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Radio labeling with pre-assigned frequencies*

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Abstract

A radio labeling of a graph G is an assignment of pairwise distinct, positive integer labels to the vertices of G such that labels of adjacent vertices differ by at least 2. The radio labeling problem (RL) consists in determining a radio labeling that minimizes the maximum label that is used (the so-called span of the labeling). RL is a well-studied problem, mainly motivated by frequency assignment problems in which transmitters are not allowed to operate on the same frequency channel. We consider the special case where some of the transmitters have pre-assigned operating frequency channels. This leads to the natural variants P-RL(l) and P-RL($*$) of RL with l pre-assigned labels and an arbitrary number of pre-assigned labels, respectively.

We establish a number of combinatorial, algorithmical, and complexity-theoretical results for these variants of radio labeling. In particular, we investigate a simple upper bound on the minimum span, yielding a linear time approximation algorithm with a constant additive error bound for P-RL($*$) restricted to graphs with girth ≥ 5. We consider the complexity of P-RL(l) and P-RL($*$) for several cases in which RL is known to be polynomially solvable. On the negative side, we prove that P-RL($*$) is

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NP-hard for cographs and for k-colorable graphs where a k-coloring is given ($k \geq 3$). On the positive side, we derive polynomial time algorithms solving p-RL($*$) and p-RL(l) for graphs with bounded maximum degree, and for solving p-RL(l) for k-colorable graphs where a k-coloring is given.

Keywords: Radio labeling, frequency assignment, graph algorithms, computational complexity, approximation algorithm, cograph, k-coloring.

AMS Subject Classifications: 05C15, 68R10, 05C78, 68W25, 94C15.

1 Introduction

The Frequency Assignment problem (FAP) is a general framework focused on point-to-point communication, e.g. in *radio* or *mobile telephone* networks. One of its main threads asks for an assignment of frequencies or frequency channels to transmitters while keeping interference at an acceptable level and making use of the available frequencies in an efficient way. Interference constraints are usually related to the use of the same or similar frequencies at locations within a certain distance (or transmitters within a certain reach) from each other. Due to the scarce resources and the increasing use of frequencies in modern wireless technology, the available frequencies should be used as efficiently as possible. There is usually a trade-off between avoiding interference and the efficient use of frequencies. We will not go deeper into the technical details here.

Graph theoretical issues come into play since possible interference between transmitters is usually modeled by a so-called *interference graph*. Each vertex of the interference graph represents a transmitter. If simultaneous broadcasting of two transmitters may cause an interference, then they are connected by an edge in the interference graph. The frequency channels are usually labeled by positive integers. Frequency channels with 'close' labels are assumed to be 'close' in the spectrum or expected to be 'more likely' to cause interference.

Regarding the assumption that a pair of 'close' transmitters should be assigned different frequencies or frequency channels, the Frequency Assignment problem is equivalent to the problem of labeling the interference graph with some constraints on the labeling. In many cases the related labeling problems are variants on what is known as the vertex coloring problem in graph theory.

However, Hale [14] observed that the signal propagation may affect the interference even in distant regions (but with decreasing intensity). Hence, not only 'close' transmitters should get different frequencies, but also frequencies used at some distance should be appropriately separated. This leads to a more detailed and complicated modeling of FAP in terms of *distance constrained labeling* of the interference graph. (See for instance the book [16] by Leese.)

In some applications the transmitters are not allowed to operate on the same frequency channel (for example, when every transmitter covers the whole area) while 'close' transmitters should use channels with sufficient separation. In this case non-reusable frequency channels should be assigned to transmitters in a proper way. This leads to the so-called Radio Labeling problem (RL), i.e. to the problem of assigning distinct labels to the vertices of a graph such that adjacent vertices get labels (positive integers) that differ by at least two. The purpose of RL is to find such a radio labeling with the smallest maximum label.

In this paper we initiate the investigation of two versions of this problem in which some of the transmitters (like military and governmental stations) already have pre-assigned labels corresponding to frequency channels which one is not allowed to change. Then the problem boils down to determining a radio labeling extending a given pre-labeling in a 'best possible' way. In this paper we consider some algorithmical, complexity-theoretical, and combinatorial aspects of these versions of the problem.

We do not want to claim that the results in the sequel have immediate practical relevance.

1.1 Definitions and preliminary observations

We denote by $G = (V, E)$ a finite undirected and simple graph. The *girth* of G is the length of a shortest cycle in G. For every nonempty $W \subseteq V$, the subgraph of $G = (V, E)$ induced by W is denoted by $G[W]$. A *cograph* is a graph containing no induced path on four vertices. The *(open) neighborhood* of a vertex v in a graph G is $N_G(v) := \{u \in V : \{u, v\} \in E\}$. The degree of a vertex v in G is $d_G(v) := |N_G(v)|$. The maximum degree of G is $\Delta(G) := \max_{v \in V} d_G(v)$. A graph G is *t-degenerate* if each of its subgraphs has a vertex of degree at most t. A *clique* C of a graph $G = (V, E)$ is a nonempty subset of V such that all the vertices of C are pairwise adjacent in G. A nonempty subset of vertices $I \subseteq V$ is *independent* in G if no two of its elements are adjacent in G. The *complement* $\overline{G}$ of $G = (V, E)$ is the graph on V with edge set $\overline{E}$ such that $\{u, v\} \in \overline{E}$ if and only if $\{u, v\} \notin E$.

A *k-coloring* of the vertices of a graph $G = (V, E)$ is a partition $I_1, I_2, \ldots, I_k$ of V into independent sets (in which some of the I_j may be empty); the k sets I_j are called the *color classes* of the k-coloring. The *chromatic number* $\chi(G)$ is the minimum value k for which a k-coloring exists. A *labeling* of the (vertex set of the) graph $G = (V, E)$ is an injective mapping $L\colon V \to \mathbf{N}^+$ (the set of positive integers). A labeling L of G is called a *radio labeling* of G if for any edge $\{u, v\} \in E$ the inequality $|L(u) - L(v)| \geq 2$ holds; the *span* of such a labeling L is $\max_{v \in V} L(v)$.

The Radio Labeling problem (RL) is defined as follows: "For a given graph G, find a radio labeling L with the smallest span." The name radio labeling was suggested by Fotakis and Spirakis in [8] but the same notion (under different names) has been introduced independently and earlier by other researchers (see, e.g. Chang & Kuo [2]). Problem RL is equivalent to the special case of the Traveling Salesman problem TSP(2,1) in which all edge weights (distances) are either one or two. The relation is as follows. For a graph $G = (V, E)$ let K_G be the complete weighted graph on V with edge weights 1 and 2 defined according to E: for every $\{u, v\} \in E$ the weight $w(\{u, v\})$ in K_G is 2 and for $\{u, v\} \notin E$ the weight $w(\{u, v\}) = 1$. The *weight* of a path in K_G is the sum of the weights of its edges. The following proposition can be found in [8, 9].

