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Edge-Isoperimetric Problems for Cartesian Powers of Regular Graphs *

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Abstract. We consider an edge-isoperimetric problem (EIP) on the cartesian powers of graphs. One of our objectives is to extend the list of graphs for whose cartesian powers the lexicographic order provides nested solutions for the EIP. We present several new classes of such graphs that include as special cases all presently known graphs with this property. Our new results are applied to derive best possible edge-isoperimetric inequalities for the cartesian powers of arbitrary regular, resp. regular bipartite, graphs with a high density.

1 Introduction

Let $G = (V_G, E_G)$ be a graph and $A, B \subseteq V_G$. Denote

$$\begin{aligned} I_G(A,B) &= \{(u,v) \in E_G \bigm| u \in A,\ v \in B\}, \\ I_G(A) &= I_G(A,A), \\ I_G(m) &= \max_{A \subseteq V_G,\ |A|=m} |I_G(A)|. \end{aligned}$$

We will often omit the index G. Our subject is the following version of the *edge-isoperimetric problem* (EIP): for a fixed m, $1 \le m \le |V_G|$, find a set $A \subseteq V_G$ such that $|A| = m$ and $|I(A)| = I(m)$. We call such a set A *optimal*. This problem is known to be NP-complete in general and has many applications in various fields of knowledge, for instance see a survey [4].

We emphasize on graphs representable as cartesian products of other graphs. Given two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$, their *cartesian product* is defined as a graph $G \times H$ with the vertex-set $V_G \times V_H$ whose two vertices (x, y) and (u, v) are adjacent iff either $x = u$ and $(y, v) \in E_H$, or $(x, u) \in E_G$ and $y = v$. The graph $G^n = G \times G \times \cdots \times G$ (n times) is called the n^{th} *cartesian power* of G.

The EIP for the cartesian powers of a graph G have been well studied for different graph classes. To present these results we need the definition of the

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lexicographic order on a set of n-tuples with integral entries. For that we say that $(x_1, \ldots, x_n)$ is greater than $(y_1, \ldots, y_n)$ iff there exists an index i, $1 \leq i \leq n$, such that $x_j = y_j$ for $1 \leq j < i$ and $x_i > y_i$. Assuming that the vertices of a graph G are totally ordered, denote by $\mathcal{F}^n(m)$ the collection of the first m vertices of G^n in the lexicographic order. We call such a set an *initial segment*. Let G be the cartesian product of n complete graphs. Then the following Theorem holds.

Theorem 1. (Harper [7] for $p_1 = \ldots = p_n = 2$, Lindsey [8] in general) *Let $n \geq 2$ and $p_1 \geq p_2 \geq \cdots \geq p_n$. Then $|I(\mathcal{F}^n(m))| \geq |I(A)|$ for any $A \subseteq K_{p_1} \times \cdots \times K_{p_n}$, $|A| = m$, and any $m \geq 1$.*

In other words, what is more commonly said, for the products of cliques there exists a *nested structure of solutions*, i.e., a family of optimal subsets $A_1, A_2, \ldots, A_m, \ldots$ such that $|A_m| = m$ for any m and $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_m \subseteq \cdots$. We call an order providing the nested structure of solutions an *optimal order* (if such exists). Existence of a nested structure of solutions provides as an immediate consequence solutions to such important problems as cutwidth and wirelength, and often allows to construct good k-partitioning of graphs and their embedding to some other graphs [4].

The classical results presented in Theorem 1 can be extended in various directions. For instance, taking an (infinite) path instead of a clique leads to a grid. In this case the EIP also has nested structure of solutions [1, 6], although the optimal order $\mathcal{G}$ is not the lexicographic one. It is further shown in [4] that the order $\mathcal{G}$ is also optimal for the products of arbitrary trees. For the definition of this order and further details, readers are referred to [1, 4, 6]. It is also known that no optimal order exists for the powers of cycles of length 5 and larger.

The order $\mathcal{G}$ and the lexicographic order were for a long time the only examples of optimal orders with respect to the EIP. However, as shown in [4], the order $\mathcal{G}$ is optimal for products of trees only. Therefore, two natural questions arise: *(i) for what other graphs is the lexicographic order on the power optimal with respect to the edge isoperimetric problems; and (ii) what other optimal orders can one expect?* In [5], we studied the cartesian powers of the Petersen graph and presented a new optimal order for them. However, it looks like this order is optimal for a rather narrow class of graphs. This, in the light of the results presented above, motivates to study further the lexicographic order.

Following this direction, Ahlswede and Cai proved in [2] that the lexicographic order is optimal for the products of complete bipartite graphs $K_{p,p}$. They also showed in [2] that this order yields the so-called *local-global principle*. That is, if the lexicographic order is optimal for G^2 then it is optimal for G^n for any $n \geq 3$ (see [4] for the local-global principles for some other orders). The main difficulty in applying this powerful result is to establish the optimality of the lexicographic order for the second cartesian power of a graph. Unfortunately, no efficient methods have been developed and the known approaches are very specific. In our paper we present new ideas in this direction and apply them for three new graph families. These new graphs include, in particular, all presently known graphs such that the lexicographic order is optimal on their cartesian powers.

