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# A DEGENERATE CENTRAL LIMIT THEOREM FOR SINGLE RESOURCE LOSS SYSTEMS

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ABSTRACT. Loss networks in heavy traffic under Kelly's scaling are analyzed. In the case of a single node and $R$ classes of calls, a degenerate diffusion approximation theorem around the corresponding fluid limit in a $R-1$ dimensional hyperplane is proved.

CONTENTS



## 1. INTRODUCTION

Loss networks can be described as sets of queues (or links) with limited capacity submitted to arrivals of jobs (calls). Each arrival (a call on a route) requires several links at the same time. A call is accepted only if there is enough room to accommodate it. These networks are described by Markov processes on a finite state space. They are reversible and their invariant measure is explicitly known.

This satisfactory picture of loss networks hides some important difficulties: First, because of the geometry of the state space, the invariant measure is expressed via a rather complicated combinatorial expression. It is then difficult to use it in practice to get estimations of simple characteristics of the network like the loss probability of a given call, or the load of the network. Second, the transient behavior of these networks is largely unknown. Kelly introduced a scaling parameter to get simple asymptotic expressions for the invariant measures. The rate of the arrivals is increased by some factor $N$ as well as the capacity of the links. When $N$ tends to infinity, it is then possible to get the asymptotic loss probabilities and some limit theorems for the invariant probabilities. (See Kelly [8]). Hunt and Kurtz [7] investigated the transient behavior of these networks. Despite these advances, many aspects of the behavior of these networks remain largely unknown, even in simple cases. The present paper establishes a non standard functional central limit result for a simple model in heavy traffic.

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The loss model studied in this paper has one link with the following characteristics.

- The capacity of the link (or a maximal number of circuits) is $\lfloor NC \rfloor$, where $N \in \mathbb{N}$ is the scaling parameter (introduced by Kelly [8]) and $C \in \mathbb{R}_+$ is the renormalized capacity of the link. ($\lfloor x \rfloor$ is the integer part of $x \in \mathbb{R}$).
- There are $R$ types of calls (or classes of customers).
- The arrival process of class $r$ customers is Poisson with parameter $N\lambda_r$. Their residence time in the network is exponential with parameter $\mu_r$.

The quantity $L_{N,r}(t)$ denotes the number of class $r$ customers in the system at time $t$. The free circuit process $(m_N(t))$ is defined by,

$$m_N(t) = \lfloor NC \rfloor - \sum_{r=1}^{R} L_{N,r}(t),$$

$m_N(t)$ is the number of empty places at time $t$.

It is known (Hunt and Kurtz [7]) that the sequence of processes

$$\left(\overline{L}_N(t)\right) = \left(\frac{L_{N,r}(t)}{N};\, r \le R\right)$$

converges to a deterministic limit, called *fluid limit*, for the Skorohod topology in the space of $\mathbb{R}^R$-valued right continuous functions with left limits (provided that the sequence of the initial conditions $\left(\overline{L}_N(0)\right)$ converges).

An important feature of these systems, remarked by Kelly [8] and shown by Hunt and Kurtz [7], is the averaging property. See Freidlin and Wentzel [4] and Kurtz [10]. The free circuit process moves much more rapidly than the process $(\overline{L}_N(t))$, as a consequence its local equilibrium determines the short term behavior of $(\overline{L}_N(t))$. It can be roughly described as follows: If at some time $t$,

$$\left(\overline{L}_N(t)\right) = x = (x_r;\, r \le R),$$

at the time scale $1/N$, the process $(m_N(s))$ is a $\mathbb{N} \cup \{+\infty\}$-valued Markov jump process $(\overline{m}_x(s))$ whose transitions are given by, for $m \in \mathbb{N} \cup \{+\infty\}$,

$$m \to \begin{cases} m+1 & \text{at rate } \mu_r x_r, \\ m-1 & \text{at rate } \lambda_r 1_{\{m \ge 1\}}. \end{cases} \tag{1}$$

Notice that the dynamic of the process is discontinuous when $m = 0$.

In this paper the perturbation around this fluid limit (which is a $R$-dimensional process) is investigated. When the initial point of the fluid limit is in the interior of the domain, it is easily shown that the components of the diffusion around the fluid limit are locally independent and related to Ornstein-Ühlenbeck processes (see Borovkov [2] and Kelly's survey [9]).

