition we obtain

$$
\begin{aligned}
& \forall i \in \mathit{view}(j) : \theta^j - p_i^j + 1 = \theta^k - p_i^k + 1 && (2) \\
\Rightarrow\ & \sum_{i\in \mathit{view}(j)} (\theta^j - p_i^j) = \sum_{i\in \mathit{view}(j)} (\theta^k - p_i^k) && (3) \\
\Leftrightarrow\ & (|\mathit{view}(j)| - 1)\theta^j = (|\mathit{view}(j)| - 1)\theta^k \\
\Leftrightarrow\ & \mathit{view}(j) = 1 \vee \theta^j = \theta^k\,. && (4)
\end{aligned}
$$

(2) and (4) yield, that, if $\mathit{view}(j) > 1$, then $p_i^j = p_i^k$ for all $i \in [n]$.

2. In this case, it follows from (3), that

$$\begin{aligned}
& (|view(j)|-1)\theta^j = |view(j)|\theta^k - \sum_{i\in view(j)} p_i^k \\
& > |view(j)|\theta^k - \sum_{i\in view(k)} p_i^k \\
& = (|view(j)|-1)\theta^k \\
\Leftrightarrow \quad & \theta^j > \theta^k.
\end{aligned} \tag{5}$$

Proof of Theorem 4.2:
Let $\mathbf{P}$ be an arbitrary Nash Equilibrium. Let $\alpha = \frac{n}{m}$ be the ratio of the number of users n and the number of links m. For all $j \in [m]$ we denote $a_j = \theta^j - \alpha$, $r_j = |view(j)|$, and $q_j = \min_{i\in[n]} \left\{ p_i^j \mid p_i^j > 0 \right\}$. We furthermore denote

$$M_1 = \{j \in [m] \mid \theta^j \leq \alpha\} \qquad \text{and} \qquad M_2 = \{j \in [m] \mid \theta^j > \alpha\}.$$

We have

$$\begin{aligned}
q_j &\leq \frac{\theta^j}{r_j} \quad \text{for all } j \in [m], \\
a_j &\leq 0 \quad \text{for all } j \in M_1, \\
a_j &> 0 \quad \text{for all } j \in M_2, \\
\sum_{j\in[m]} a_j &= 0.
\end{aligned}$$

In order to prove the claim, we first compute the social cost of $\mathbf{P}$. Because of Proposition 3.4 we have

$$\mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{P}) \leq n + S_1 + S_2$$

with

$$S_i = \sum_{j\in M_i} \frac{r_j - 1}{r_j} \cdot (\alpha + a_j)^2$$

for $i = 1, 2$. If $a_j = 0$ for all $j \in [m]$, then

$$\begin{aligned}
\mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{P}) &\leq n + \sum_{j\in[m]} \frac{r_j - 1}{r_j} \cdot \alpha^2 \\
&\leq n + \sum_{j\in[m]} \frac{n-1}{n} \cdot \alpha^2 = \mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{F}).
\end{aligned}$$

Now assume $a_j \neq 0$ for some $j \in [m]$. Denote $a = |\min_{j\in[m]} a_j|$. Then $a > 0$, and for all $j \in [m]$, the Nash conditions imply

$$\begin{aligned}
\theta^j - q_j \leq \alpha - a \quad &\Rightarrow \quad \alpha + a_j - q_j \leq \alpha - a \\
&\Rightarrow \quad a_j + a \leq q_j \leq \frac{\alpha + a_j}{r_j} \\
&\Rightarrow \quad a_j \leq \frac{\alpha - r_j a}{r_j - 1} \\
&\Rightarrow \quad q_j \leq \frac{\alpha + \frac{\alpha - r_j a}{r_j - 1}}{r_j} = \frac{\alpha - a}{r_j - 1}.
\end{aligned} \tag{6}$$

S_1 can only be increased by replacing r_j by n for all $j \in [m]$. Thus

$$S_1 \leq \frac{n-1}{n}\left(|M_1|\alpha^2 - 2\alpha B + \sum_{j\in M_1} a_j^2\right)$$

with $B = |\sum_{j\in M_1} a_j|$. It is $a = \max_{j\in M_1} |a_j|$ and therefore

$$\sum_{j\in M_1} a_j^2 \leq a \cdot \sum_{j\in M_1} |a_j| = a \cdot B.$$

Thus,

$$S_1 \leq \frac{n-1}{n}\left(|M_1|\alpha^2 - 2\alpha B + a\cdot B\right).$$

Furthermore,

$$S_2 = \sum_{j\in M_2} \frac{r_j - 1}{r_j}\left(\alpha^2 + 2\alpha a_j + a_j^2\right),$$

and $\sum_{j\in[m]} a_j = 0$ implies

$$\sum_{j\in M_2} a_j = B.$$

Thus,

$$\mathsf{QSC}\left(\mathbf{w},\mathbf{c},\mathbf{P}\right) \leq n + |M_1| \cdot \frac{n-1}{n}\cdot\alpha^2 + \sum_{j\in M_2}\Delta_j,$$

where

$$\Delta_j = \frac{r_j-1}{r_j}\cdot\alpha^2 + 2\alpha a_j\left(\frac{r_j-1}{r_j} - \frac{n-1}{n}\right) + a_j\left(\frac{r_j-1}{r_j}a_j + \frac{n-1}{n}\cdot a\right).$$

Inequality (6) implies $a_j \leq q_j - a$. Thus,

$$\Delta_j \leq \frac{r_j-1}{r_j}\cdot\alpha^2 + \left(\frac{r_j-1}{r_j} - \frac{n-1}{n}\right)(2\alpha - a)a_j + \frac{r_j-1}{r_j}\cdot a_j q_j.$$

Lemma 4.1(2) implies $r_j \leq n-1$ for all $j \in M_2$. This is true, since $r_j = n$ implies $\theta^k \geq \theta^j > \alpha$ for all $k \in [m]$, which is a contradiction.
Assume $r_j \geq 2$ for all $j \in M_2$. We will prove the theorem by showing that $\Delta_j \leq \frac{n-1}{n}\cdot\alpha^2$ holds for all $j \in M_2$. We distinguish between two cases:

- $\underline{r_j \leq \frac{n+1}{2}}$:
  If $a > \frac{\alpha}{2}$, then there exists a link in M_1 with load less the $\frac{\alpha}{2}$. Thus, all users on links in M_2 are unsatisfied since there are at least 2 users on each of these links, and the load is at least α.