The paper is organized as follows. In Section 2 we introduce new graph classes $H_{p,i}$ and $B_{p,i}$ and present some basic definitions and lemmas which are used in the sequel. A motivation for studying these graph classes is provided by Lemma 1, which establishes best-possible isoperimetric inequalities for powers of arbitrary regular, resp. regular bipartite, graphs. In Section 3 we prove the optimality of the lexicographic order for the cartesian powers of $H_{p,i}$. The approach used in this proof also works well for the powers of complete bipartite graphs and complete t-partite graphs and generalize the result of Ahlswede and Cai [2] concerning the complete bipartite graphs. Concluding remarks and open questions in Section 5 complete the paper.

2 Definitions and auxiliary lemmas

Here we introduce three new families of graphs, for whose cartesian powers the lexicographic order is optimal. The graph $H_{p,i}$, can be constructed from K_{2p} by partitioning its vertex set into two parts V' and V'' with $|V'| = |V''| = p$ and removing i disjoint perfect matchings between these sets. More exactly,

$$\begin{aligned}
V_{H_{p,i}} &= V' \cup V'', \quad V' = \{0, 1, \ldots, p-1\},\ V'' = \{p, p+1, \ldots, 2p-1\}, \\
E_{H_{p,i}} &= \{(u,v) \mid u, v \in V',\ \text{or } u, v \in V''\} \cup \\
&\qquad \{(u,v) \mid u \in V',\ v \in V'',\ u + v \neq (2p - j) \bmod 2p,\ j = 1, \ldots, i\}.
\end{aligned}$$

An example of $H_{3,1}$ is shown in Fig. 1(a). The vertices belonging to V' and V'' are put into ovals. Note that various ways of deleting i perfect matchings from K_{2p} may lead to non-isomorphic graphs. However, we do not worry much about that, because the function $I(m)$ for all these graphs is the same, as it follows from the proof of Lemma 3. This function is mostly important for our approach rather than the structural properties of the underlying graphs. Another graph we study in our paper, $B_{p,i}$, is the one obtained from $K_{2p,2p}$ for an even p by removing i disjoint perfect matchings (cf. Fig. 1(b) and the conclusion for a precise definition). The lexicographic order is also optimal for $B^n_{p,i}$ for any $n \geq 2$ and $i \leq \frac{p}{4}$. This can be proved by following the lines of the proof of Theorem 2 in the next section. The interest to these graphs is provided by the following lemma, whose proof is presented at the end of this section after establishing some auxiliary results.

Lemma 1.

a. Let G be a regular graph with $|V_G| = V_{H_{p,i}}$ and $\deg(G) = \deg(H_{p,i})$, that admits an optimal order in the EIP. Then $I_{H^n_{p,i}}(m) \geq I_{G^n}(m)$ for any $n \geq 1$ and $m = 1, \ldots, (2p)^n$.

b. Let G be a regular bipartite graph with both independent sets of size p (p even) and $\deg(G) = \deg(B_{p,i})$, that admits an optimal order in the EIP. Then $I_{B^n_{p,i}}(m) \geq I_{G^n}(m)$ for any $n \geq 1$ and $m = 1, \ldots, (2p)^n$.

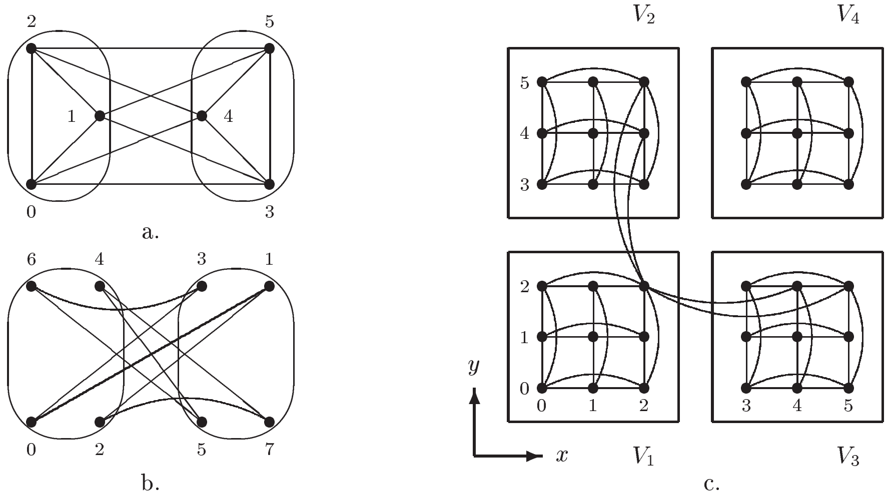

Fig. 1. Representations of $H_{3,1}$ (a), $B_{4,2}$ (b), and $H^2_{3,1}$ (c)

This provides best-possible isoperimetric inequalities for powers of regular, resp. regular bipartite, graphs. The class of regular graphs is important, because in the light of (1) the isoperimetric problem for regular graphs is equivalent to the problem of minimization of $|\theta(A)|$, where $\theta(A) = I(A, \overline{A})$ and $\overline{A} = V_G \setminus A$. The function $\theta(A)$ appears in most applications of the edge-isoperimetric inequalities [4]. For a non-regular graph G a solution to the EIP on G provides a lower bound for this function following from the identity $I(A) \cup I(\overline{A}) \cup \theta(A) = E_G$. Moreover, we conjecture that the regularity of G is important for the lexicographic order to be optimal for G^n (cf. the conclusion).