The interesting case is when the fluid limit is on the boundary of the domain and stays on it for a while (otherwise it is similar to the previous case). In this situation, a diffusion approximation picture turns out to be much more delicate to obtain because of the reflection on the boundary of the process at the normal scale, even for the simple model considered in this paper. It is interesting to note that one does not get a reflected diffusion process in the limit as one might think at first sight. The perturbation around this fluid limit is shown to be a degenerate $R{-}1$-dimensional diffusion related to an Ornstein-Ühlenbeck process. One of the main ingredients for the proof of the convergence is Proposition 2. Note that this is not the only one, a central limit theorem for the time spent on the boundary is also necessary, although somewhat hidden in our setting. An attempt to show an analogous result in the case where the calls require several links reveals that such a central limit theorem has to be shown. For the moment, we cannot prove such a result.

## 2. A degenerate central limit theorem

*Notations and assumptions.* It is assumed that, for $x \geq 0$, $\mathcal{N}_x$ is a Poisson process with parameter $x$, $\mathcal{N}_x(dy)$ is the infinitesimal increment of the associated counting measure and $\mathcal{N}_x(]0,t])$ denotes the number of points of this process in the interval $]0,t]$. With an upper index $(\mathcal{N}_x^k)$, it denotes an i.i.d. sequence of Poisson processes with parameter $x$. The renormalized capacity $C$ can be assumed to be 1 without any loss of generality.

Heavy traffic is assumed: that is, if $\rho_r = \lambda_r/\mu_r$, for $r \leq R$,

$$\sum_{r=1}^{R} \rho_r > 1. \tag{2}$$

This hypothesis implies that the saturation of the queue occurs with probability 1.

For $1 \leq r \leq R$ and $t \geq 0$, $L_{N,r}(t)$ is the number of class $r$ customers at time $t$ in the system. The initial conditions satisfy

$$\lim_{N \to +\infty} \overline{L}_{N,r}(0) = \overline{l}_r, \tag{3}$$

for $r = 1, \ldots, R$ with $(\overline{l}_r) \in \mathbb{R}_+^R$, $\overline{l}_1 + \cdots + \overline{l}_R \leq 1$.

*The equations of evolution.* It is easily seen that the process $(L_{N,r}(t))$ has the same distribution as the solution of the following stochastic differential equation,

$$\begin{aligned} L_{N,r}(t) = L_{N,r}(0) + \int_0^t 1_{\left\{\sum_{q=1}^R L_{N,q}(s-)<N\right\}} \mathcal{N}_{N\lambda_r}(ds) \\ - \sum_{k=1}^{+\infty} \int_0^t 1_{\{k \leq L_{N,r}(s-)\}} \mathcal{N}_{\mu_r}^k(ds). \end{aligned} \tag{4}$$

For $r = 1, \ldots, R$, the martingales associated to $(L_{N,r}(t))$ are defined by

$$\begin{aligned} M_{1,r}^N(t) &= \int_0^t 1_{\left\{\sum_{q=1}^R L_{N,q}(s-)<N\right\}} \left(\mathcal{N}_{N\lambda_r}(ds) - N\lambda_r\, ds\right), \\ M_{2,r}^N(t) &= -\sum_{k=1}^{+\infty} \int_0^t 1_{\{k \leq L_{N,r}(s-)\}} \left(\mathcal{N}_{\mu_r}^k(ds) - \mu_r\, ds\right); \end{aligned}$$

and their increasing processes (see Ethier and Kurtz [3]) are given by

$$\left\langle M_{1,r}^N \right\rangle(t) = N\lambda_r \int_0^t 1_{\left\{\sum_{q=1}^R L_{N,q}(s)<N\right\}}\, ds, \tag{5}$$

$$\left\langle M_{2,r}^N \right\rangle(t) = \mu_r \int_0^t L_{N,r}(s)\, ds. \tag{6}$$

Equation (4) can be written as

$$\begin{aligned}(7)\quad L_{N,r}(t) = L_{N,r}(0) + M_{1,r}^{N}(t) + M_{2,r}^{N}(t) \\ + N\lambda_r \int_0^t 1_{\left\{\sum_{q=1}^R L_{N,q}(s)<N\right\}}\,ds - \mu_r \int_0^t L_{N,r}(s)\,ds.\end{aligned}$$

2.1. **The fluid limits.** If $(X_N(t))$ is a sequence of processes on $\mathbb{R}_+$, one defines the renormalized sequence of processes of $(X_N(t))$ by $\overline{X}_N(t) = X_N(t)/N$, for $t \geq 0$.