  So, without loss of generality we may assume $a \leq \frac{\alpha}{2}$. The fact that $a_j \leq q_j \leq \frac{\alpha}{r_j-1}$ implies

$$\begin{aligned}
\Delta_j &\leq \frac{r_j-1}{r_j}\cdot\alpha^2 + \left(\frac{r_j-1}{r_j} + \frac{n-1}{n}\right)\frac{3\alpha}{2}\cdot a_j + \frac{r_j-1}{r_j}\cdot a_j q_j \\
&= \frac{r_j-1}{r_j}\cdot\alpha^2 + \left(\frac{1}{n} - \frac{1}{r_j}\right)\frac{3\alpha}{2}\cdot a_j + \frac{r_j-1}{r_j}\cdot a_j q_j \\
&\leq \frac{r_j-1}{r_j}\cdot\alpha^2 + \left(\frac{3}{2n} - \frac{1}{2r_j}\right)\alpha a_j \\
&\leq \frac{r_j-1}{r_j}\cdot\alpha^2 + \left(\frac{3}{2n} - \frac{1}{2r_j}\right)\frac{\alpha^2}{r_j-1}.
\end{aligned}$$

If $r_j \leq \frac{n}{3}$, then the claim obviously holds. So assume $r_j > \frac{n}{3}$. It remains to show that

$$\frac{1}{n} + \left(\frac{3}{2n} - \frac{1}{2r_j}\right) \cdot \frac{1}{r_j - 1} \leq \frac{1}{r_j},$$

that is,

$$r(r-1) + \frac{3}{2}r - \frac{1}{2}n \quad \leq \quad n(r-1). \tag{7}$$

Denote

$$f(x) = x^2 - x + \frac{3}{2}x - \frac{1}{2}n - nx + n = x^2 + \frac{1}{2}x - nx + \frac{1}{2}n.$$

The function $f(x)$ has the roots

$$\begin{aligned} x_{1,2} &= \frac{n - \frac{1}{2}}{2} \pm \sqrt{\frac{1}{4}\left(n^2 - n + \frac{1}{4}\right) - \frac{1}{2}n} \\ &= \frac{n}{2} - \frac{1}{4} \pm \frac{1}{2}\sqrt{n^2 - 3n + \frac{1}{4}}. \end{aligned}$$

Since $f(\infty) = f(-\infty) = \infty$, (7) holds if and only if $x_1 \leq \frac{n}{3}$ and $x_2 \geq \frac{n+1}{2}$. We have

$$\begin{aligned} x_1 \leq \frac{n}{3} \quad &\Leftrightarrow \quad \frac{n}{6} - \frac{1}{4} \leq \frac{1}{2}\sqrt{n^2 - 3n + \frac{1}{4}} \\ &\Leftrightarrow \quad \frac{n^2}{9} - \frac{n}{3} + \frac{1}{4} \leq n^2 - 3n + \frac{1}{4} \\ &\Leftrightarrow \quad \frac{8}{3}n \leq \frac{8}{9}n^2, \end{aligned}$$

and this holds for $n \geq 3$. Furthermore,

$$\begin{aligned} x_2 \geq \frac{n+1}{2} \quad &\Leftrightarrow \quad \frac{1}{2}\sqrt{n^2 - 3n + \frac{1}{4}} \geq \frac{3}{4} \\ &\Leftrightarrow \quad n^2 - 3n + \frac{1}{4} \geq \frac{9}{4}, \end{aligned}$$

and this holds for $n \geq 4$. If $n = 3$, then the only possible case with $r_j = 2$ is included in the case that $r_j \geq \frac{n+1}{2}$. Moreover, we can assume $n \geq 3$ since $n = 2$ and $r_j = 2$ for any $j \in [m]$ implies $\theta^j = \alpha$ for all $j \in [m]$.

- $\underline{r_j \geq \frac{n+1}{2}}$:
  We use $a_j \leq q_j - a$ and $q_j \leq \frac{\alpha}{r_j - 1}$. It is

$$\Delta_j \leq \frac{r_j - 1}{r_j} \cdot \alpha^2 + F_j \cdot a_j,$$

where

$$F_j = \left(\frac{1}{n} - \frac{1}{r_j}\right)(2\alpha - a) + \frac{\alpha}{r_j - 1} = \left(\frac{2}{n} - \frac{1}{r_j}\right)\alpha + \left(\frac{1}{r_j} - \frac{1}{n}\right) \cdot a.$$

It is $F_j \geq 0$ and therefore

$$F_j \cdot a_j \leq (B + C \cdot a)(D - a)$$

with

$$\begin{aligned} B &= \alpha\left(\frac{2}{n}-\frac{1}{r_j}\right) \\ C &= \left(\frac{1}{r_j}-\frac{1}{n}\right) \\ D &= \frac{\alpha}{r_j-1}. \end{aligned}$$

The function $f(x) = (B + C \cdot x)(D - x)$ has its maximum at $x_0 = \frac{C \cdot D - B}{2C}$, and it holds

$$\begin{aligned} C \cdot D - B &= \left(\frac{1}{r_j}-\frac{1}{n}\right)\cdot\frac{\alpha}{r_j-1} - \left(\frac{2}{n}-\frac{1}{r_j}\right)\cdot\alpha \\ &= \frac{\alpha}{n \cdot r_j}\left(\frac{n-r_j}{r_j-1} - 2r_j + n\right) \\ &= \frac{\alpha}{n \cdot (r_j-1)}(n+1-2r_j) \le 0. \end{aligned}$$

Therefore, for $a = x \ge 0$ the maximum value is obtained for $a = 0$. This implies

$$F_j \cdot a_j \le B \cdot D = \alpha^2 \cdot \left(\frac{2}{n}-\frac{1}{r_j}\right)\cdot\frac{1}{r_j-1}.$$

Thus,

$$\begin{aligned} \Delta_j &\le \alpha^2 \cdot \left(\frac{r_j-1}{r_j} - \frac{1}{r_j\cdot(r_j-1)} + \frac{2}{n\cdot(r_j-1)}\right) \\ &\le \alpha^2 \cdot \left(\frac{r_j - 2 + \frac{2}{n}}{r_j-1}\right) \\ &\le \alpha^2 \cdot \frac{n-3+\frac{2}{n}}{n-2}, \end{aligned}$$

and

$$\frac{n-3+\frac{2}{n}}{n-2} \le \frac{n-1}{n}$$

holds for all n.

Now assume that there exists a link $j \in [m]$ with $r_j = 1$. Let ν be the number of users being alone on a link, and let μ be the number of links occupied only by one user. Then $\mu \ge \nu$. Set $n_1 = n - \nu$ and $m_1 = m - \mu$. Let U_1 be the set of remaining users, and let L_1 be the set of remaining links. Then $r_j \ge 2$ holds for all $j \in L_1$. The above proof showed

$$\begin{aligned} \mathsf{QSC}\left(\mathbf{w}, \mathbf{c}, \mathbf{P}\right) &\le \nu + \frac{n_1}{m_1}(n_1 + m_1 - 1) \\ &= \nu + \frac{n-\nu}{m-\mu}(n-\nu+m-\mu-1) \\ &= n + \frac{n-\nu}{m-\mu}(n-\nu-1). \end{aligned}$$

We want to show

$$\mathsf{QSC}\left(\mathbf{w}, \mathbf{c}, \mathbf{P}\right) \le \mathsf{QSC}\left(\mathbf{w}, \mathbf{c}, \mathbf{F}\right) = \frac{n}{m}(n+m-1)$$

which is equivalent to

$$m(n-\nu)(n-\nu-1) \le (m-\mu)n(n-1).$$

For $\mathbf{P}$ to be a Nash Equilibrium some relation between n, m, ν, μ has to hold. There is one user $i \notin U_1$ occupying at least $\lceil \frac{\mu}{\nu} \rceil$ links, i.e. the probabilities on one of these links is at most $\frac{1}{\lceil \frac{\mu}{\nu} \rceil} \leq \frac{\nu}{\mu}$. Therefore, for all $k \in U_1$ and $j \in L_1$ with $p_k^j > 0$

$$\theta^j - p_k^j + 1 \leq 1 + \frac{\nu}{\mu}$$

has to hold. Summing up over all links from L_1 gives

$$n_1 - 1 \leq \frac{\nu}{\mu} \cdot m_1.$$

So, we have to show that

$$m(n-\nu)(n-\nu-1) \leq (m-\mu)n(n-1)$$

holds under the condition that $\nu \leq \mu$, $\mu < m$, $\nu < n$, and $n \leq \frac{\nu}{\mu}m + 1$. Denote

$$f(n) = (m-\mu)n(n-1) - m(n-\nu)(n-\nu-1).$$

Since $f(\infty) = f(-\infty) = -\infty$, we only have to check if $f(n) \geq 0$ for $n = \nu + 1$ and $n = \frac{\nu}{\mu}m + 1$. If $n = \nu + 1$, then

$$f(n) = f(\nu + 1) = (m-\mu)n(n-1) \geq 0.$$

If $n = \frac{\nu}{\mu}m + 1$, then

$$f(n) = f(\frac{\nu}{\mu}m + 1) = (m-\mu)\left(\frac{\nu}{\mu}m + 1\right)\frac{\nu}{\mu}m - m\left(\frac{\nu}{\mu}m - \nu + 1\right)\left(\frac{\nu}{\mu}m - \nu\right) \geq 0.$$

This completes the proof of the claim.