The third graph class we consider here is given by the complete t-partite graphs $K_{p,p,\ldots,p}$. It can be also shown that the lexicographic order is optimal for its cartesian powers too. This extends a result of [2] for complete bipartite graphs. We also develop a new method for solving the EIP for the product of two graphs. This method works well for all graphs mentioned above.

We will need the following two auxiliary propositions for the sequel.

Lemma 2. *Let $G = (V_G, E_G)$ be a regular graph and $A \subseteq V_G$. Then A is an optimal set iff $\overline{A} = V_G \setminus A$ is optimal.*

Proof. Let $\deg(v) = k$ for any $v \in V_G$. Then

$$\begin{aligned} 2|I(A)| + |I(A, \overline{A})| &= k|A| \qquad (1) \\ 2|I(\overline{A})| + |I(A, \overline{A})| &= k|\overline{A}| \ = \ k(|V_G| - |A|) \end{aligned}$$

The above equations imply $2|I(\overline{A})| = k(|V_G| - 2|A|) + 2|I(A)|$. Thus, $|I(A)|$ and $|I(\overline{A})|$ differ on an additive constant for any $A \subseteq V_G$. Hence, if one of these sets is optimal, then the other one is optimal too. $\square$

Denote $V = V_{H_{p,i}}$ for brevity and let $\mathcal{F}(m) = \{0, 1, \ldots, m-1\}$ for $m = 0, 1, \ldots, 2p$.

Lemma 3. $|I(\mathcal{F}(m))| \geq |I(A)|$ *for any* $A \subseteq V$ *with* $|A| = m$.

Proof. The assertion is obvious if $m \leq p$, because $H_{p,i}$ contains K_p, whose vertices are numbered first. Now if $m > p$ then the assertion follows from Lemma 2. □

Let $n \geq 2$ and $A \subseteq V^n$. For $1 \leq i \leq n$ and $\mathbf{x} = (x_1, \ldots, x_n) \in V^n$ denote

$$A_i(\mathbf{x}) = \{y \bigm| (x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) \in A\}.$$

The set $A \subseteq V^n$ is called *compressed* if $A_i(\mathbf{x}) = \{0, 1, \ldots, |A_i(\mathbf{x})|\}$ for any $i = 1, \ldots, n$ and any $\mathbf{x} \in A$. Standard arguments (see [2,4]) provide that if a graph G admits an optimal order, then for any $m \geq 1$ there exist compressed optimal sets in V_{G^n}. This observation is a key point for the proof of Lemma 1.

Proof of Lemma 1.
For a graph G from the statement and $j = 1, \ldots, |V_G|$, let us denote

$$\Delta_G(j) = I_G(j) - I_G(j-1) \quad \text{with} \quad \Delta_G(0) = 0.$$

Taking into account that G admits an optimal order, denote the vertices of G by $0, 1, \ldots, 2p-1$ according to their positions in this order. This provides $V_G = V$.

We prove the first proposition of the lemma by induction on n. For $n = 1$ we can use the same arguments as in the proof of Lemma 3. Note that $I_G(m) = \sum_{j=0}^{m-1} \Delta_G(j)$ for any graph G that admits an optimal order.

Now for $n \geq 2$ let $A \subseteq V_{G^n}$ with $|A| = m$ be an optimal compressed set. Then A can be viewed as a subset of V^n. Note (see e.g. [2] for (2)) that for any compressed set $A \subseteq V_{G^n} = V^n$ one has

$$|I_{G^n}(A)| = \sum_{(x_1,\ldots,x_n)\in A} \sum_{i=1}^{n} \Delta_G(x_i) \tag{2}$$

$$= \sum_{j=0}^{p-1} \left[\sum_{(j,x_2,\ldots,x_n)\in A} \sum_{i=2}^{n} \Delta_G(x_i) \right] + \sum_{(x_1,\ldots,x_n)\in A} \Delta_G(x_1). \tag{3}$$

Similar formulas are also valid for compressed subsets $B \subseteq V_{H^n_{p,i}} = V^n$. Note that A and B are the subsets of the same vertex set. Set $B = A$. Now for any fixed j in (3) the induction hypothesis implies

$$\sum_{(j,x_2,\ldots,x_n)\in A} \sum_{i=2}^{n} \Delta_G(x_i) \ \leq \sum_{(j,x_2,\ldots,x_n)\in B} \sum_{i=2}^{n} \Delta_{H_{p,i}}(x_i). \tag{4}$$

Furthermore, for a fixed $\mathbf{x} \in V$ and a compressed set $A = B$ (with $A_1(\mathbf{x}) = B_1(\mathbf{x})$) Lemma 3 implies

$$\sum_{x_1 \in A_1(\mathbf{x})} \Delta_G(x_1) = I_G(|A_1(\mathbf{x})|) \leq I_{H_{p,i}}(|B_1(\mathbf{x})|) = \sum_{x_1 \in B_1(\mathbf{x})} \Delta_{H_{p,i}}(x_1) \tag{5}$$

Summing up (4) for $j = 0, \dots, p-1$ and (5) for $x_2, \dots, x_n$ such that $\mathbf{x} = (x_1, x_2, \dots, x_n) \in A$ and using (3) one gets

$$I_{G^n}(m) = |I_{G^n}(A)| \le |I_{H^n_{p,i}}(B)| \le \max_{B \subseteq V^n,\, |B|=m} |I_{H^n_{p,i}}(B)| = I_{H^n_{p,i}}(m),$$

and the induction goes through.