It is well known (See Hunt and Kurtz [7]) that the process $(\overline{L}_{N,r}(t);\, r = 1,\dots,R)$ converges in the Skorohod topology to the fluid limit $(\overline{l}(t)) = (\overline{l}_r(t);\, r = 1,\dots,R)$ which is the unique solution of the ordinary differential equation

$$(8)\qquad \overline{l}_r'(t) = \begin{cases} \dfrac{\lambda_r}{\Lambda}\left(\langle \mu, \overline{l}(t)\rangle \wedge \Lambda\right) - \mu_r \overline{l}_r(t), & \text{if } \displaystyle\sum_1^R \overline{l}_k(t) = 1; \\ \lambda_r - \mu_r \overline{l}_r(t), & \text{if } \displaystyle\sum_1^R \overline{l}_k(t) < 1; \end{cases}$$

with $\overline{l}_r(0) = \overline{l}_r$; $\Lambda = \sum_1^R \lambda_r$, $\mu = (\mu_r)$, $a \wedge b = \min(a,b)$ and $\langle \cdot,\cdot \rangle$ is the usual scalar product in $\mathbb{R}^R$. The dynamical system $(\overline{l}_r(t);\, r = 1,\dots,R)$ lives in the region

$$\mathcal{D} = \{y \in \mathbb{R}_+^R : y_1 + \cdots + y_R \leq 1\}.$$

It is easily seen that Condition (2) implies that,

$$(9)\qquad \lim_{t\to+\infty} \left(\overline{l}_r(t)\right) \stackrel{\text{def.}}{=} \overline{l}(\infty) = \left(\frac{\rho_r}{\sum_1^R \rho_k}\right),$$

therefore $\overline{l}(\infty)$ is the stable point of $(\overline{l}_r(t);\, r = 1,\dots,R)$, it lies on the boundary of $\mathcal{D}$. See Zachary [13] for a general result on the stable points of fluid limits of loss networks.

Denote by

$$\Delta = \{y \in \mathcal{D} : y_1 + \cdots + y_R = 1\}, \quad \Delta^+ = \{y \in \Delta : \langle \mu, y\rangle < \Lambda\},$$

$\Delta^+$ is the set of points of the boundary of $\mathcal{D}$ at which the dynamical system is not pushed into the interior of $\mathcal{D}$: if $(\overline{l}_r) \in \Delta^+$, Equations (8) give

$$\sum_r \overline{l}_r'(0) = \sum_r \mu_r \overline{l}_r - \sum_r \mu_r \overline{l}_r = 0.$$

2.2. **A central limit theorem on $\Delta^+$.** From now on, it is assumed that Condition (3) is satisfied and

$$(10)\qquad \lim_{N\to+\infty} \sqrt{N}\left(\overline{L}_{N,r}(0) - \overline{l}_r\right) = v_r,$$

with $(v_r) \in \mathbb{R}^R$,

$$(11)\qquad \overline{l}(0) = \left(\overline{l}_r\right) \in \Delta^+ \quad \text{and} \quad v_1 + \cdots + v_R = 0,$$

i.e. the process $(L_{N,r}(t);\, r = 1,\dots,R)$ is very close to saturation at the origin when $N$ is sufficiently large.

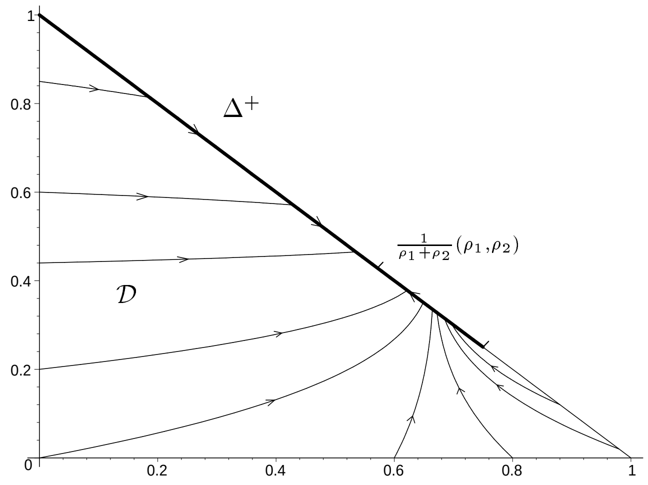


FIGURE 1. The fluid limits of a loss system with $R = 2$

Since $\bar{l}(0) \in \Delta^+$, it is easily seen that there exists some $T$ such that $\bar{l}(s) \in \Delta^+$ for all $s \in [0, T]$, i.e.

$$\sum_r \bar{l}_r(s) = 1 \text{ and } \sup_{0\leq s\leq T} \langle \mu, \bar{l}(s)\rangle < \Lambda. \tag{12}$$

Notice that in the case $R = 2$, one can take $T = +\infty$ (See Figure 1). This is not the case in general, for $R = 3$ for example, but there is a region of $\Delta^+$ containing $\bar{l}(\infty)$ (see Equation (9)) such that if $\bar{l}(0)$ is in this region, then $T$ can be taken as infinity (see Bean *et al.* [1]).