Proof or Theorem 5.1:

Let there be n users and m links. If $n \leq m$, then every pure Nash equilibrium has optimal social cost. Now assume $n > m$. Let $\mathbf{P}$ be any pure Nash equilibrium. Let us first assume that $w_i \leq \frac{W}{m}$ holds for all users $i \in [n]$. Let $B = \min_{j \in [m]} \delta^j$ be the minimum traffic on any of the links. Then $B > 0$, and it has been shown in [17], that on every link the load is bounded by $2B$.
So,

$$\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P}) = \sum_{j \in [m]} (\delta^j)^2, \text{ with } B \leq \delta^j \leq 2B \text{ for all } j \in [m].$$

Now we use some iterative procedure to compute an upper bound for $\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P})$. Note, that $x_k \geq x_j$ implies $(x_k + \varepsilon)^2 + (x_j - \varepsilon)^2 > x_k^2 + x_j^2$ for all $\varepsilon > 0$. We compute artificial load values $x_j, j \in [m]$, by the following procedure:

For all $j \in [m]$ do $x_j \leftarrow \delta^j$
While there exist $j, k \in [m]$ with $j \neq k$ and $B < x_j \leq x_k < 2B$
  $\triangle = \min\{x_j - B, 2B - x_k\}$
  $x_j \leftarrow x_j - \triangle$
  $x_j \leftarrow x_k + \triangle$

Note, that in each iteration step $\sum_{i \in [m]} x_i^2$ increases and that $W = \sum_{j \in [m]} \delta^j = \sum_{j \in [m]} x_j$ remains unchanged. When the algorithm terminates, then there exists at most one link $j \in [m]$ with $B < x_j < 2B$. So we know that there exists some $k \in [m]$, such that

- $\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P}) \leq \sum_{j \in [m]} x_j^2$,
- $x_j = 2B$ for k links,
- $x_j = B + x$, $0 \leq x \leq B$, for one link, and
- $x_j = B$ for $m - k - 1$ links.

Note, that

$$\mathsf{QOPT}(\mathbf{w},\mathbf{c}) \geq \frac{W^2}{m} = \frac{(mB+kB+x)^2}{m}$$

and therefore

$$\frac{\mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{P})}{\mathsf{QOPT}(\mathbf{w},\mathbf{c})} \leq \frac{((3k+m-1)B^2+(B+x)^2)m}{(mB+kB+x)^2} = f(k)\,.$$

We now maximize the function $f(k)$. The first and the second derivative of $f(k)$ is

$$\begin{aligned} f'(k) &= \frac{3B^2m}{(mB+kB+x)^2} - \frac{2((3k+m-1)B^2+(B+x)^2)mB}{(mB+kB+x)^3} \quad \text{and} \\ f''(k) &= -\frac{12B^3m}{(mB+kB+x)^3} + \frac{6((3k+m-1)B^2+(B+x)^2)mB^2}{(mB+kB+x)^4}\,. \end{aligned}$$

The only root of $f'(k)$ is

$$r = \frac{B^2m - Bx - 2x^2}{3B^2}.$$

Furthermore,

$$f''(r) = -\frac{81B^6m}{16B^2+8Bx-8x^2} < 0.$$

Thus, $f(k)$ is maximal if $k = r$, that is,

$$f(k) \leq f(r) = \frac{9mB^2}{8mB^2+4Bx-x^2} \overset{x\leq B}{\leq} \frac{9mB^2}{8mB^2} = \frac{9}{8},$$

showing the upper bound for the case, that $w_i > \frac{W}{m}$ for all $i \in [n]$.

We still have to consider the case that $w_i > \frac{W}{m}$ holds for some user $i \in [n]$. In every Nash equilibrium $\mathbf{P}$ such a user is alone on its link. Let I be the set of these users. Let w', c', P' be defined by omitting users $i \in I$ and the links they are occupying. Then,

$$\frac{\mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{P})}{\mathsf{QOPT}(\mathbf{w},\mathbf{c})} = \frac{\mathsf{QSC}(\mathbf{w}',\mathbf{c}',\mathbf{P}') + \sum_{i\in I} w_i^2}{\mathsf{QOPT}(\mathbf{w}',\mathbf{c}') + \sum_{i\in I} w_i^2} \leq \frac{9}{8}\,,$$

which completes the proof of $\frac{9}{8}$ as an upper bound. To show tighness, consider the following instance: two identical users with $w_1 = w_2 = 1$ and $\frac{2}{\varepsilon}$ users with traffic ε on $m = 3$ identical links for $\varepsilon > 0$ with $\frac{1}{\varepsilon} \in \mathbb{N}$, and $(\frac{1}{\varepsilon} - 1)/3 \in \mathbb{N}$. Consider the pure Nash Equilibrium $\mathbf{P}$ in which users 1 and 2 are assigned to the same link, and the remaining users are evenly distributed to the other 2 links. We have

$$\mathsf{QSC}\left(\mathbf{w},\mathbf{c},\mathbf{P}\right) = 2^2 + 2\cdot 1^2 = 6,$$

whereas

$$\mathsf{OPT}\left(\mathbf{w},\mathbf{c}\right) = 3\cdot\left(\frac{4}{3}\right)^2 = \frac{16}{3}.$$

Thus,

$$\frac{\mathsf{QSC}\left(\mathbf{w},\mathbf{c},\mathbf{P}\right)}{\mathsf{OPT}\left(\mathbf{w},\mathbf{c}\right)} = \frac{9}{8}\,.$$

Proof of Theorem 5.2:
We first prove that the Quadratic Coordination Ratio is at most $\frac{4}{3}$. Assume, by way of contradiction, that it is *not*; hence, there exists an instance $\langle \mathbf{w},\mathbf{c},\mathbf{P}\rangle$, where $\mathbf{P}$ is a pure Nash equilibrium, such that

$$\frac{\mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{P})}{\mathsf{QOPT}(\mathbf{w},\mathbf{c})} > \frac{4}{3}\,. \tag{8}$$

We may assume, that $(\mathbf{w}, \mathbf{c})$ describes an instance with minimum number of links fulfilling (8), and that all traffics equal 1, i.e., $\mathbf{w} = \{1\}^n$, because the traffic size has no impact on the coordination ratio. Furthermore, let $\mathbf{P}$ be a worst case Nash equilibrium (a Nash equilibrium with maximum quadratic social cost) for that instance, and let $\mathbf{Q}$ denote an optimal assignment (an assignment with minimum quadratic social cost). For every $j \in [m]$ denote the traffic on link j according to $\mathbf{P}$ by $\delta^j(\mathbf{P})$ and according to the optimum assignment by $\delta^j(\mathbf{Q})$. Note, that the traffic on any link is deterministic for the model under consideration, and that $\delta^j(\mathbf{P}), \delta^j(\mathbf{Q}) \in \mathbb{N}_0$ for all $j \in [m]$.