The second proposition can be proved similarly by taking into account that an analog of Lemma 3 is valid for the graph $B_{p,i}$. □

By using a similar argument the following result can be proved, that can be considered as a best-possible edge-isoperimetric inequality for the cartesian powers of general graphs.

Corollary 1. *Let G be a graph with $|V_G| = p$ that admits nested solutions in the EIP. Then $I_{K_p^n} \ge I_{G^n}(m)$ for any $n \ge 1$ and $m = 1, \dots, p^n$.*

3 EIP for the cartesian powers of $H_{p,i}$

One of the main results of the paper is formulated in the following theorem.

Theorem 2. *If $p \ge 3$ and $1 \le i \le p - i$ then $|I(\mathcal{F}^2(|A|))| \ge |I(A)|$ for any $A \subseteq V \times V$.*

Proof. Let $A \subseteq V \times V$ be an optimal compressed set non-isomorphic to $\mathcal{F}^2(|A|)$ (if such exists). Denote

$$\begin{aligned}
V_1 &= \{(x,y) \mid 0 \le x < p,\ 0 \le y < p\},\\
V_2 &= \{(x,y) \mid 0 \le x < p,\ p \le y < 2p\},\\
V_3 &= \{(x,y) \mid p \le x < 2p,\ 0 \le y < p\},\\
V_4 &= \{(x,y) \mid p \le x < 2p,\ p \le y < 2p\},
\end{aligned}$$

and let $A_i = A \cap V_i$, $i = 1,2,3,4$. Obviously, for any i the subgraph induced by the vertex set V_i is isomorphic to $K_p \times K_p$. These sets are shown in Fig. 1(c) for the graph $H^2_{3,1}$. The graphs induced by vertices located in one row (or in one column) in this figure are isomorphic to $H_{p,i}$. We will often use the fact that any vertex of V_2 (and V_3) is connected with $p - i$ vertices of V_1 and versa. In Fig. 1(c) such connections are shown for the vertex $(2,2) \in V_1$.

Case 1. Assume $A \subseteq V_1 \cup V_2$. Since A is compressed, then $|A_1| \ge |A_2|$. Thus, due to Theorem 1 we can assume that A_1 and A_2 are initial segments of V_1 and V_2 respectively (see Fig. 2). Let $|A_1| = k_1 p + \delta_1$ and $|A_2| = k_2 p + \delta_2$ for some integer $k_1, k_2, \delta_1, \delta_2$ with $0 \le k_1, k_2 \le p - 1$ and $0 \le \delta_1, \delta_2 < p$. If $|A_1| = p^2$, we assume $k_1 = p - 1$ and $\delta_1 = p$. Now $|A_1| \ge |A_2|$ implies $k_1 \ge k_2$. Moreover, if $k_1 = k_2$ then $\delta_1 = p$ or $\delta_2 = 0$ due to the compression, hence $A = \mathcal{F}^2(|A|)$. So we can assume $k_1 > k_2$. *Case 1a.* Assume $k_1 - k_2$ is even (and positive). First consider the case $\delta_1 + \delta_2 \le p$. We transform A into $B = \mathcal{F}^2(|A|)$ in two steps. First, we construct a set C by moving $(k_2, y) \in A_2$ for $p \le y < p + \delta_2$ to $(k_1, y + \delta_1 - p) \in V_1 \setminus A_1$. It is easily shown that

$$|I(C)| - |I(A)| = \delta_2(\delta_1 + k_1 - k_2) - \delta_2(p - i). \tag{6}$$

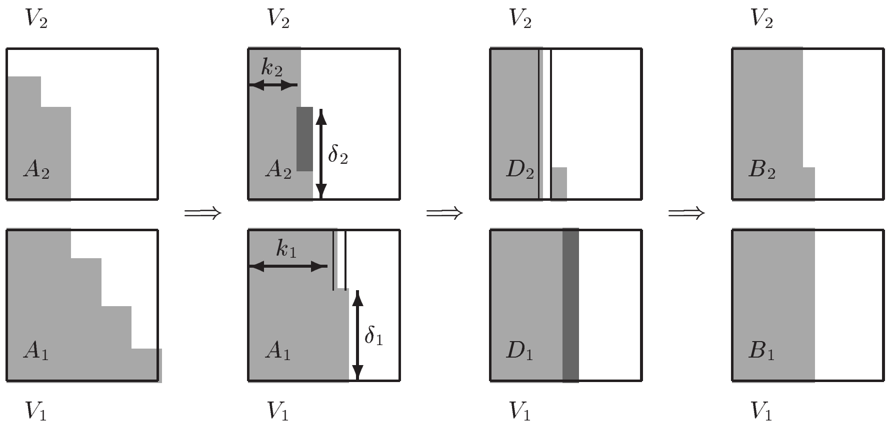

Fig. 2. Compression of A in case 1 and transformation for $\delta_1 + \delta_2 > p$ in case 1a.