Proposition 1.1. *There is a radio labeling of G with span k if and only if there is a Hamiltonian path (i.e. a path on $|V|$ vertices) of weight $k-1$ in K_G.*

Another equivalent formulation of this problem, which was extensively studied in the literature, is the HAMILTONIAN PATH COMPLETION problem (HPC), i.e. the problem of partitioning the vertex set of a graph G into the smallest possible number of sets which are spanned by paths in G. This equivalence is expressed in the following well-known proposition.

Proposition 1.2. *There is a radio labeling of G with span $\le k$ if and only if there is a partition of V into $\le k$ sets, such that each of these sets induces a subgraph in $\overline{G}$ that contains a Hamiltonian path.*

Now let us turn to the versions of RL with pre-assigned labels. For a graph $G = (V, E)$ a *pre-labeling* L' of a subset $V' \subset V$ is an injective mapping $L' \colon V' \to \mathbf{N}^+$. We say that a labeling L of G *extends* the pre-labeling L' if $L(u) = L'(u)$ for every $u \in V'$. We study the following two problems:

- P-RL($*$): RADIO LABELING WITH AN ARBITRARY NUMBER OF PRE-LABELED VERTICES.
  For a given graph G and a given pre-labeling L' of G, determine a radio labeling of G extending L' with the smallest span.
- P-RL(l): RADIO LABELING WITH A FIXED NUMBER OF PRE-LABELED VERTICES.
  For a given graph $G = (V, E)$, a subset $V' \subseteq V$ with $|V| \le l$, and a pre-labeling $L' \colon V' \to \mathbf{N}^+$, determine a radio labeling of G extending L' with the smallest span.

1.2 Earlier results

As we mentioned above the TRAVELING SALESMAN problem TSP(2,1) (which is equivalent to RL without any pre-labeling) is a well-studied problem. Papadimitriou & Yannakakis [17] proved that this problem is MAX SNP-hard, but gave an approximation algorithm for TSP(2,1) which finds a solution not worse than 7/6 times the optimum solution. Later Engebretsen [4] improved their result by showing that the problem is not approximable within $5381/5380 - \varepsilon$ for any $\varepsilon > 0$.

Damaschke, Deogun, Kratsch & Steiner [3] proved that the HAMILTONIAN PATH COMPLETION problem HPC can be solved in polynomial time on cocomparability graphs (complements of comparability graphs). To obtain this result they used a reduction to the problem of finding the bump number of a partial order. (The bump number of a poset P and its linear extension L is the number of neighbors in L which are comparable in P.) It was proved by Habib, Möhring & Steiner [13] and by Schäffer & Simons [19] that the BUMP NUMBER problem can be solved in polynomial time. By Proposition 1.2, the result of Damaschke, Deogun, Kratsch & Steiner yields that RL is polynomial time solvable for comparability graphs. Later, this result was rediscovered by Chang & Kuo [2] but under the name of $L'(2,1)$-labeling and only for cographs, a subclass of the class of comparability graphs. Notice that RL is NP-hard for cocomparability graphs because the HAMILTONIAN PATH problem is known to be NP-hard for bipartite graphs which form a subclass of comparability graphs. Recently, Fotakis and Spirakis [8] proved that RL can be solved in polynomial time within the class of graphs for which a k-coloring can be obtained in polynomial time (for some fixed k). Note that, for example, this class of graphs includes the well-studied classes of planar graphs and graphs with bounded treewidth.

We are not aware of any existing results concerning the pre-labeling versions P-RL($*$) and P-RL(l) of RL.

We complete this subsection by mentioning some results concerning the related notion of radio coloring (also known as $L(2,1)$-labeling, $\lambda_{2,1}$-coloring and $\chi_{2,1}$-labeling). A *radio coloring* of a graph $G = (V, E)$ is a function $f\colon V \to \mathbf{N}^+$ such that $|f(u) - f(v)| \geq 2$ if $\{u, v\} \in E$ and $|f(u) - f(v)| \geq 1$ if the distance between u and v in G is 2. Here the *distance* between two vertices u and v in a connected graph G is the smallest number of edges in a path of G between u and v. So, the difference between radio coloring and radio labeling is that in a radio coloring vertices at distance at least three may have equal labels (or colors). The notion of radio coloring was introduced by Griggs & Yeh [12] under the name $L(2,1)$-labeling. As with radio labeling the span of a radio coloring f of G is $\max_{v \in V} f(v)$.

The problem of determining a radio coloring with minimum span has received a lot of attention. For various graph classes the problem was studied by Sakai [18], Bodlaender, Kloks, Tan & van Leeuwen [1], Van den Heuvel, Leese & Shepherd [15], and others. NP-hardness results for this RADIO COLORING problem (RC) restricted to planar, split, or cobipartite graphs were obtained by Bodlaender, Kloks, Tan & van Leeuwen [1]. Fixed-parameter tractability properties of RC are discussed by Fiala, Kratochvíl and Kloks [6]. Fiala, Fishkin & Fomin [5] study on-line algorithms for RC. For only very few graph classes the problem is known to be polynomially solvable. Chang & Kuo [2] obtained a polynomial time algorithm for RC restricted to trees and cographs. The complexity of RC even for graphs of treewidth 2 is a long standing open question. An interesting direction of research was initiated by Fiala, Kratochvíl & Proskurowski [7]. They consider a precolored version of the RADIO COLORING problem, i.e. a version in which some colors are pre-assigned to some vertices. They proved that RC with a given precoloring can be solved in polynomial time for trees. Recently Golovach [11] proved that RC with a given precoloring is NP-hard for graphs of treewidth 2.

1.3 Our results and organization of the paper

We study algorithmical, complexity-theoretical, and combinatorial aspects of radio labeling with pre-labeled vertices. In Section 2 we give some simple combinatorial bounds for the minimum span of such labelings: We introduce an easy-to-compute lower bound M on the minimum span of a labeling extending a pre-labeling, and we show that there always exists such a radio labeling

- with span $\leq \lfloor (7M - 2)/3 \rfloor$, for arbitrary graphs

- with span $\leq \lfloor (5M+2)/3 \rfloor$, for graphs of girth at least 4
- with span $\leq M+3$, for graphs of girth at least 5.

All these bounds are best possible. The third bound is even best possible for paths, i.e. for graphs of infinite girth. In Section 3, we derive similar results for t-degenerate graphs. In Section 4 we obtain polynomial time algorithms for graphs with bounded degree; these algorithms are based on the results of Section 2.

Sections 5 and 6 are devoted to the algorithmical study of the radio labeling problem with pre-assigned labels. We study these problems restricted to graphs for which a k-coloring is given, and restricted to cographs, two graph classes for which the radio labeling problem without pre-labeling is known to be solvable in polynomial time. Known and new results on these radio labeling problems are summarized in the following table.

	graphs with a bounded Δ	graphs with a given k-coloring	Cographs
RL	P [8]	P [8]	P [3, 2]
P-RL(l)	P [$*$]	P [$*$]	???
P-RL($*$)	P [$*$]	NPC for $k \geq 3$ [$*$]	NP [$*$]

In this table, an entry P denotes solvable in polynomial time, NPC denotes NP-complete, [$*$] denotes a contribution from this paper, and the sign ??? marks an open problem.