Due to Lemma 6.3 we may assume, that for some link k the traffic according to $\mathbf{P}$ exceeds the traffic according to $\mathbf{Q}$ by $m-1$, whereas the traffic of any other link according to $\mathbf{P}$ is by one lower than according to $\mathbf{Q}$, and that all the latter links have equal speed, say c, and equal traffic according to $\mathbf{Q}$, say $\delta(\mathbf{Q})$. Without loss of generality, we may assume $k = 1$ and $c = 1$. Thus,

$$\begin{aligned}
&\delta^1(\mathbf{P}) = \delta^1(\mathbf{Q}) + m - 1, \quad \text{and}\\
&\delta^j(\mathbf{P}) = \delta^j(\mathbf{Q}) - 1 = \delta(\mathbf{Q}) - 1,\ c_j = 1 \quad \text{for } 2 \leq j \leq m\,.
\end{aligned}$$

For the social costs of $\mathbf{P}$ and $\mathbf{Q}$ we have

$$\begin{aligned}
\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P}) &= \frac{(\delta^1(\mathbf{Q}) + m - 1)^2}{c_1} + (m-1)(\delta(\mathbf{Q}) - 1)^2 \quad \text{and}\\
\mathsf{QOPT}(\mathbf{w}, \mathbf{c}) &= \frac{(\delta^1(\mathbf{Q}))^2}{c_1} + (m-1)(\delta(\mathbf{Q}))^2\,.
\end{aligned}$$

Hence, $\frac{\mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{P})}{\mathsf{QOPT}(\mathbf{w},\mathbf{c})}$ is bounded from above by the result of the following optimization problem with three unknowns δ^1, δ and c_1:

$$\text{maximize } \frac{\frac{(\delta^1+m-1)^2}{c_1} + (m-1)(\delta-1)^2}{\frac{(\delta^1)^2}{c_1} + (m-1)(\delta)^2} \quad \text{w.r.t.} \tag{9}$$

(a) $\delta^1, \delta \in \mathbb{N}_0, \delta \geq 1, c_1 \in \mathbb{R}_{>0}$
(b) $\frac{(\delta^1+1)^2}{c_1} + (\delta-1)^2 \geq \frac{(\delta^1)^2}{c_1} + (\delta)^2$
(c) $\frac{(\delta^1-1)^2}{c_1} + (\delta+1)^2 \geq \frac{(\delta^1)^2}{c_1} + (\delta)^2$
(d) $\frac{\delta^1+m-1}{c_1} \leq \delta$

δ^1 and δ correspond to $\delta^1(\mathbf{Q})$ and $\delta(\mathbf{Q})$. Because of $0 \leq \delta^j(\mathbf{P}) = \delta(\mathbf{Q}) - 1$ for $j \geq 2$, $\delta(\mathbf{Q})$ cannot be zero.

Because of Lemma 6.4, (b) and (c) are necessary and sufficient for the optimality of $\mathbf{Q}$. (d) is the Nash equilibrium condition for $\mathbf{P}$.
Setting $A = (\delta^1+1)^2, B = (\delta^1)^2, C = (m-1)(\delta-1)^2$ and $D = (m-1)(\delta)^2$, we can rewrite the fraction to be maximized as $\frac{A+c_1C}{B+c_1D}$. Because $\frac{A}{B} > 1 > \frac{C}{D}$, c_1 has to be minimized in order to maximize the fraction. (c) and (d) yield lower bounds on c_1:
(c) $\Leftrightarrow c_1 \geq \frac{2\delta^1-1}{2\delta+1}$,
(d) $\Leftrightarrow c_1 \geq \frac{\delta^1}{\delta} + \frac{m-1}{\delta}$.
As the last lower bound on c_1 is stronger, (d) implies (c). Thus, we can drop (c) and set $c_1 = \frac{\delta^1+m-1}{\delta}$. This simplifies the optimization problem. The fraction in (9) becomes

$$\begin{aligned}
&\frac{(\delta^1+m-1)^2\delta + (m-1)(\delta-1)^2(\delta^1+m-1)}{(\delta^1)^2\delta + (m-1)(\delta)^2(\delta^1+m-1)}\\
=&1 + \frac{(m-1)^2\delta + 2\delta^1(m-1)\delta + (m-1)(-2\delta+1)(\delta^1+m-1)}{(\delta^1)^2\delta + (m-1)(\delta)^2(\delta^1+m-1)}\\
=&1 + \frac{-(m-1)^2\delta + (m-1)(\delta^1+m-1)}{(\delta^1)^2\delta + (m-1)(\delta)^2(\delta^1+m-1)}
\end{aligned}$$

Thus, the problem is now:

$$\text{maximize } 1 + \frac{-(m-1)^2 + (m-1)(\delta^1 + m - 1)/\delta}{(\delta^1)^2 + (m-1)(\delta^1 + m - 1)\delta} \text{ w.r.t.}$$

(a) $\delta^1, \delta \in \mathbb{N}_0, \delta \geq 1$
(b) $\delta^1 \geq \delta(2m-3) - (m-1)$.
(b) implies, that the numerator of the fraction is nonnegative for all $m \geq 2$. Hence, in order to maximize the fraction, δ has to be minimized. The lowest possible value for δ (because of (a)) is $\delta = 1$. This yields

$$\text{maximize } 1 + f(\delta^1) \text{ w.r.t.}$$

(a) $\delta^1 \in \mathbb{N}_0$
(b) $\delta^1 \geq m - 2$,
where

$$f(\delta^1) = \frac{(m-1)\delta^1}{(\delta^1)^2 + (m-1)(\delta^1 + m - 1)}.$$

To maximize $f(\delta^1)$ we compute the first and second order derivative. $\frac{\partial f(\delta^1)}{\partial \delta^1} = 0$ implies $(\delta^1)^2 = (m-1)^2$. Evaluating the second order derivative for $\delta^1 = m-1$ and $\delta^1 = -(m-1)$ yields $\frac{\partial^2 f(\delta^1)}{(\partial \delta^1)^2}|_{\delta^1 = m-1} < 0$ and $\frac{\partial^2 f(\delta^1)}{(\partial \delta^1)^2}|_{\delta^1 = -(m-1)} > 0$. I.e., $f(\delta^1)$ has a local maximum at $\delta^1 = m-1$ and a local minimum at $\delta^1 = -(m-1)$ and no further extrema. Thus, the global maximum of $f(\delta^1)$ in the feasible interval is determined by the local maximum $f(m-1) = \frac{1}{3}$. Hence, $\frac{\mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{P})}{\mathsf{QOPT}(\mathbf{w},\mathbf{c})} \leq 1 + f(m-1) = \frac{4}{3}$, contradicting the existence of an instance with $\frac{\mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{P})}{\mathsf{QOPT}(\mathbf{w},\mathbf{c})} > \frac{4}{3}$.