Denote $C_i = C \cap V_i$, $D_i = D \cap V_i$ and $B_i = B \cap V_i$, for $i = 1, 2, 3, 4$. Now we move $(x, y) \in C_1$ with $\frac{k_1+k_2}{2} \leq x < k_1$, $0 \leq y < p$ to $(x - (k_1 - k_2)/2, y + p) \in V_2 \setminus C_2$ and $(x, y) \in C_1$ with $x = k_1$, $0 \leq y < \delta_1 + \delta_2$ to $((k_1 + k_1)/2, y) \in V_1 \setminus C_1$. For the resulting set D one has

$$|I(D)| - |I(C)| = \frac{p(k_1 - k_2)}{2}\left(p - i - \frac{k_1 - k_2}{2}\right) - (\delta_1 + \delta_2)\frac{k_1 - k_2}{2}. \qquad (7)$$

Note that $B_2 = D_2$ and $|B_1| = |D_1|$. Since the subgraphs induced by V_1 and V_2 are isomorphic to $K_2 \times K_2$, then $|I(D)| = |I(B)|$. Therefore, from (6) and (7) for $\Delta = |I(B)| - |I(A)|$ one gets

$$\Delta = \frac{p(k_1 - k_2)}{2}\left(p - i - \frac{k_1 - k_2}{2}\right) + \delta_2\left(\delta_1 + \frac{k_1 - k_2}{2}\right) - \delta_1\frac{k_1 - k_2}{2} - \delta_2(p - i). \quad (8)$$

Since $i \leq p - i$, then $p/2 \geq i$. Furthermore, $k_2 \geq 0$ and $k_1 \leq p - 1$ imply $k_1 - k_2 \leq p - 1$. These inequalities imply $p - i \geq p/2 > (p - 1)/2 \geq (k_1 - k_2)/2$. Therefore, $p - i - (k_1 - k_2)/2 \geq 1$, and from (8) it follows

$$\Delta \geq \frac{(p - \delta_1)(k_1 - k_2)}{2} - \delta_2 + \delta_1\delta_2. \qquad (9)$$

Since $p \geq \delta_1$ and $\delta_2\delta_1 \geq \delta_2$ for $\delta_1 \neq 0$, then $\Delta \geq 0$. If $\delta_1 = 0$, then taking into account $k_1 - k_2 \geq 2$ and (9), we get

$$\Delta \geq \frac{p(k_1 - k_2)}{2} - \delta_2 \geq p - \delta_2 > 0.$$

Now consider the case $\delta_1 + \delta_2 > p$. Then $\delta_2 > 0$. We transform A into $B = \mathcal{F}^2(|A|)$ in three steps. First construct the set C by moving $p - \delta_1$ vertices of the form $(k_2, y) \in A_2$ for $\delta_1 + \delta_2 \le y < \delta_2 + p$ to $(k_1, y - \delta_2) \in V_1 \setminus A_1$. One has

$$|I(C)| - |I(A)| = (p - \delta_1)(\delta_1 + k_1 - k_2 - (\delta_1 + \delta_2 - p) - (p - i)). \tag{10}$$

Next, we move the vertices $(k_2, y) \in C_2$ for $p \le y < \delta_1 + \delta_2$ to $((k_1 + k_2)/2, y)$. For the resulting set D one has $|I(D)| = |I(C)|$. Finally, we move $(x, y) \in D_1$ with $\frac{k_1+k_2}{2} < x \le k_1$ and $0 \le y < p$ to $(x + 1 - (k_1 - k_2)/2, y + p) \in V_2 \setminus D_2$. The resulting set is B. One has

$$|I(B)| - |I(D)| = \frac{p(k_1 - k_2)}{2} \left(p - i - \frac{k_1 - k_2}{2} - 1 \right) + (\delta_1 + \delta_2 - p) \frac{k_1 - k_2}{2}. \tag{11}$$

It follows from (10) and (11) that for $\Delta = |I(B)| - |I(A)|$ one has

$$\begin{aligned} \Delta = {} & \frac{p(k_1 - k_2)}{2} \left(p - i - \frac{k_1 - k_2}{2} \right) - \delta_1 \frac{k_1 - k_2}{2} \\ & + (p - \delta_1)(\delta_1 - \delta_2 + (p - \delta_1) - (p - i)) + \delta_2 \frac{k_1 - k_2}{2} \\ = {} & \frac{p(k_1 - k_2)}{2} \left(p - i - \frac{k_1 - k_2}{2} \right) - \delta_1 \frac{k_1 - k_2}{2} \\ & + (p - \delta_1)(i - \delta_2) + \delta_2 \frac{k_1 - k_2}{2}. \end{aligned} \tag{12}$$

Similarly as above, $p - 1 - (k_1 - k_2)/2 \ge 1$. Now if $i \ge \delta_2$, then

$$\frac{p(k_1 - k_2)}{2} \left(p - i - \frac{k_1 - k_2}{2} \right) - \delta_1 \frac{k_1 - k_2}{2} \ge \frac{(p - \delta_1)(k_1 - k_2)}{2} \ge 0, \tag{13}$$

which implies $\Delta > 0$.