For the results in the middle column, we assume that k is a fixed integer that is not part of the input. Note that the class of graphs with a given k-coloring contains important and well-studied graph classes such as the class of planar graphs and the class of graphs with bounded treewidth.

2 Upper bounds for the minimum span

Let $G = (V, E)$ denote a graph on n vertices, and let $V' \subseteq V$ and $L' \colon V' \to \mathbf{N}^+$ be a fixed subset of V and a pre-labeling for V', respectively. We define the parameter M which will be very useful in the rest of the paper:

$$M \ := \ \max\{n, \ \max_{v \in V'} L'(v)\}.$$

Clearly, M is straightforward to compute if G and L' are known. And clearly, M is a lower bound on the span of any radio labeling in G extending the pre-labeling L' of G. A natural question is how far M can be away from the minimum span of such a labeling. We will show that the answer to this question heavily relies on the girth of the graph G:

Theorem 2.1. *Consider a graph G on $n \geq 7$ vertices, and a pre-labeling L' of G. Then there is a radio labeling in G extending L'*

(a) with span $\leq \lfloor (7M-2)/3 \rfloor$

(b) with span $\leq \lfloor (5M+2)/3 \rfloor$ if G has girth at least 4

(c) with span $\leq M+3$ if G has girth at least 5.

All these bounds are best possible. The third bound is even best possible for the class of paths.

Proof. Let us start the proof of Theorem 2.1 by showing that all the stated bounds indeed are best possible: For (a), let x be a positive integer and y an integer such that $M = 3x+y$ and $-1 \leq y \leq 1$. We consider the complete graph on M vertices in which x vertices are pre-labeled with labels $2, 5, 8, \ldots, 3x-1$, whereas the remaining $2x+y$ vertices are unlabeled. Since we cannot use the labels $1, 2, \ldots, 3x$ at the unlabeled vertices, and the labels at these vertices have to differ by at least two, the span of any radio labeling extending the pre-labeling is at least $3x + 2(2x+y) - 1 = 7x + 2y - 1 = \lfloor (7M-2)/3 \rfloor$.

For (b), let x be a positive integer and y an integer such that $M = 3x+y$ and $0 \leq y \leq 2$. We consider the complete bipartite graph $K_{x+1,2x+y-1}$ on M vertices. The $x+1$ vertices in the first part of the bipartition are pre-labeled with the labels $2, 5, 8, \ldots, 3x+2$, whereas the $2x+y-1$ vertices in the second part of the bipartition are unlabeled. Note that the pre-labeling forbids the labels $1, \ldots, 3x+y+1$ for the unlabeled vertices. Thus the span of any radio labeling extending the pre-labeling is at least $5x + 2y = \lfloor (5M+2)/3 \rfloor$.

Finally, for (c) we consider the path $v_1 - v_2 - \cdots - v_n$ on $n \geq 7$ vertices. The pre-labeling assigns $L'(v_1) = 1$, $L'(v_3) = 5$, $L'(v_5) = 3$, $L'(v_7) = 7$, and $L'(v_k) = k$ for $k \geq 8$. Then $M = n$, and all but three vertices (v_2, v_4, v_6) are pre-labeled. Since we cannot use the labels 2, 4, and 6 at these three vertices, every radio labeling of this path extending the pre-labeling L' has a span of at least $n+3 = M+3$.

The greedy preprocessing step. In several of our proofs, we use the same preprocessing step. We denote this step as *greedy preprocessing*.

Greedy preprocessing is done as follows. We extend the pre-labeling by assigning labels from $\{1, \ldots, M\}$ to unlabeled vertices, such that the conditions that all labels are different, and that adjacent vertices have labels that differ by at least two keep being met. This preprocessing step terminates, when we get stuck: then either all vertices have been labeled, or for every unused label $c \in \{1, \ldots, M\}$ and for every unlabeled vertex v, v is adjacent to a vertex labeled with $c-1$ or v is adjacent to a vertex labeled with $c+1$. Note that this greedy preprocessing step does not change the value of M.

Proof of Theorem 2.1.(a). It is sufficient to consider the case where G is the complete graph on n vertices. Assume that there are $l = |V'|$ pre-labeled vertices and that we start with the available labels $1, 2, \ldots, N = \lfloor (7M-2)/3 \rfloor$. Call a label *blocked*, if it is used in the pre-labeling or if it is adjacent to a (vertex labeled with a) label in the pre-labeling (and thus cannot be used). Hence there are at most $\min\{3l, M+1\}$ blocked labels. Order the remaining available (nonblocked) labels increasingly, and assign the labels at the odd positions in the ordering to the remaining $n-l$ vertices. Note that we need at most $\min\{3l, M+1\}+2(n-l)-1$ labels. If $3l \le M+1$, then this number is at most

$$3l+2(n-l)-1 \;=\; 2n+l-1 \;\le\; 2M+\frac{M+1}{3}-1 \;=\; \frac{7M-2}{3}.$$

If $3l \ge M+1$, then this number is at most

$$M+1+2(n-l)-1 \;=\; M+2n-2l \;\le\; M+2M-2\frac{M+1}{3} \;=\; \frac{7M-2}{3}.$$

So, in both cases we obtain a feasible radio labeling with span at most $\lfloor (7M-2)/3 \rfloor$.

Proof of Theorem 2.1.(b). We start with $N = \lfloor (5M+2)/3 \rfloor$ available labels $1, 2, \ldots, N$, perform the greedy preprocessing described above, and from now on consider the labeling thus obtained. If all vertices are labeled, then we are done and there is nothing to show. Otherwise, consider some fixed label $c \in \{1, \ldots, M\}$ that is not used in this labeling. Then every unlabeled vertex v must be adjacent to a vertex labeled with $c+1$ or to a vertex labeled with $c-1$. Denote the set of unlabeled vertices adjacent to $c-1$ by A, and denote the set of unlabeled vertices that are adjacent to $c+1$ but not to $c-1$ by B. Then A must be an independent set. (Any edge in A together with the vertex labeled with $c-1$ would induce a triangle, and bring the girth of G down to 3.) And also B must be an independent set.

Assume that l vertices are labeled. Since every label blocks at most two other labels, there are at most $\min\{3l, M+1\}$ blocked labels and we have at least

$$\lfloor (5M+2)/3 \rfloor - \min\{3l, M+1\} \;\ge\; n-l$$

available labels for the remaining $n-l$ vertices. The displayed inequality can be seen as follows. If $3l \le M+1$, then $\lfloor (2M+2)/3 \rfloor \ge 2l$. Adding $M-3l \ge n-3l$ yields the desired inequality in this case. If $3l \ge M+2$, then $n-l \le M - \lceil (M+2)/3 \rceil$. Together with $M - \lceil (M+2)/3 \rceil \le \lfloor (5M+2)/3 \rfloor - M - 1$ we get the desired inequality also for this second case.