Finally, we give an example, that proofs the bound tight. Let $m \geq 2$ and set $c_1 = 2(m-2), c_2 = \cdots = c_m = 1$. Let $\mathbf{Q}$ and $\mathbf{P}$ be assignments with

$$\begin{aligned} \delta^1(\mathbf{Q}) &= m-1, & \delta^2(\mathbf{Q}) &= \cdots = \delta^m(\mathbf{Q}) = 1, \\ \delta^1(\mathbf{P}) &= 2(m-1), & \delta^2(\mathbf{Q}) &= \cdots = \delta^m(\mathbf{Q}) = 0. \end{aligned}$$

$\mathbf{P}$ is a Nash equilibrium, because

$$\frac{\delta^j(\mathbf{P}) + 1}{c_j} \geq \frac{\delta^1(\mathbf{P})}{c_1} \quad \text{for } 2 \leq j \leq m.$$

For the ratio of the quadratic social cost we get

$$\frac{\mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{P})}{\mathsf{QOPT}(\mathbf{w},\mathbf{c})} = \frac{\frac{(2(m-1))^2}{2(m-1)}}{\frac{(m-1)^2}{2(m-1)} + (m-1)} = \frac{4}{3}.$$

Lemma 6.1 *Let $(\mathbf{w},\mathbf{c})$, $\mathbf{Q}$, $\mathbf{P}$, $\delta^j(\mathbf{Q})$ and $\delta^j(\mathbf{P})$ be as in the proof of Theorem 5.2. Then, $\delta^j(\mathbf{Q}) - \delta^j(\mathbf{P}) \leq 1$ for all $j \in [m]$.*

Proof: Assume the claim is violated for some link k, i.e.,

$$\delta^k(\mathbf{P}) = \delta^k(\mathbf{Q}) - x, x \in \mathbb{N}, x \geq 2. \tag{10}$$

Due to Lemma 6.4, optimality of $\mathbf{Q}$ implies for all $j \in [m]$

$$\begin{aligned} & \frac{(\delta^j(\mathbf{Q})+1)^2}{c_j} + \frac{(\delta^k(\mathbf{Q})-1)^2}{c_k} \geq \frac{(\delta^j(\mathbf{Q}))^2}{c_j} + \frac{(\delta^k(\mathbf{Q}))^2}{c_k} \\ \Leftrightarrow & \frac{2\delta^j(\mathbf{Q})+1}{c_j} \geq \frac{2\delta^k(\mathbf{Q})-1}{c_k} \\ \Leftrightarrow & \frac{\delta^j(\mathbf{Q})}{c_j} \geq \frac{2\delta^k(\mathbf{Q})-1}{2c_k} - \frac{1}{2c_j}. \end{aligned} \tag{11}$$

As $\mathbf{P}$ is a Nash equilibrium,

$$\frac{\delta^j(\mathbf{P})}{c_j} \leq \frac{\delta^k(\mathbf{P})+1}{c_k} \text{ for all } j \in [m]. \tag{12}$$

Now, let j be some link with

$$\delta^j(\mathbf{P}) \geq \delta^j(\mathbf{Q})+1.$$

Such link exists, because $\sum_i \delta^i(\mathbf{Q}) = \sum_i \delta^i(\mathbf{P}) = n$. Then,

$$\begin{aligned}
&\frac{2\delta^k(\mathbf{Q})-1}{2c_k} - \frac{1}{2c_j} + \frac{1}{c_j} \overset{(11)}{\leq} \frac{\delta^j(\mathbf{Q})+1}{c_j} \leq \frac{\delta^j(\mathbf{P})}{c_j} \\
&\overset{(12)}{\leq} \frac{\delta^k(\mathbf{P})+1}{c_k} \overset{(10)}{=} \frac{\delta^k(\mathbf{Q})-x+1}{c_k} \leq \frac{\delta^k(\mathbf{Q})-1}{c_k} \\
\Rightarrow &\frac{1}{2c_k} + \frac{1}{2c_j} \leq 0,
\end{aligned}$$

a contradiction. ∎

Lemma 6.2 *Let $(\mathbf{w},\mathbf{c})$, $\mathbf{Q}$, $\mathbf{P}$, $\delta^j(\mathbf{Q})$ and $\delta^j(\mathbf{P})$ be as in the proof of Theorem 5.2. Then, $\delta^k(\mathbf{P}) = \delta^k(\mathbf{Q})+m-1$ for some $k \in [m]$, and $\delta^j(\mathbf{P}) = \delta^j(\mathbf{Q})-1$ for all $j \in [m]\backslash\{k\}$.*

Proof: First, let us assume, there exists some link k with $\delta^k(\mathbf{P}) = \delta^k(\mathbf{Q})$. Then, removing that link and all associated users yields a new instance with quadratic coordination ratio

$$\rho' = \frac{\mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{P}) - (\delta^k(\mathbf{Q}))^2/c_k}{\mathsf{QOPT}(\mathbf{w},\mathbf{c}) - (\delta^k(\mathbf{Q}))^2/c_k}.$$

$\rho' < \rho := \frac{\mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{P})}{\mathsf{QOPT}(\mathbf{w},\mathbf{c})}$ would imply

$$\begin{aligned}
&(\mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{P}) - (\delta^k(\mathbf{Q}))^2/c_k)\mathsf{QOPT}(\mathbf{w},\mathbf{c}) < \\
&\quad < \mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{P})(\mathsf{QOPT}(\mathbf{w},\mathbf{c}) - (\delta^k(\mathbf{Q}))^2/c_k) \\
\Leftrightarrow\ &\mathsf{QOPT}(\mathbf{w},\mathbf{c}) > \mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{P}) \\
\Leftrightarrow\ &\rho < 1,
\end{aligned}$$

but we know, that $\rho \geq 1$. Hence, the new instance has at least the same coordination ratio but fewer links, which contradicts the assumption, that $(\mathbf{w},\mathbf{c})$ is a smallest (according to m) example that fulfils (8).
Now, assume more than one link have $\delta^j(\mathbf{P}) \geq \delta^j(\mathbf{Q})+1$ and let link k be one of them. By Lemma 6.1 and the first part of this proof, all links with $\delta^j(\mathbf{P}) \not\geq \delta^j(\mathbf{Q})+1$ have $\delta^j(\mathbf{P}) = \delta^j(\mathbf{Q})-1$. Then

$$\rho = \frac{\sum_{j\in[m]} \frac{(\delta^j(\mathbf{P}))^2}{c_j}}{\sum_{j\in[m]} \frac{(\delta^j(\mathbf{Q}))^2}{c_j}}.$$

Let S be a set of $(\delta^k(\mathbf{P}) - \delta^k(\mathbf{Q}))$ links with $\delta^j(\mathbf{P}) = \delta^j(\mathbf{Q})-1$. Setting

$$\begin{aligned}
A &= \sum_{j\in[m]\backslash(\{k\}\cup S)} \frac{(\delta^j(\mathbf{P}))^2}{c_j}, \\
B &= \sum_{j\in[m]\backslash(\{k\}\cup S)} \frac{(\delta^j(\mathbf{Q}))^2}{c_j}, \\
C &= \sum_{j\in S\cup\{k\}} \frac{(\delta^j(\mathbf{P}))^2}{c_j} \text{ and} \\
D &= \sum_{j\in S\cup\{k\}} \frac{(\delta^j(\mathbf{Q}))^2}{c_j},
\end{aligned}$$

we can rewrite the quadratic coordination ratio of $\mathbf{P}$ as

$$\rho = \frac{A+C}{B+D}\,.$$

Either $\frac{A}{B} > \frac{C}{D}$, or $\frac{A}{B} \le \frac{C}{D}$. In the first case we can remove the links in $S \cup \{k\}$ together with the users assigned to them, obtaining an example with even higher quadratic coordination ratio and fewer links. In the second case, we can remove all links in $[m] \backslash (S \cup \{k\})$ and their associated users, obtaining a counterexample with fewer links. Both cases contradict the assumption, that $(\mathbf{w}, \mathbf{c})$ has minimum number of links among all counterexamples. ■