Note that $p - \delta_1 < \delta_2$. Thus, if $i < \delta_2$ and $(k_1 - k_2)/2 > \delta_2 - i$, then (12) and (13) imply

$$\begin{aligned} \Delta & > \frac{p(k_1 - k_2)}{2} \left(p - i - \frac{k_1 - k_2}{2} \right) - \delta_1 \frac{k_1 - k_2}{2} - \delta_2(\delta_2 - i) + \delta_2 \frac{k_1 - k_2}{2} \\ & \ge \frac{p(k_1 - k_2)}{2} \left(p - i - \frac{k_1 - k_2}{2} \right) - \delta_1 \frac{k_1 - k_2}{2} \ge 0. \end{aligned}$$

Now if $i < \delta_2$ and $\delta_2 - i \ge (k_1 - k_2)/2$, then (12) implies

$$\begin{aligned} \Delta & = \frac{p(k_1 - k_2)}{2} \left(p - i - \frac{k_1 - k_2}{2} \right) + \delta_1 \left(\delta_2 - i - \frac{k_1 - k_2}{2} \right) - p(\delta_2 - i) + \delta_2 \frac{k_1 - k_2}{2} \\ & = \left[\frac{p(k_1 - k_2)}{2} \left(p - i - \frac{k_1 - k_2}{2} \right) - \delta_2 \left(p - i - \frac{k_1 - k_2}{2} \right) \right] \\ & \quad + \delta_1 \left(\delta_2 - i - \frac{k_1 - k_2}{2} \right) + (p - \delta_2) i. \end{aligned} \tag{14}$$

Since $p > \delta_2$ and $k_1 - k_2 \ge 2$, then the last term of (14) and the term in brackets are positive. Therefore, $\Delta > 0$.

Case 1b. In this case we assume that $k_1 - k_2 - 1$ is even. The proof of this case is similar to case 1a. and it is omitted because of space limitations.

Case 2. Assume $A \subseteq V_1 \cup V_2 \cup V_3$ and $A_2 \neq \emptyset$ and $A_3 \neq \emptyset$. Without loss of generality we can assume that A_2 is an initial segment (in "columns" of V_2) and so is A_3 (in "rows" of V_3) (cf. Fig. 3). Indeed, after such a transformation the number of inner edges in A_2 and A_3 cannot decrease by Theorem 1 and the number of edges between A_1 and A_2 (also between A_1 and A_3) remains the same. Let $|A_2| = k_2 p + \delta_2$ and $|A_3| = k_3 p + \delta_3$ with $0 \leq \delta_2, \delta_3 < p$. We can assume $k_2 \geq k_3$. Now we consider an $(p - k_2) \times (p - k_3)$ rectangle $V_5 \subseteq V_1$ with

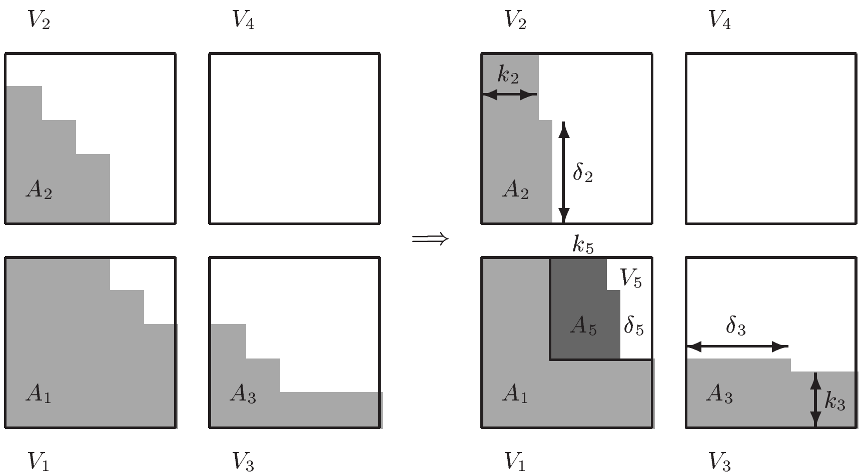

Fig. 3. Compression of A in case 2

$V_5 = \{(x, y) \in V_1 \mid k_2 \leq x < p,\ k_3 \leq y < p\}$ and replace $A_5 = A \cap V_5$ with the initial segment in V_5 of the same size. This initial segment is shown by the dark gray area in Fig. 3. By Theorem 1 (applied for V_5), the resulting set will be optimal too. This is true because the number of inner edges within A_5 will not decrease, and the number of edges between A_5 and $V_1 \setminus V_5$ remains the same. Let $|A_5| = (p - k_3)k_5 + \delta_5$ with $0 \leq \delta_5 < p - k_3$.