Now we distinguish three subcases. In the first subcase, there are vertices $a \in A$, $b \in B$, such that a is not adjacent to b. We assign the $|A|$ smallest of the $n-l$ available labels to the vertices in A, and the $|B|$ largest of these labels to the vertices in B. This is done in such a way that vertex a receives the largest label in A, and such that vertex b receives the smallest label in B. This gives a radio labeling with span $\le \lfloor (5M+2)/3 \rfloor$.

In the second subcase, either A or B is empty. In this case, we use the $n-l$ available labels on the $n-l$ vertices in A or B.

In the third subcase, we assume that none of A and B is empty and that they span a complete bipartite graph, i.e. each vertex in A is adjacent to each vertex in B. Consider an arbitrary label $d \in \{1, \ldots, M\}$ that is not used in the pre-labeling. Then every unlabeled vertex $v \in A \cup B$ must be adjacent to the vertex labeled with $d+1$ or to the vertex labeled with $d-1$. There are only two possibilities for this: either all vertices in A are adjacent to the vertex labeled with $d+1$ and all vertices in B are adjacent to the vertex labeled with $d-1$, or all vertices in A are adjacent to the vertex labeled with $d-1$ and all vertices in B are adjacent to the vertex labeled with $d+1$. As a consequence, there are at most $\min\{2l, M+1\}$ blocked labels in this subcase, and at least

$$\lfloor (5M+2)/3 \rfloor - \min\{2l, M+1\} \;\geq\; n-l+1$$

available labels for the remaining $n-l$ vertices. We assign the $|A|$ smallest of these labels arbitrarily to the vertices in A, and the $|B|$ largest of these labels arbitrarily to the vertices in B.

Proof of Theorem 2.1.(c). We start with the greedy preprocessing described above, and from now on consider the labeling thus obtained. Denote by $C = \{c_1, \ldots, c_k\}$ with $c_1 < c_2 < \cdots < c_k$ the set of all unused labels from $\{1, \ldots, M\}$ in the labeling obtained after the preprocessing. Denote by U the set of unlabeled vertices. It is clear that $|U| \leq |C|$. We prove that the labeling obtained by the greedy preprocessing can always be extended to a radio labeling of G using at most three additional labels $M+1$, $M+2$, and $M+3$.

Without loss of generality, we assume that after the greedy preprocessing there is a vertex of G labeled with M. (If this is not the case, the same proof produces a span $\leq M+2$.) For the sake of convenience we shall identify each labeled vertex with its label. If $|U| = 1$, we just label the only unlabeled vertex with label $M+2$. If $|U| = 2$, then the two unlabeled vertices are either nonadjacent and we can label them with $M+2$ and $M+3$, or at least one of them is not adjacent to M (since G contains no 3-cycles), and we label this vertex with $M+1$ and the other vertex with $M+3$.

From now on we assume $|U| \geq 3$. If C contains two consecutive labels d and $d+1$, then every vertex in U must be adjacent to the vertices labeled with $d-1$ and $d+2$. This yields a cycle of length four in G, and contradicts the assumption on the girth of G. Therefore, there are no consecutive labels in C, and in particular $c_3 - 1 > c_1 + 1$.

Next, we first discuss the case $|U| \geq 4$. Then $|C| \geq 4$ and $c_3 < M$. Suppose for the sake of contradiction that $c_1 = 1$. Then all vertices in U are adjacent to label 2, and at least two vertices of U are adjacent to $c_3 - 1$ or to $c_3 + 1$. This would yield a cycle of length four. This contradiction shows $c_1 \geq 2$. There cannot be more than two vertices of U adjacent to $c_1 - 1$ because otherwise at least two of them would be adjacent to $c_3 + 1$ or to $c_3 - 1$, and we again would obtain a 4-cycle. Similarly, we see that at most two vertices of U are adjacent to $c_1 + 1$. As a consequence, each of the vertices $c_1 - 1$ and $c_1 + 1$ has exactly two neighbors in U. This implies that $|U| = 4$, and that each vertex of U is adjacent to exactly one of $c_1 - 1$ and $c_1 + 1$. Denote the four vertices in U by u_1, u_2, u_3, u_4 such that u_1 and u_2 are the neighbors of $c_1 - 1$, and u_3 and u_4 are the neighbors of $c_1 + 1$. Moreover, we may assume that u_1 and u_3 are adjacent to $c_3 - 1$ and that u_2 and u_4 are adjacent to $c_3 + 1$; see Figure 1.

For every label $c \in C \backslash \{c_1, c_3\}$ we have that $|\{c-1, c+1\} \cap \{c_1-1, c_1+1, c_3-1, c_3+1\}| = 0$. Otherwise, we either obtain a 4-cycle or can use the label c at one of the vertices in U; as an example, consider, e.g. the case that $c - 1 = c_1 + 1$: then we can use the label c

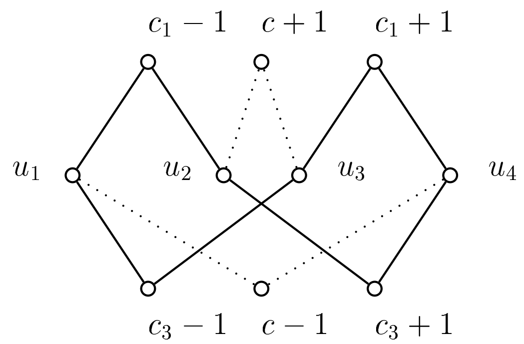

Figure 1: A step in the proof of Theorem 2.1.(c).

at u_1 or u_2 unless both are adjacent to $c_1 + 3$, yielding a 4-cycle with $c_1 - 1$; the other cases are similar. But then there are only two possibilities for the vertices in U to be the neighbor of $c - 1$ or $c + 1$ without creating a 4-cycle: u_1 and u_4 should be adjacent to one of these labels and u_2 and u_3 to the other. But since $|C \setminus \{c_1, c_3\}| \geq 2$ and C does not contain consecutive integers, we obtain a 4-cycle in G, which is a contradiction. So the case $|U| \geq 4$ can not occur at all.

We are left with the case that $|U| = 3$. By similar arguments as above, we may assume that $c_1 - 1$ is adjacent to u_1, that $c_1 + 1$ is adjacent to u_2 and u_3, that $c_3 + 1$ is adjacent to u_1 and u_2, and that $c_3 - 1$ is adjacent to u_3 (the other cases are analogous). The girth condition implies that the graph induced by the vertices $\{u_1, u_2, u_3\}$ has at most one edge, the edge $\{u_1, u_3\}$. If such an edge exists, then M is nonadjacent to one of these three vertices, and we label this vertex with $M + 1$ and the other one with $M + 3$; in that case the vertex u_2 is labeled with $M + 2$. If there is no such edge, then M is nonadjacent to at least one of the vertices u_i (otherwise we obtain a 4-cycle) and we label this vertex with $M + 1$ and the other two vertices with $M + 2$ and $M + 3$. This completes the proof of Theorem 2.1.(c). $\square$

Note that the proof of Theorem 2.1.(c) yields a linear time approximation algorithm determining a radio labeling extending a pre-labeling in graphs with girth ≥ 5 with a span that is at most an additive 3 away from the minimum span.