Lemma 6.3 *Let $(\mathbf{w}, \mathbf{c})$, $\mathbf{Q}$, $\mathbf{P}$, $\delta^j(\mathbf{Q})$ and $\delta^j(\mathbf{P})$ be as in the proof of Theorem 5.2. Then, there exists an instance $(\tilde{\mathbf{w}}, \tilde{\mathbf{c}})$ that has the same number m of links as $(\mathbf{w}, \mathbf{c})$ with optimal assignment $\tilde{\mathbf{Q}}$, Nash equilibrium assignment $\tilde{\mathbf{P}}$ and corresponding traffics $\delta^j(\tilde{\mathbf{Q}})$ and $\delta^j(\tilde{\mathbf{P}})$, such that $\delta^k(\tilde{\mathbf{P}}) = \delta^k(\tilde{\mathbf{Q}}) + m - 1$ for some link $k \in [m]$, $\delta^i(\tilde{\mathbf{Q}}) = \delta^j(\tilde{\mathbf{Q}}) = \delta^i(\tilde{\mathbf{P}}) + 1 = \delta^j(\tilde{\mathbf{P}}) + 1$ and $\tilde{c_i} = \tilde{c_j}$ for all $i, j \in [m] \backslash \{k\}$ and $\frac{\mathsf{QSC}(\tilde{\mathbf{w}}, \tilde{\mathbf{c}}, \tilde{\mathbf{P}})}{\mathsf{QOPT}(\tilde{\mathbf{w}}, \tilde{\mathbf{c}})} \ge \frac{\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P})}{\mathsf{QOPT}(\mathbf{w}, \mathbf{c})}$.*

Proof: Let $k \in [m]$ be the link with $\delta^k(\mathbf{P}) = \delta^k(\mathbf{Q}) + m - 1$ due to Lemma 6.2. Let $S \subsetneq [m] \backslash \{k\}$ be a maximum set of links, such that $\delta^i(\mathbf{Q}) = \delta^j(\mathbf{Q})$ and $c_i = c_j$ for all $i, j \in S$, and let $s = |S|$ be the cardinality of S. Let $i \in [m] \backslash (S \cup \{k\})$ be arbitrary. We now obtain two new instances $(\mathbf{w}', \mathbf{c}')$ and $(\mathbf{w}'', \mathbf{c}'')$ from $(\mathbf{w}, \mathbf{c})$ by
a) setting the traffic and speed of link i to the traffic and speed of the links in S (this yields $(\mathbf{w}', \mathbf{c}')$), and
b) setting the traffic and speed of the links in S to the traffic and speed of link i (this yields $(\mathbf{w}'', \mathbf{c}'')$).
Let $\mathbf{Q}', \mathbf{P}'$ and $\mathbf{Q}'', \mathbf{P}''$ be the modified assignments for $(\mathbf{w}', \mathbf{c}')$ and $(\mathbf{w}'', \mathbf{c}'')$, respectively. We now show, that either

$$\frac{\mathsf{QSC}(\mathbf{w}', \mathbf{c}', \mathbf{P}')}{\mathsf{QSC}(\mathbf{w}', \mathbf{c}', \mathbf{Q}')} \ge \frac{\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P})}{\mathsf{QOPT}(\mathbf{w}, \mathbf{c})} \text{ or } \frac{\mathsf{QSC}(\mathbf{w}'', \mathbf{c}'', \mathbf{P}'')}{\mathsf{QSC}(\mathbf{w}'', \mathbf{c}'', \mathbf{Q}'')} \ge \frac{\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P})}{\mathsf{QOPT}(\mathbf{w}, \mathbf{c})}\,.$$

Set

$$\begin{aligned}
A &= \sum_{j \in [m] \backslash (S \cup \{i\})} \frac{(\delta^j(\mathbf{P}))^2}{c_j}, \\
B &= \sum_{j \in [m] \backslash (S \cup \{i\})} \frac{(\delta^j(\mathbf{Q}))^2}{c_j}, \\
v &= \frac{(\delta^i(\mathbf{Q}) - 1)^2}{c_i}, \\
w &= \frac{(\delta^i(\mathbf{Q}))^2}{c_i}, \\
x &= \frac{(\delta^j(\mathbf{Q}) - 1)^2}{c_j} \forall j \in S \text{ and} \\
y &= \frac{(\delta^j(\mathbf{Q}))^2}{c_j} \forall j \in S\,.
\end{aligned}$$

Then,

$$\begin{aligned}
\frac{\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P})}{\mathsf{QOPT}(\mathbf{w}, \mathbf{c})} &= \frac{A + v + sx}{B + w + sy} \\
\frac{\mathsf{QSC}(\mathbf{w}', \mathbf{c}', \mathbf{P}')}{\mathsf{QSC}(\mathbf{w}', \mathbf{c}', \mathbf{Q}')} &= \frac{A + (s+1)x}{B + (s+1)y} \\
\frac{\mathsf{QSC}(\mathbf{w}'', \mathbf{c}'', \mathbf{P}'')}{\mathsf{QSC}(\mathbf{w}'', \mathbf{c}'', \mathbf{Q}'')} &= \frac{A + (s+1)v}{B + (s+1)w}\,.
\end{aligned}$$

Assume,

$$\frac{\mathsf{QSC}(\mathbf{w}', \mathbf{c}', \mathbf{P}')}{\mathsf{QSC}(\mathbf{w}', \mathbf{c}', \mathbf{Q}')} < \frac{\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P})}{\mathsf{QOPT}(\mathbf{w}, \mathbf{c})} \text{ and } \frac{\mathsf{QSC}(\mathbf{w}'', \mathbf{c}'', \mathbf{P}'')}{\mathsf{QSC}(\mathbf{w}'', \mathbf{c}'', \mathbf{Q}'')} < \frac{\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P})}{\mathsf{QOPT}(\mathbf{w}, \mathbf{c})}\,.$$

This yields

$$
\begin{aligned}
& (A+(s+1)x)(B+w+sy) < (A+v+sx)(B+(s+1)y) \\
& \wedge (A+(s+1)v)(B+w+sy) < (A+v+sx)(B+(s+1)w) \\
\Leftrightarrow\ & Aw+Bx+(s+1)wx < Ay+Bv+(s+1)vy \\
& \wedge sAy+sBv+s(s+1)vy < sAw+sBx+s(s+1)wx \\
\Rightarrow\ & Aw+Bx+(s+1)wx < Aw+Bx+(s+1)wx\,,
\end{aligned}
$$

a contradiction. $\mathbf{P}'$ is a Nash equilibrium for $(\mathbf{w}',\mathbf{c}')$, because the Nash equilibrium condition holds for the links in S (no user can move from or to one of those links) and therefore is also valid for link i, which is just a copy of the links in S in the modified instance. Furthermore, as in $\mathbf{Q}$ no user can be shifted from or to one of the links in S thereby decreasing $\mathsf{QOPT}(\mathbf{w},\mathbf{c})$, this also holds in $\mathbf{Q}'$ for link i and hence for all links. That is,

$$\frac{(\delta^r(\mathbf{Q}')+w)^2}{c_r}+\frac{(\delta^t(\mathbf{Q}')-w)^2}{c_t}\geq\frac{(\delta^r(\mathbf{Q}'))^2}{c_r}+\frac{(\delta^t(\mathbf{Q}'))^2}{c_t}$$

for all $r,t\in[m]$. Due to Lemma 6.4, $\mathbf{Q}'$ is an optimal assignment for $(\mathbf{w}',\mathbf{c}')$. The same reasoning holds for $(\mathbf{w}'',\mathbf{c}'')$, i.e., $\mathbf{Q}''$ is an optimal assignment and $\mathbf{P}''$ is a Nash equilibirum for $(\mathbf{w}'',\mathbf{c}'')$. In both modified instances, the number of equal links (the links in S) has increased to $s+1$. Thus, by applying this step repeatedly, we eventually obtain an instance as stated in the claim of the Lemma. Note, that the number of links remains the same, while the number of users can be increased, but not more than by a factor of m. ■