Without loss of generality we can assume $\delta_2 = 0$. Indeed, otherwise if $\delta_3 > p - \delta_2$ then we move $(x, k_3) \in A_3$ for $\delta_2 + \delta_3 \leq x < p + \delta_3$ to $(k_2, x - \delta_2 + p) \in V_2 \setminus A_2$. For the resulting set C one has

$$|I(C)| - |I(A)| = (p - \delta_2)(\delta_2 - ((\delta_2 + \delta_3 - p) + k_2 - k_3)) > 0.$$

If $0 < \delta_3 \leq p - \delta_2$ then we move $(x, k_3) \in A_3$ for $p \leq x < p + \delta_3$ to $(k_2, x + \delta_2) \in V_2 \setminus A_2$. For the resulting set C one has

$$|I(C)| - |I(A)| = \delta_3(\delta_2 + k_2 - k_3) > 0.$$

If $\delta_3 = 0$ then $A_3 \neq \emptyset$ implies $k_3 > 0$. We move $(x, k_3 - 1) \in A_3$ for $p + \delta_2 \leq x < 2p$ to $(k_2, x) \in V_2 \setminus A_2$. For the resulting set C one has

$$|I(C)| - |I(A)| = (p - \delta_2)(k_2 - k_3 + 1) > 0.$$

In all cases we get a contradiction with the optimality of A.

Case 2a. Assume $k_5 > 0$. If $k_5 > k_3$ or $k_5 = k_3$ and $\delta_3 = 0$, then $|A| \leq 2p^2$. We move $(x, y) \in A_3$ to $(y + k_2, x) \in V_2 \setminus A_2$. This results in a set $C \subseteq V_1 \cup V_2$ with $|I(C)| \geq |I(A)|$, and we apply the arguments of case 1 to further transform C to $\mathcal{F}^2(|A|)$.

If $1 \leq k_5 < k_3$ or $1 \leq k_5 = k_3$ and $\delta_3 > 0$, then we move $(x, y) \in A_3$ for $p \leq x < 2p$ and $0 \leq y < k_5$ to $(y + k_2, x) \in V_2 \setminus A_2$. For the resulting set C one has

$$|I(C)| - |I(A)| = k_5 p(k_2 - (k_3 - k_5)) - \delta_3 k_5 \geq k_5(p - \delta_3) \geq 0.$$

Moreover, $|C_3| < |A_3|$, and C satisfies the assumptions of case 2. Applying similar arguments to C we obtain a set D either with $D_3 = \emptyset$ or with $k_5 = 0$. In the first case we apply to D the arguments of case 1 and transform it to $\mathcal{F}^2(|A|)$.

Case 2b. Assume $k_5 = 0$ and $k_3 > 0$. We show that in this case A is not optimal. We start with interchanging $(x, 0) \in A_3$ for $p + \delta_3 \leq x < 2p$ and $(x, k_3) \in V_3 \setminus A_3$. This transformation does not affect the number of inner edges in a set. First assume $k_2 < p - 1$. If $\delta_5 > 0$ we move $(k_2, y) \in A_1$ for $k_3 < y \leq k_3 + \delta_5$

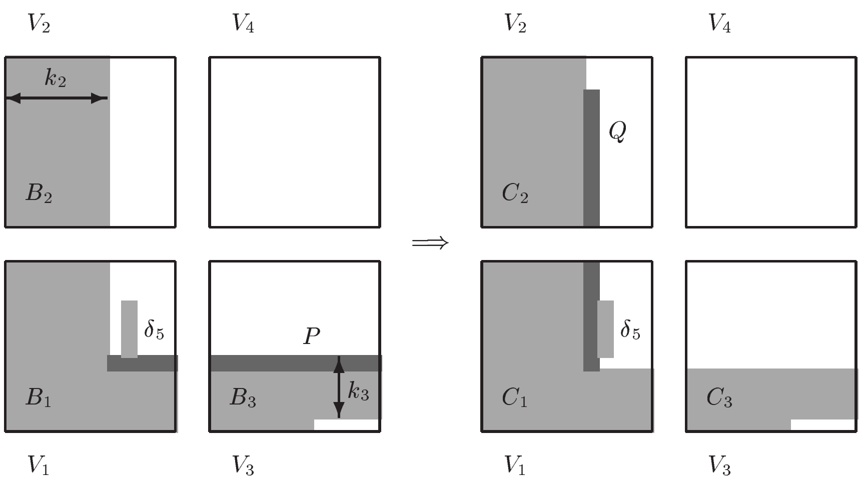

Fig. 4. Transformation of A into B and C in case 2b

to $(k_2 + 1, y)$. The resulting set B has the same number of inner edges. If $\delta_5 = 0$ we set $B = A$. The set B is shown in the left part of Fig. 4. Denote

$$P = \{(x, k_3 - \epsilon) \in A \;\big|\; k_2 \leq x < 2p\},$$

$$Q = \{(k_2, y) \notin A \mid k_3 - \epsilon \leq y < 2p + k_3 - k_2 - \epsilon\}$$

where $\epsilon = 0$ if $\delta_3 > 0$ and $\epsilon = 1$ if $\delta_3 = 0$. Now $|P| = |Q| = 2p - k_2$ and we replace P with Q (cf. Fig 4). We show that for the obtained set C one has $|I(C)| > |I(B)|$. Indeed, if $\delta_3 > 0$ then for $\Delta = |I(C)| - |I(B)|$ one has

$$\begin{aligned}
\Delta &= [|I(Q)| + |I(Q, A)|] - [|I(P)| + |I(P, A \setminus P)|] \\
&= \left[\binom{p}{2} - \binom{k_3}{2} + (2p - k_2)k_2 + \binom{p + k_3 - k_2}{2} + (p + k_3 - k_2)(p - i)\right] \\
&\quad - \left[(p - k_2)(k_3 + k_2) + \binom{p - k_2}{2} + p(p - i) + \binom{p}{2} + k_3\delta_3 + (p - \delta_3)(k_3 - 1)\right] \\
&= pk_2 + (k_3 - k_2)(p - i) - pk_3 - p + \delta_3 = i(k_2 - k_3) + (p - \delta_3) > 0.
\end{aligned}$$