3 Graphs with bounded degeneracy

In this section, we apply the techniques from the proof of Theorem 2.1 to t-degenerate graphs, i.e. graphs with the property that each of their subgraphs has a vertex of degree at most t. We will also use the easy, well-known fact that a t-degenerate graph on n vertices has at most $t(n-t)+t(t-1)/2$ edges. First we consider the case without any pre-labeling, and obtain the following result which is easy to prove. This result will be used in the case where a pre-labeling is assumed.

Lemma 3.1. *If G is a t-degenerate graph on n vertices, then it has a radio labeling with span $\leq n + 2t$.*

Proof. Let G be a counterexample to Lemma 3.1 with the least number of vertices. Remove a vertex v of minimum degree from G and consider a radio labeling with span $\leq n-1+2t$ in $G-v$ (which exists by the choice of G). If we started with the $n+2t$ available labels $1,2,\ldots,n+2t$, there are at least $2t+1$ unused labels left. Since v has degree at most t and each neighbor of v forbids the use of at most two labels at v, we can choose a suitable label for v. $\square$

We now turn to the variant in which a pre-labeling is assumed. We obtain an upper bound for the minimum span of a radio labeling extending the pre-labeling in a t-degenerate graph, depending on M and t only.

Theorem 3.2. *If G is a t-degenerate graph and L' is a pre-labeling of G, then there exists a radio labeling extending L' with span $\leq M+(4+\sqrt{3})t+1$.*

Proof. We again start with the greedy preprocessing step as described in the preceding section, and with available labels $1,2,\ldots,M+(4+\sqrt{3})t+1$. Denote by U the set of unlabeled vertices after this step. We only consider the case where $|U|=2p$ for some positive integer p; the case with odd $|U|$ is similar and left to the reader. Denote by $C=\{c_1,\ldots,c_{2p}\}$ with $c_1<\ldots<c_{2p}$ the set of the first $2p$ unused labels. It is clear that C cannot contain three consecutive labels $c-1,c,c+1$ (otherwise, we can use the label c at some vertex of U). Thus for every i we have $c_i+1<c_{i+2}-1$. Consider the set C' of vertices c_i+1 and c_i-1 for every odd $i=1,3,\ldots,2p-1$. Then C' contains exactly p pairs of vertices and each unlabeled vertex of U must be adjacent to at least one vertex from every pair. The graph induced by the set $U\cup C'$ has $4p$ vertices. Denote the number of edges of this graph by x. We have $2p^2\leq x\leq(4p-t)t+(t-1)t/2=4pt-t^2/2-t/2$. Thus $(p-t)^2\leq 3t^2/4-t/4\leq 3t^2/4$, yielding $2p\leq(2+\sqrt{3})t$. Using Lemma 3.1 and the fact that $G[U]$ is t-degenerate, we can label the unlabeled vertices of U with the labels $\{M+2,\ldots,M+(4+\sqrt{3})t+1\}$. This completes the proof. $\square$

The above results imply a polynomial time approximation algorithm for solving the radio labeling problem (with pre-labeling) in t-degenerate graphs. Note, that the bound in Theorem 3.2 can possibly be improved considerably. We pose the following conjecture.

Conjecture. If G is a t-degenerate graph and L' is a pre-labeling of G, then there exists a radio labeling extending L' with span $\leq M+3t$.

The upper bound in the above conjecture cannot be improved. For $t=1$ this is clear from earlier examples. (See the discussion on case (c) of Theorem 2.1.) Let us show it here for general t by the following example. Let V be the union of disjoint sets A, B, C and D, with $|A|=|B|=|C|=t$, $|D|=t+1$. Let A be a clique, and assume that each vertex of $B\cup C$ is adjacent to every vertex from A. Finally assume $B\cup D$ induces a complete bipartite graph. Let G be the graph on V with the described edges. One easily checks that G is a t-degenerate graph. Pre-label the vertices of C with labels $3,7,\ldots,4t-1$ and of D with labels $1,5,\ldots,4t+1$. Then $M=4t+1$ and none of the even labels less than M can be used for labeling $A\cup B$. Since each vertex of A is adjacent to all vertices of $A\cup B$, we lose t labels more, and the largest label we have to use is at least $7t+1=M+3t$.

4 Graphs with a bounded maximum degree

We now turn to graphs with a bounded maximum degree. Using similar proof techniques as in the previous section, we will prove that P-RL(∗) is polynomially solvable within this class of graphs. We first prove the next technical result on t-degenerate graphs.

Theorem 4.1. *Let $G = (V, E)$ be a t-degenerate graph with maximum degree Δ and let $V' \subseteq V$ be the set of vertices that is pre-labeled by L'. If the number of unlabeled vertices $p = |V \setminus V'| \geq 4\Delta(t+1)$, then L' can be extended to a radio labeling of G with span M.*

Proof. Let $H = G[V \setminus V']$ be the graph induced by the unlabeled vertices, and start with the available labels $1, 2, \ldots, M$. As long as H has edges we will apply the following labeling procedure. Consider a vertex v in H of minimum positive degree. It has at most t neighbors. We can label them all with unused labels because there are at most $2\Delta - 2$ blocked labels among the unused labels and the number of available labels in every step is at least the number of unlabeled vertices at that moment, which is always at least 4Δ. Adapt H by removing the neighbors of v. In each step we reduce the number of unlabeled vertices by at most t and increase the number of isolated vertices by one. Since $p \geq 4\Delta(t+1)$, we have $q \geq 4\Delta$ isolated vertices in H when H becomes edgeless.

We now show that the set Q of the remaining q isolated vertices of H can be labeled with unused labels from the set $\{1, \ldots, M\}$. Denote the set of such labels by C (it is clear that $|C| \geq q$) and consider the auxiliary bipartite graph G' with vertex partition $Q \cup C$ where an edge $\{v, c\}$ exists if and only if the vertex v of Q can be labeled with the label c. It is sufficient to show that G' has a matching saturating all vertices of Q. Suppose that there is no such matching. Then by standard matching theory there is a set $A \subset Q$ such that $|A| = a$ and $|N(A)| \leq a - 1$ (where $N(A)$ is the neighborhood of A in G'). Let $B = C \setminus N(A)$. We have $|B| \geq q - a + 1$. Note that each label could be forbidden for at most 2Δ vertices and, vice versa, that for every vertex at most 2Δ labels could be forbidden. Therefore, since there are no edges between A and B, we have $a \leq 2\Delta$ and $q - a + 1 \leq |B| \leq 2\Delta$. But this implies $q \leq 4\Delta - 1$, a contradiction. This completes the proof. □

From Theorem 4.1 we easily obtain the following complexity result for graphs with a bounded maximum degree.