Lemma 6.4 *Let $\mathbf{Q}$ be any pure assignment for an instance $(\mathbf{w},\mathbf{c})$ of the model of identical traffics and related links, let $\mathbf{w}=(w,\ldots,w)$, and let for every $j\in[m]$ $\delta^j(\mathbf{Q})$ denote the traffic on link j according to $\mathbf{Q}$. Then, $\mathbf{Q}$ is optimal, i.e., $\mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{Q})=\mathsf{QOPT}(\mathbf{w},\mathbf{c})$, if and only if for every pair of links $i,j\in[m]$*

$$\frac{(\delta^i(\mathbf{Q})+w)^2}{c_i}+\frac{(\delta^j(\mathbf{Q})-w)^2}{c_j}\geq\frac{(\delta^i(\mathbf{Q}))^2}{c_i}+\frac{(\delta^j(\mathbf{Q}))^2}{c_j}\,.$$

Proof: 1. '$\Rightarrow$': Let $\mathbf{Q}$ be optimal. If we create a new assignment $\mathbf{Q}'$ by shifting one user from link j to link i for some $i,j\in[m]$, we get

$$\mathsf{QSC}(\mathbf{w}',\mathbf{c}',\mathbf{Q}')=\mathsf{QOPT}(\mathbf{w},\mathbf{c})+\frac{(\delta^i(\mathbf{Q})+w)^2}{c_i}+\frac{(\delta^j(\mathbf{Q})-w)^2}{c_j}-\frac{(\delta^i(\mathbf{Q}))^2}{c_i}-\frac{(\delta^j(\mathbf{Q}))^2}{c_j}$$

for the social cost of $\mathbf{Q}'$. Now, $\mathsf{QSC}(\mathbf{w}',\mathbf{c}',\mathbf{Q}')\geq\mathsf{QOPT}(\mathbf{w},\mathbf{c})$ yields the above inequality.
2. '$\Leftarrow$': Let $\mathbf{R}$ be an optimal assignment, and let $\mathbf{Q}$ be any assignment, for which the above inequality holds. Let $\delta^j(\mathbf{Q})$ and $\delta^j(\mathbf{R})$ be the traffic of link j according to $\mathbf{Q}$ and $\mathbf{R}$, respectively. Because of 1. we have

$$
\begin{aligned}
& \frac{(\delta^j(\mathbf{R})+w)^2}{c_j}+\frac{(\delta^i(\mathbf{R})-w)^2}{c_i}\geq\frac{(\delta^i(\mathbf{R}))^2}{c_i}+\frac{(\delta^j(\mathbf{R}))^2}{c_j} \\
\Leftrightarrow & \frac{2\delta^j(\mathbf{R})w+w^2}{c_j}\geq\frac{2\delta^i(\mathbf{R})w-w^2}{c_i}
\end{aligned}
\tag{13}
$$

By assumption,

$$
\begin{aligned}
& \frac{(\delta^i(\mathbf{Q})+w)^2}{c_i}+\frac{(\delta^j(\mathbf{Q})-w)^2}{c_j}\geq\frac{(\delta^i(\mathbf{Q}))^2}{c_i}+\frac{(\delta^j(\mathbf{Q}))^2}{c_j} \\
\Leftrightarrow & \frac{2\delta^i(\mathbf{Q})w+w^2}{c_i}\geq\frac{2\delta^j(\mathbf{Q})w-w^2}{c_j}\,.
\end{aligned}
\tag{14}
$$

If $\delta^j(\mathbf{Q})=\delta^j(\mathbf{R})$ for all $j\in[m]$, then $\mathbf{Q}$ is optimal and nothing is to show. Hence, we may assume that there exist links $i,j\in[m]$ with $\delta^i(\mathbf{R})-\delta^i(\mathbf{Q})>0$ and $\delta^j(\mathbf{Q})-\delta^j(\mathbf{R})>0$. I.e., link i has more traffic according to

$\mathbf{R}$ than according to $\mathbf{Q}$, and link j has more traffic according to $\mathbf{Q}$ than to $\mathbf{R}$. As the traffic on any link is a multiple of w, we have

$$\delta^i(\mathbf{R}) - \delta^i(\mathbf{Q}) \geq w \tag{15}$$
$$\delta^j(\mathbf{Q}) - \delta^j(\mathbf{R}) \geq w \tag{16}$$

But this yields

$$\begin{aligned} \frac{2\delta^j(\mathbf{R})w + w^2}{c_j} &\overset{(13)}{\geq} \frac{2\delta^i(\mathbf{R})w - w^2}{c_i} \overset{(15)}{\geq} \frac{2\delta^i(\mathbf{Q})w + w^2}{c_i} \\ &\overset{(14)}{\geq} \frac{2\delta^j(\mathbf{Q})w - w^2}{c_j} \overset{(16)}{\geq} \frac{2\delta^j(\mathbf{R})w + w^2}{c_j} \end{aligned}$$

Notice, that the left and right side of the inequality chain are identical and therefore all terms must be equal. On the one hand this implies $\delta^j(\mathbf{Q}) = \delta^j(\mathbf{R}) + w$ and $\delta^i(\mathbf{R}) = \delta^i(\mathbf{Q}) + w$. That is, for each link, the traffics according to $\mathbf{Q}$ and $\mathbf{R}$ can differ by at most one user. This implies that the number of links with $\delta^i(\mathbf{R}) - \delta^i(\mathbf{Q}) > 0$ and of those with $\delta^j(\mathbf{Q}) - \delta^j(\mathbf{R}) > 0$ are equal. On the other hand we can follow

$$\begin{aligned} &\Rightarrow \frac{2\delta^j(\mathbf{R})w + w^2}{c_j} = \frac{2\delta^i(\mathbf{Q})w + w^2}{c_i} \\ &\Leftrightarrow \frac{(\delta^i(\mathbf{Q}))^2}{c_i} + \frac{(\delta^j(\mathbf{R}) + w)^2}{c_j} = \frac{(\delta^i(\mathbf{Q}) + w)^2}{c_i} + \frac{(\delta^j(\mathbf{R}))^2}{c_j} \\ &\Leftrightarrow \frac{(\delta^i(\mathbf{Q}))^2}{c_i} + \frac{(\delta^j(\mathbf{Q}))^2}{c_j} = \frac{(\delta^i(\mathbf{R}))^2}{c_i} + \frac{(\delta^j(\mathbf{R}))^2}{c_j}. \end{aligned} \tag{17}$$