In the following the statements concerning the convergence in distribution of processes will refer to the Skorohod topology on the space of real valued functions on $[0, T[$ which are right continuous with left limits.

**Proposition 1.** *When $N$ goes to infinity the martingale*

$$\left(\left(\sqrt{N}\,\overline{M}^N_{1,r}(t);\, r = 1, \dots, R\right), \left(\sqrt{N}\,\overline{M}^N_{2,r}(t);\, r = 1, \dots, R\right)\right)$$

*converges in distribution to $(B_{1,r}(\gamma_{1,r}(t));\, 1 \leq r \leq R), B_{2,r}(\gamma_{2,r}(t));\, 1 \leq r \leq R))$, where $B_{1,r}$, $B_{2,r}$, $r = 1, \dots, R$ are independent standard Brownian motions on $\mathbb{R}$ and for $r = 1, \dots, R$,*

$$\gamma_{1,r}(t) = \lambda_r \int_0^t \frac{\langle \mu, \bar{l}(s)\rangle}{\Lambda}\, ds$$

$$\gamma_{2,r}(t) = \mu_r \int_0^t \bar{l}_r(s)\, ds.$$

*Proof.* The increasing processes of $\left(\sqrt{N}\,\overline{M}_{1,r}^N(t)\right)$ and $\left(\sqrt{N}\,\overline{M}_{2,r}^N(t)\right)$ are given by

$$\left\langle \sqrt{N}\,\overline{M}_{1,r}^N \right\rangle(t) = \lambda_r \int_0^t 1_{\left\{\sum_{q=1}^R \overline{L}_{N,q}(s)<1\right\}}\,ds, \tag{13}$$

$$\left\langle \sqrt{N}\,\overline{M}_{2,r}^N \right\rangle(t) = \mu_r \int_0^t \overline{L}_{N,r}(s)\,ds. \tag{14}$$

According to Hunt and Kurtz [7], when $N$ goes to infinity, the right hand side of Equation (13) converges in distribution to

$$\lambda_r \int_0^t \frac{\langle \mu, \overline{l}(s)\rangle}{\Lambda}\,ds,$$

and the right hand side of Equation (14) to

$$\mu_r \int_0^t \overline{l}_r(s)\,ds.$$

For $q$, $r\in\{1,\dots,R\}$, $q\neq r$, the processes

$$\left(\left\langle \overline{M}_{1,r}^N, \overline{M}_{2,r}^N\right\rangle(t)\right),\ \left(\left\langle \overline{M}_{1,q}^N, \overline{M}_{2,r}^N\right\rangle(t)\right),$$
$$\left(\left\langle \overline{M}_{1,q}^N, \overline{M}_{1,r}^N\right\rangle(t)\right) \text{ and } \left(\left\langle \overline{M}_{2,q}^N, \overline{M}_{2,r}^N\right\rangle(t)\right),$$

are identically 0 since the martingales $(M_{1,r}^N(t))$ and $(M_{2,r}^N(t))$ are stochastic integrals with respect to martingales associated with independent Poisson processes.

To conclude, a classical result is applied (see Theorem 1.4 page 339 of Ethier and Kurtz [3] for example). The proposition is therefore proved. □

For $t<T$, if $Z_N(t)$ denotes the empty space in the queue at time $t$, i.e.

$$Z_N(t) = N - (L_{N,1}(t)+\cdots+L_{N,R}(t)),$$

from Relation (7) one gets the identity

$$\overline{Z}_N(t) = \overline{Z}_N(0) - \sum_{r=1}^R \left(\overline{M}_{1,r}^N(t) + \overline{M}_{2,r}^N(t)\right) - \Lambda\int_0^t 1_{\left\{\overline{Z}_N(s)>0\right\}}\,ds + \int_0^t \left\langle \mu, \overline{L}_N(s)\right\rangle ds. \tag{15}$$

Condition (12) implies that

$$\left(\overline{Z}_N(t);\, t\leq T\right) \xrightarrow{\text{dist.}} 0,$$

as $N$ goes to infinity; the following proposition shows that Condition (11) entails a stronger statement.