Corollary 4.2. *Let k be a fixed positive integer. For every graph G with maximum degree $\Delta \leq k$ and pre-labeling L',* P-RL(∗) *can be solved in polynomial time.*

Proof. Each graph with maximum degree Δ is clearly Δ-degenerate. If at most $4\Delta(\Delta+1)$ vertices of G are not pre-labeled by L', then one can use a brute force algorithm to find a radio labeling extending L' with a minimum span, e.g. by checking all admissible labelings. The time complexity of such a brute force algorithm is $O(n^{4\Delta(\Delta+1)})$. If more than $4\Delta(\Delta+1)$ vertices are unlabeled, then by Theorem 4.1 there is a radio labeling extending L' with span M (which is clearly the minimum), and from the proof of Theorem 4.1 it is not difficult to check that such a labeling can be found in polynomial time. □

The above corollary shows that P-RL(l) and P-RL($*$) have the same complexity behavior as RL for graphs with a bounded maximum degree, i.e. all three of the problems can be solved in polynomial time. This picture changes if we restrict ourselves to graphs which are k-colorable and for which a k-coloring is given (as part of the input) for some fixed positive integer k. This is the topic of the next section.

5 Graphs with a bounded chromatic number

In this section we concentrate on radio labeling algorithms for graphs with a bounded chromatic number, in particular for the case where a k-coloring of the graph (for a fixed constant k) is provided as part of the input.

Related to Proposition 1.1 we discussed the useful equivalence between RL and the Traveling Salesman problem TSP(2,1). We now adapt this equivalence to capture the restrictions of the pre-labeling problem. Let L be a labeling of a graph $G = (V, E)$ on n vertices. The path $P = (v_1, v_2, \ldots, v_n)$ *corresponding* to L visits the vertices by increasing labels, i.e. for all $1 \leq a < b \leq n$ we have $L(v_a) < L(v_b)$. P is a path in the complete graph K_G; its *weight* $w(P)$ is measured according to the edge weights w in K_G as introduced in the paragraph preceding Proposition 1.1.

Lemma 5.1. *Let L' be a pre-labeling of a subset $V' \subseteq V$ of a graph $G = (V, E)$, and let $P = (v_1, v_2, \ldots, v_n)$ be an ordering of the vertices of G. Then the path P in K_G corresponds to some extension L of L' to V, if and only if the following two conditions are satisfied:*

(T1) For any $1 \leq a < b \leq n$ with $v_a, v_b \in V'$, the weight $w(v_a, v_{a+1}, \ldots, v_b)$ of the subpath from v_a to v_b is at most $L'(v_b) - L'(v_a)$.

(T2) Let c be the smallest index with $v_c \in V'$. If $c \neq 1$, then the weight $w(v_1, v_2, \ldots, v_c)$ of the subpath from v_1 to v_c is at most $L'(v_c) - 1$.

For any path P that satisfies (T1) and (T2), a labeling L extending L' with the smallest possible span can be computed in polynomial time $O(n)$.

Proof. Suppose P corresponds to a labeling L that extends L'. Let $a < b$, v_a, $v_b \in V'$. Consider the path $v_a, v_{a+1}, \ldots, v_b$. By induction, we see that for all i with $0 \leq i \leq b - a$, $L(v_{a+i})$ is at least the sum of $L(v_a)$ and the weight $w(v_a, v_{a+1}, \ldots, v_{a+i})$ of the subpath from v_a to v_{a+i}. So, $L(v_b) \geq L(v_a) + w(v_a, v_{a+1}, \ldots, v_b)$, and (T1) follows. In a similar way, it can be shown that (T2) must hold.

It remains to prove the 'if'-statement. Consider a path P that satisfies (T1) and (T2). We construct the following labeling L: If $v_1 \in V'$ then we set $L(v_1) = L'(v_1)$, and if $v_1 \notin V'$ then we set $L(v_1) = 1$. For $i \geq 2$ and $v_i \in V'$, we set $L(v_i) = L'(v_i)$. Finally for $i \geq 2$ and $v_i \notin V'$, we set $L(v_i) = L(v_{i-1}) + 1$ if $\{v_i, v_{i-1}\} \notin E$ and we set $L(v_i) = L(v_{i-1}) + 2$ otherwise.

It can be seen that the above procedure in fact computes a labeling L with the smallest possible span for P among all labelings that extend L'. $\square$

Now let $G = (V, E)$ be a graph with a given k-coloring with color classes $I_1, I_2, \ldots, I_k$. Let L' be a pre-labeling of a subset $V' \subseteq V$ with $|V'| = l$. Let L be a radio labeling of G that extends L', and let $P = (v_1, v_2, \ldots, v_n)$ be the path corresponding to L. Consider two consecutive vertices v_a and v_{a+1} along this path with $v_a \in I_i$ and $v_{a+1} \in I_j$. If $i = j$, then v_a and v_{a+1} form a *monochromatic* edge. If $i \neq j$, then v_a and v_{a+1} form a *bichromatic* edge of type (i, j).

Lemma 5.2. *Let $G = (V, E)$ be a graph with a k-coloring with color classes $I_1, I_2, \ldots, I_k$. Let L' be a pre-labeling of a subset $V' \subseteq V$ with $|V'| = l$. Then there exists a radio labeling L of G extending L' with the smallest possible span, such that its corresponding path P satisfies the following two conditions:*

(P1) For any bichromatic edges $\{v_a, v_{a+1}\}$ and $\{v_b, v_{b+1}\}$ of type (i, j) with $a < b$ and $i \neq j$, there exists a pre-labeled vertex $v_c \in V'$ on the subpath $(v_{a+1}, \ldots, v_b)$.

(P2) P contains at most $(l + 1)k(k - 1)$ bichromatic edges.

Proof. Among all radio labelings of G that extend L' with the smallest possible span, consider the labeling L for which the corresponding path $P = (v_1, v_2, \ldots, v_n)$ contains the largest possible number of monochromatic edges.

Suppose that this path P violates property (P1). Then there exist two bichromatic edges $\{v_a, v_{a+1}\}$ and $\{v_b, v_{b+1}\}$ of type (i, j) with $a < b$ and $i \neq j$, such that none of the vertices on the subpath $(v_{a+1}, \ldots, v_b)$ is pre-labeled; see Figure 2. We replace the subpath $(v_a, v_{a+1}, v_{a+2}, \cdots, v_{b-1}, v_b, v_{b+1})$ of P by $(v_a, v_b, v_{b-1}, v_{b-2}, \cdots, v_{a+1}, v_{b+1})$, and thus produce a new path P^*. Since the edges $\{v_a, v_b\}$ and $\{v_{a+1}, v_{b+1}\}$ are monochromatic, whereas the edges $\{v_a, v_{a+1}\}$ and $\{v_b, v_{b+1}\}$ were not, the new path P^* has more monochromatic edges than P. Moreover, if we compute a labeling L^* for P^* as indicated in the proof of Lemma 5.1, then the span of L^* is at most the span of L. This contradicts our choice of the labeling L. Hence, P satisfies property (P1).

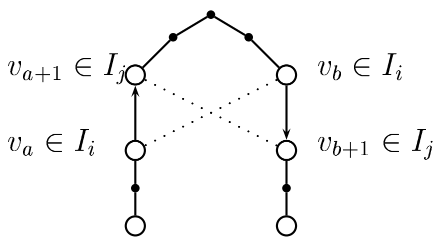

Figure 2: A step in the proof of property (P1) in Lemma 5.2.