(17) tells us, that the contribution of links i and j to the social cost is the same for $\mathbf{Q}$ and $\mathbf{R}$. As for every link j that has a greater traffic according to $\mathbf{Q}$ than to $\mathbf{R}$ there exists some link i that has a lower traffic according to $\mathbf{Q}$ than to $\mathbf{R}$, we have $\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{Q}) = \mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{R})$. ∎

Proof of Theorem 5.3:
Due to Corollary 3.6 the social cost of the fully mixed Nash Equilibrium is

$$\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{F}) = W_1 + 2 \cdot \frac{W_2}{m},$$

where $W_1 = \sum_{1 \leq i \leq n} w_i^2$ and $W_2 = \sum_{1 \leq i < j \leq n} w_i w_j$. We distinguish between two cases:

1. $\underline{w_i \leq \frac{W}{m} \text{ for all } i \in [n]}$: This implies

$$\mathsf{QOPT}(\mathbf{w}, \mathbf{c}) \geq \frac{W^2}{m} = \frac{W_1 + 2W_2}{m}.$$

Thus,

$$\Delta = \frac{\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{F})}{\mathsf{QOPT}(\mathbf{w}, \mathbf{c})} \leq \frac{mW_1 + 2W_2}{W_1 + 2W_2} = 1 + (m-1) \cdot \frac{W_1}{W_1 + 2W_2} = 1 + (m-1) \cdot \frac{x}{2+x},$$

where $x = \frac{W_1}{W_2}$. Denote $f(x) = \frac{x}{2+x}$. Since $f(x) \geq 0$ for all $x \geq 0$, and

$$f'(x) = \frac{(2+x) - x}{(2+x)^2} > 0,$$

$f(x)$ attains its maximum value for largest $x = \frac{W_1}{W_2}$. Now consider w_i, w_j with $\frac{W}{m} > w_i \geq w_j > 0$, and choose some x with $x \leq w_j$, $x \leq \frac{W}{m} - w_i$. Form a new instance by setting

$$\begin{aligned}
\tilde{w}_i &= w_i + x \\
\tilde{w}_j &= w_j - x \\
\tilde{w}_k &= w_k \quad \text{for all } k \neq i, j.
\end{aligned}$$

Then,

$$\begin{aligned}
\tilde{W}_1 &= (w_i + x)^2 + (w_j - x)^2 + \sum_{k \neq i,j} w_k^2 = W_1 + 2x(w_i - w_j) + 2x^2 > W_1 \\
\tilde{W}_2 &= (w_i + x) \sum_{k \neq i,j} w_k + (w_j - x) \sum_{k \neq i,j} w_k + (w_i + x)(w_j - x) + \sum_{k<l;k,l \neq i,j} w_k w_l \\
&= W_2 - (w_i - w_j)x - x^2 < W_2.
\end{aligned}$$

This shows that $\frac{\tilde{W}_1}{\tilde{W}_2} \geq \frac{W_1}{W_2}$,proving that $\frac{W_1}{W_2}$ is maximal for $w_i = \frac{W}{m}$ for $1 \leq i \leq m$ and $w_i = 0$ for $i > m$. The claim follows with Theorem 5.4.

2. $\underline{w_1 > \frac{W}{m}}$: Define r such that

$$w_i > \frac{W - w_1 - \cdots - w_{i-1}}{m + 1 - i} \quad \text{for } 1 \leq i \leq r$$

and

$$w_{r+1} \leq \frac{W - w_1 - \cdots - w_r}{m - r}.$$

In any pure Nash Equilibrium, each w_i, $1 \leq i \leq r$, is the only traffic on its link. Since there exists a pure Nash Equilibrium with social cost $\mathsf{QOPT}\,(\mathbf{w}, \mathbf{c})$, it is

$$\mathsf{QOPT}\,(\mathbf{w}, \mathbf{c}) \geq \sum_{1 \leq i \leq r} w_i^2 + (m - r)\frac{(W - w_1 - \cdots - w_r)^2}{(m - r)^2}.$$

Denote $\alpha = \frac{W - w_1 - \cdots - w_r}{(m-r)}$. Let $\alpha > w_i \geq w_j > 0$. We again introduce a new instance

$$\begin{aligned}
\tilde{w}_i &= w_i + x \\
\tilde{w}_j &= w_j - x \\
\tilde{w}_k &= w_k \quad \text{for all } k > r, k \neq i, j.
\end{aligned}$$

In the same way as in the first case, it follows that we can assume $n = m$ and $w_i = \alpha$ for $r + 1 \leq i \leq m$, yielding

$$\begin{aligned}
\mathsf{QOPT}\,(\mathbf{w}, \mathbf{c}) &\geq \sum_{1 \leq i \leq r} w_i^2 + (m - r)\alpha^2 = W_1, \\
W_2 &= \sum_{1 \leq i < j \leq r} w_i w_j + (m - r)\left(\sum_{1 \leq i \leq r} w_i\right)\alpha + \binom{m - r}{2}\alpha^2.
\end{aligned}$$

This implies

$$\Delta = \frac{\mathsf{QSC}\,(\mathbf{w}, \mathbf{c}, \mathbf{F})}{\mathsf{QOPT}\,(\mathbf{w}, \mathbf{c})} \leq \frac{W_1 + \frac{2}{m}W_2}{W_1} = 1 + \frac{2}{m}\frac{W_1}{W_2}.$$

Again, this maximizes if all traffics are equal, proving the claim.

Furthermore, we now construct an instance with m identical links and traffic vector $\mathbf{w}$, such that the ratio of the quadratic social cost of the fully mixed Nash equilibrium $\mathbf{F}$ and the optimum quadratic social cost is arbitrarily close to $2 - \frac{1}{m}$. Therefore, let there be m users with traffic size 1 and $r \in m\mathbb{N}$ users with traffic size $\varepsilon > 0$. Due to Theorem 3.2 we have $QSC(\mathbf{w}, \mathbf{c}, \mathbf{F}) = m + r\varepsilon^2 + (m + r\varepsilon)^2/m - 1 - r\varepsilon^2/m$. Distributing all traffics evenly yields the optimum quadratic social cost $QOPT(\mathbf{w}, \mathbf{c}) = (m + r\varepsilon)^2/m$. For $\varepsilon \to 0$, $\frac{QSC(\mathbf{w},\mathbf{c},\mathbf{F})}{QOPT(\mathbf{w},\mathbf{c})}$ tends to $2 - \frac{1}{m}$.

Proof of 5.4:

From Theorem 4.2 we get, that the quadratic social cost of any Nash equilibrium is bounded by the quadratic social cost of the fully mixed Nash equilibrium. Due to Corollary 3.6 the quadratic social cost of the fully mixed Nash Equilibrium is

$$\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{F}) = \frac{n(n+m-1)}{m}.$$

1. Using the lower bound $\frac{n^2}{m}$ on $\mathsf{QOPT}(\mathbf{w}, \mathbf{c})$, we get

$$\frac{\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{F})}{\mathsf{QOPT}(\mathbf{w}, \mathbf{c})} \leq \frac{n(n+m-1)}{m} \cdot \frac{m}{n^2} = \frac{n+m-1}{n}.$$

2. Using the lower bound n on $\mathsf{QOPT}(\mathbf{w}, \mathbf{c})$, we get

$$\frac{\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{F})}{\mathsf{QOPT}(\mathbf{w}, \mathbf{c})} \leq \frac{n+m-1}{m}.$$