Similarly, if $\delta_3 = 0$ then $\Delta = i(k_2 - k_3 + 1) > 0$.

Now assume $k_2 = p - 1$. In this case we construct the set C by replacing P with the set $R = \{(p-1, y) \notin A \mid k_3 + \delta_5 - \epsilon < y < p + k_3 + \delta_5 - \epsilon\}$ with ϵ defined above. For $\Delta = |I(C)| - |I(B)|$ one has

$$\begin{aligned}
\Delta &= [|I(R)| + |I(R, A)|] - [|I(P)| + |I(P, A \setminus P)|] \\
&= \binom{p - \delta_5 - k_3 - 1}{2} + \binom{k_3 + 1 + \delta_5}{2} \\
&\quad + p(p-1) + (p - \delta_5 - k_3 - 1)(k_3 + 1 + \delta_5) + (k_3 + 1 + \delta_5)(p - i) \\
&\quad - \left[\binom{p}{2} + k_3\delta_3 + (p - \delta_3)(k_3 - 1) + p(p - i)\right] \\
&= 2(p - \delta_5 - k_3 - 1)(k_3 + 1 + \delta_5 + i/2) + (p - \delta_3) + p\delta_5 \geq p - \delta_3 > 0.
\end{aligned}$$

In all cases we get a contradiction with the optimality of A.

Case 2c. We assume that $k_5 = 0$, $k_3 = 0$ and $\delta_3 > 0$. Similar arguments to the case 2b. can be used to show contradiction to the optimality in this case. We omit the proof because of space limitations.

Case 3. Assume $V_1 \subset A$. Then $\overline{A} = V \setminus A \subseteq V_2 \cup V_3 \cup V_4$, and we can apply the arguments of case 1 and case 2 to $\overline{A}$ and transform it to $\mathcal{F}^2(|\overline{A}|)$. Since, by Lemma 3, both A and $\overline{A}$ are optimal sets then the proof is completed. □

Due to the local-global principle for the lexicographic order [2], we have the following result:

Corollary 2. *If $1 \leq i \leq p - i$, then $|I(\mathcal{F}^n(|A|))| \geq |I(A)|$ for any $A \subset V^n$ and $n \geq 1$.*

The next theorem shows that the assumption concerning i is important for the existence of an optimal order.

Theorem 3. *There exist no optimal order for $H_{p,i} \times H_{p,i}$ if $p \geq 3$ and $p - i < i < p - 1$.*

We omit the proof of this Theorem due to space limitations. Note, that Theorem 3 is not true for $i = p-1$. In this case $H_{p,i}$ (resp. $H^2_{p,i}$) is isomorphic to $K_p \times K_2$ (resp. $K_p \times K_p \times K_2 \times K_2$) and the existence of an optimal order follows from Theorem 1. Furthermore, Theorem 3 is also not true for $i = p$. In this case $H_{p,i}$ (resp. $H^2_{p,i}$) is non-connected and is isomorphic to two disjoint copies of K_p (resp. to four disjoint copies of $K_p \times K_p$). In this case the lexicographic order is also optimal as it follows from [3].

4 Concluding remarks

Results similar to Theorems 2 and 3 can be obtained for the graph $B_{p,i} = (V_{B_{p,i}}, E_{B_{p,i}})$, where

$$\begin{aligned} V_{B_{p,i}} &= V' \cup V'', \quad V' = \{0, 2, \ldots, 2p-2\},\ V'' = \{1, 3, \ldots, 2p-1\}, \\ E_{B_{p,i}} &= \{(u,v) \bigm| u \in V',\ v \in V'', \quad u + v \neq (2p-j) \bmod 2p,\ j = 1, \ldots, i\}. \end{aligned}$$

It is easily shown that for any m the set $\{0, 1, \ldots, m-1\}$ is a solution to the EIP on $B_{p,i}$. With this numbering the lexicographic order on $V = V_{B_{p,i}} \times V_{B_{p,i}}$ is well-defined.

Theorem 4. *Let $i \leq p/2 - i$ and let $p \geq 4$ be even. Then $|I(\mathcal{F}^2(|A|))| \geq |I(A)|$ for any $A \subseteq V \times V$.*

The local-global principle [2] implies that a similar result holds for any cartesian power of $B_{p,i}$. We also have some preliminary results concerning the EIP for the cartesian powers of $K_{p,\ldots,p}$ (t parts) with up to two t-factors deleted. In these cases the lexicographic order is also optimal.

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