Property (P2) is a simple quantitative consequence of property (P1): The pre-labeled vertices split P into $l+1$ subpaths, and any such subpath contains at most one bichromatic edge of type (i, j). Since there are only $k(k - 1)$ possible types of bichromatic edges, the bound in property (P2) follows. $\square$

Theorem 5.3. *Let $G = (V, E)$ be a graph with a given k-coloring with color classes $I_1, I_2, \ldots, I_k$. Let L' be a pre-labeling of a subset $V' \subseteq V$ with $|V'| = l$. Then a radio labeling L of G extending L' with the smallest possible span can be computed in time $O(n^{4(l+1)k(k-1)})$.*

Proof. The proof is based on Lemma 5.2. The idea is to use a brute force algorithm that enumerates all possible subsets of bichromatic edges as described in property (P2), and then tries to extend them to a radio labeling. This is done in the following way.

1. Compute all possible subsets S of at most $(l+1)k(k-1)$ bichromatic edges.

2. For each such subset S, compute all possible functions $f : S \to \{1, \ldots, n-1\}$. The interpretation of $f(v_a, v_{a+1}) = m$ is that v_a is the mth vertex and that v_{a+1} is the $(m+1)$th vertex on the path.

3. Compute all possible functions $g : V' \to \{1, \ldots, n\}$. The interpretation of $g(v_c) = m$ is that v_c is the mth vertex on the path.

4. For each subset S and all functions f and g, determine whether there exists a compatible path. If such a path exists, then compute a corresponding labeling with the smallest possible span according to Lemma 5.1.

5. Output the detected labeling with the minimum span.

There are $O(n^{2(l+1)k(k-1)})$ possibilities for subset S in the first step. There are $O(n^{(l+1)k(k-1)})$ possible functions f in the second step and $O(n^l)$ possible functions g in the third step. Below, we will show that the fourth step can be performed in $O(n^2)$ time. This yields an overall time complexity of at most $O(n^{4(l+1)k(k-1)})$.

So how do we check compatibility in the fourth step? If any vertex is assigned to two or more distinct positions (for instance, since it is incident with two or more bichromatic edges in S), then there clearly exists no compatible path. Also, if there are two or more distinct vertices assigned to the same position (for instance, by f and by g), then there clearly exists no compatible path. So from now on we will assume that every vertex goes to at most one position, and any position receives at most one vertex. Then the functions f and g fully determine the positions of all pre-labeled vertices, and the positions of all vertices on bichromatic edges in the corresponding path. What about the empty positions? Consider a maximal piece Z of empty positions, and let z_ℓ and z_r be the vertices assigned to the position immediately to the left of Z and immediately to the right of Z, respectively. Then z_ℓ and z_r must belong to the same color class I_j. Otherwise, there is a missing bichromatic edge, and there can not be a compatible path. So we will assume that z_ℓ and z_r belong to the same color class I_j; then the vertices that go into positions in Z must all be in I_j. This determines the color class for all empty positions.

We randomly assign the remaining vertices to the empty positions, subject to the condition that every empty position receives a vertex of the right color class. If no such assignment exists, then there can not be a compatible path. Otherwise, all these random assignments yield a path with the same sequence of edge weights w in K_G. We finally check whether conditions (T1) and (T2) of Lemma 5.1 are satisfied by the resulting path, and use this lemma to compute a corresponding labeling with the smallest possible span. $\square$

For each of the graph classes in the following corollary, it is possible to construct a vertex coloring with a constant number of colors in polynomial time. Hence:

Corollary 5.4. *The radio labeling problem* P-RL(l) *is polynomially solvable*

- *on the class of planar graphs,*
- *on any class of graphs of bounded treewidth,*
- *on the class of bipartite graphs.*

The above results show that P-RL(l) is solvable in polynomial time for graphs with a bounded chromatic number and a given coloring. This result does not carry over to the more general labeling problem P-RL$(*)$ where the number of pre-labeled vertices is part of the input. We next show that P-RL$(*)$ is NP-hard even when restricted to 3-colorable graphs with a given 3-coloring; this result then easily generalizes to k-colorable graphs ($k \geq 4$) with a given k-coloring. We use a polynomial time transformation from the following problem.

PARTITION INTO TRIANGLES
Instance: A graph $G = (V, E)$, with $|V| = 3q$ for a positive integer q.
Question: Is there a partition of V into triangles, i.e. into $V_1, V_2, \ldots, V_q$ such that $G[V_i] = K_3$?

This problem remains NP-hard even when the graph $G = (V, E)$ is 3-colorable and a partition of V into independent sets I_1, I_2, I_3 with $|I_1| = |I_2| = |I_3|$, is given. This can be proved by replacing EXACT COVER BY 3-SETS in the reduction on p. 68–69 of Garey & Johnson [10] by 3-DIMENSIONAL MATCHING.

Theorem 5.5. *For any fixed $k \geq 3$, problem* P-RL$(*)$ *is NP-hard even when the input is restricted to graphs with a given k-coloring.*

Proof. Let us first prove the theorem for $k = 3$. Let $G = (V, E)$ be a graph and let I_1, I_2, I_3, $|I_1| = |I_2| = |I_3| = q$, be a partition of V into three independent sets. We construct a graph $F = (V_F, E_F)$ and a pre-labeling L' of F such that there is a radio labeling of F with span $\leq K$ extending L' if and only if V can be partitioned into triangles.

The graph F is obtained as follows: V_F is the disjoint union of four independent sets V_1^A, V_1^B, V_2, V_3 of cardinality q. (The vertices of V_1^A, V_1^B correspond to I_1, the vertices of

V_2 to I_2 and those of V_3 to I_3. Also for simplicity we denote vertices from V_F and the corresponding vertices from V by the same letters.)

$$\begin{aligned} E_F \;=\; & \{\{u,v\}\colon u \in V_1^A, v \in V_2 \text{ and } \{u,v\} \notin E\} \cup & (1)\\ & \{\{u,v\}\colon u \in V_1^B, v \in V_3 \text{ and } \{u,v\} \notin E\} \cup & (2)\\ & \{\{u,v\}\colon u \in V_2, v \in V_3 \text{ and } \{u,v\} \notin E\} \cup & (3)\\ & \{\{u,v\}\colon u \in V_1^A, v \in V_3\} \cup & (4)\\ & \{\{u,v\}\colon u \in V_1^B, v \in V_2\}. & (5) \end{aligned}$$

The vertices of V_1^A are pre-labeled with integers $4i-3$ and the vertices of V_1^B with $4i$, where $1 \le i \le q$.

Notice that F is 3-colorable because the sets $V_1^A \cup V_1^B, V_2, V_3$ are independent. We claim that there is a radio labeling with span at most $K = 4q$ in F extending the pre-labeling if and only if the vertex set V of G can be partitioned into q triangles.

If there is such a radio labeling in F with span K, then every label from $\{1, 2, \ldots, K\}$ should be used in this labeling. Then for every i with $1 \le i \le q$ the vertices labeled with $4i-3$ and $4i$ are in V_1^A and V_1^B (because of the pre-labeling). Then by (5), the vertices labeled with $4i-2$ should be in V_2, and by (4) the vertices labeled with $4i-1$ should be in V_3. Then by (1), (2) and (3), the vertices in G corresponding to vertices labeled with $4i-3, 4i-2, 4i-1, 4i$ induce a triangle in G. Hence, V can be partitioned into q triangles.

For the converse, consider a partition into triangles of V in G. For every triangle with vertex set $\{v_1, v_2, v_3\}$, $v_i \in I_i$, $1 \le i \le 3$, the corresponding four vertices $v_1^A \in V_1^A, v_1^B \in V_1^B, v_2 \in V_2, v_3 \in V_3$ in F can be labeled with four consecutive labels. The vertices v_1^A, v_1^B are pre-labeled with $4j-3$ and $4j$ for some j; by (1), vertex v_2 can be labeled with $4j-2$, and by (2), (3), vertex v_3 can be labeled with $4j-1$. Therefore, F has a radio labeling extending the pre-labeling with at most $4q = K$ labels. This settles the case for 3-colorable graphs with a given 3-coloring.

To prove the theorem for any fixed $k \ge 3$, we add to the graph F two disjoint cliques of size k and pre-label the vertices of one clique with labels $\{4q+1, 4q+3, \ldots, 4q+2k-1\}$ and of the other clique with $\{4q+2, 4q+4, \ldots, 4q+2k\}$. Then the new graph is k-colorable, and using the previous arguments it is clear that this graph has a radio labeling with $4q+2k$ labels extending the pre-labeling if and only if the graph G has a partition into triangles. This completes the proof. $\square$

6 NP-completeness results for cographs

We now turn to the last class of graphs for which RL is known to be polynomially solvable, namely the class of cographs, i.e. graphs without an induced path on four vertices. Using an easy reduction from 3-PARTITION we show that P-RL($*$) is NP-hard for cographs.

Theorem 6.1. *Problem* P-RL($*$) *is NP-hard for cographs.*

Proof. We use a reduction from the problem 3-PARTITION stated below to P-RL($*$).

3-PARTITION
Instance: A set A of non-negative integers $a_1, \ldots, a_{3m}$, and a bound B, such that for all i with $1 \le i \le 3m$, $(B+1)/4 < a_i < B/2$ and $\sum_{1\le i\le 3m} a_i = mB$.
Question: Can A be partitioned into m disjoint sets $A_1, A_2, \ldots, A_m$ such that $\sum_{a_i \in A_j} a_i = B$ for every j with $1 \le j \le m$?

3-PARTITION is NP-complete in the strong sense (Problem SP15 in Garey & Johnson [10]).

Let the set $A = \{a_1, \ldots, a_{3m}\}$ and the bound B be an instance of 3-PARTITION. Consider a complete $(3m+1)$-partite graph G_A with vertex partition $V = V_0 \cup V_1 \cup \ldots \cup V_{3m}$, where $|V_0| = m-1$ and $|V_i| = a_i$ for every i with $i = 1, \ldots, 3m$. This means there is an edge between a vertex $v \in V_i$ and a vertex $w \in V_j$, if and only if $i \neq j$. Clearly, G_A is a cograph because it obviously does not contain a path on four vertices as an induced subgraph. Pre-label the vertices $v_1, \ldots, v_{m-1}$ of V_0 using the labels $L'(v_j) = j(B+5)-1$, where $j = 1, \ldots, m-1$ and let $K = m(B+5)-3$. We show that the pre-labeling L' of G_A can be extended to a radio labeling with span $\le K$ if and only if A can be 3-partitioned.

If there is a 3-partition $A_1, \ldots, A_m$, then each of the sets A_i contains exactly three elements a_{i_1}, a_{i_2}, and a_{i_3}. Consider then for every $i = 1, \ldots, m$ the following labeling:

- Vertices of the set V_{i_1} are labeled with $\{(i-1)(B+5)+1, \ldots, (i-1)(B+5)+a_{i_1}\}$;
- Vertices of the set V_{i_2} are labeled with $\{(i-1)(B+5)+a_{i_1}+2, \ldots, (i-1)(B+5)+a_{i_1}+a_{i_2}+1\}$;
- Vertices of the set V_{i_3} are labeled with $\{(i-1)(B+5)+a_{i_1}+a_{i_2}+3, \ldots, (i-1)(B+5)+a_{i_1}+a_{i_2}+a_{i_3}+2\}$.

Since $a_{i_1}+a_{i_2}+a_{i_3} = B$, this produces a radio labeling extending the pre-labeling L' with span $\le K$.

For the converse, suppose that a radio labeling of G_A extending the pre-labeling L' with span $\le K$ exists. For $j = 1, \ldots, m$, let $B_j = \{(j-1)(B+5)+1, \ldots, j(B+5)-3\}$ where $j = 1, \ldots, m$. We call B_j a *range*. Vertices in $V_1 \cup \ldots \cup V_{3m}$ must get a label in a range B_j, $1 \le j \le m$. Note that for each range B_j, $|B_j| = B+2$. Each range must have at least two unused labels: labels given to vertices in different sets V_i must differ by at least two, and each $|V_i| = a_i < B/2$, thus each range contains either vertices from at least three sets V_i (and hence at least two unused labels), or vertices from at most two sets V_i, and hence at least $B+2-2(B/2-1) = 4$ unused labels. Since $\sum_{i=1}^{3m} |V_i| = \sum_{i=1}^{3m} a_i = mB = \sum_{j=1}^{m}(|B_j|-2)$, there are exactly two unused labels in each range B_j. For each i, $1 \le i \le 3m$, consider the highest label ℓ_i given to a vertex in V_i. The label $\ell_i + 1$ either does not belong to a range (this is true for exactly m such labels, as there are m ranges), or is an unused label in a range. Thus, every unused label in a range is one larger than the highest label among all vertices in a set V_i, for some i, $1 \le i \le 3m$. As a consequence, all labels of a set V_i must be assigned to the same range. Now, let A_j be the set of values a_i such that vertices of V_i have their label in B_j. As B_j contains exactly B used labels, we have for each j, $\sum_{a_i \in A_j} a_i = B$. This completes the proof. □

We do not know whether P-RL(l) is NP-hard for cographs, and leave it as one of the open problems in the next section.

7 Open problems

In this paper we initiated the study of two versions of the radio labeling problem in which a pre-labeling is assumed. Many questions remain open, a few of which are listed below:

- We leave the complexity of any of the variants of RADIO LABELING (RL, P-RL(l) and P-RL($*$)) for interval graphs as open problems.
- Another open problem concerns the computational complexity of P-RL(l) for cographs. Recall that NP-hardness of P-RL($*$) restricted to cographs is proved in this paper, and that RL is polynomial for cographs.
- Our results imply that P-RL(l) is polynomial for bipartite graphs. On the other hand, we proved that P-RL($*$) is NP-hard for 3-partite graphs even if a 3-coloring of the graph is given. The complexity of P-RL($*$) for bipartite graphs is open.
- By Theorem 5.3, P-RL(l) is polynomial for planar graphs and graphs of bounded treewidth. The complexity of P-RL($*$) for these graph classes is open.

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