**Proposition 2.** *The process*

$$\left(\sqrt{N}\,\overline{Z}_N(t);\, 0\leq t<T\right) = \left(\frac{N-L_{N,1}(t)-\cdots-L_{N,R}(t)}{\sqrt{N}};\, 0\leq t<T\right)$$

*converges in distribution to* 0 *as* $N$ *goes to infinity.*

*Proof.* Since the limit $(\overline{l}(t))$ of the process $(\overline{L}_{N,r}(t);\, r=1,\ldots,R)$ is continuous, for $\varepsilon>0$ and $\eta>0$ there exists $N_0\in\mathbb{N}$ such that for $N\geq N_0$,

$$\mathbb{P}\left(\sup_{0\leq t\leq T}\left|\left\langle \mu,\overline{L}_N(t)\right\rangle-\left\langle \mu,\overline{l}(t)\right\rangle\right|\geq\eta\right)\leq\varepsilon.$$

Condition (12) implies that there exists $\eta>0$ such that for $\varepsilon>0$, there is $N_0\in\mathbb{N}$ satisfying the following inequality: for $N\geq N_0$,

$$\text{if } \mathcal{H}_N=\left\{\sup_{0\leq t\leq T}\left\langle \mu,\overline{L}_N(t)\right\rangle\leq\Lambda-\eta\right\} \text{ then } \mathbb{P}\left(\mathcal{H}_N\right)\geq 1-\varepsilon. \tag{16}$$

The process $(Z_N(t))$ satisfies the stochastic integral equation

$$Z_N(t)=Z_N(0)-\sum_{r=1}^{R}\int_0^t 1_{\{Z_N(s-)>0\}}\mathcal{N}_{\lambda_r N}(ds)+\sum_{r=1}^{R}\sum_{k=1}^{+\infty}\int_0^t 1_{\{k\leq L_{N,r}(s-)\}}\mathcal{N}^k_{\mu_r}(ds),$$

it has the same distribution as the solution of the equation (with the same notation $Z_N$)

$$Z_N(t)=Z_N(0)-\sum_{r=1}^{R}\int_0^t 1_{\{Z_N(s-)>0\}}\mathcal{N}_{\lambda_r N}(ds) + \mathcal{N}_1\left(\left]0,N\int_0^t\left\langle \mu,\overline{L}_N(s)\right\rangle\,ds\right]\right). \tag{17}$$

The process $(Z_N(t))$ can be viewed as the number of customers of an $M(t)/M/1$ queue with the service rate $\Lambda N$ and $N\left\langle \mu,\overline{L}_N(t)\right\rangle$ as the instantaneous arrival rate at time $t$. The process $(\widetilde{Z}_N(t))$ is constructed with a coupling : $\widetilde{Z}_N(0)=Z_N(0)$ and

$$\widetilde{Z}_N(t)=\widetilde{Z}_N(0)-\sum_{r=1}^{R}\int_0^t 1_{\{\widetilde{Z}_N(s-)>0\}}\mathcal{N}_{\lambda_r N}(ds)+\mathcal{N}_1\left(]0,N(\Lambda-\eta)t]\right); \tag{18}$$

the Poisson processes $\mathcal{N}_{\lambda_r N}$, $r=1,\ldots,R$ and $\mathcal{N}_1$ in the above equation are the same as in Identity (17).

The Markov process $(X_N(t))=(\widetilde{Z}_N(t/N))$ has the same distribution as the number of customers in an $M/M/1$ queue with arrival rate $\Lambda-\eta$ and service rate $\Lambda$. If $H(K)$ denotes the hitting time of the level $K$ starting from 0 by this process, it is well known that if $\kappa=(\Lambda-\eta)/\Lambda$, the variable $\kappa^K H(K)$ converges in distribution to an exponentially distributed random variable with parameter $\eta^2/\Lambda$ as $K$ goes to infinity.

Equations (17) and (18) show that on the event $\mathcal{H}_N$ defined by Relation (16), the inequality $Z_N(t)\leq\widetilde{Z}_N(t)$ holds for all $t\leq T$, thus for $a>0$ and $\varepsilon>0$,

$$\begin{aligned}\mathbb{P}\left(\sup_{0\leq t\leq T}\sqrt{N}\,\overline{Z}_N(t)\geq a\right)&=\mathbb{P}\left(\sup_{0\leq t\leq T}Z_N(t)\geq a\sqrt{N}\right)\\&\leq\varepsilon+\mathbb{P}\left(\sup_{0\leq t\leq T}\widetilde{Z}_N(t)\geq a\sqrt{N}\right)\\&=\varepsilon+\mathbb{P}_{Z_N(0)}\left(\sup_{0\leq t\leq NT}X_N(t)\geq a\sqrt{N}\right),\end{aligned}$$

where $\mathbb{P}_b$ is the conditional probability $\mathbb{P}(\,\cdot\,|X(0)=b)$, hence using the strong Markov property of $(X_N(t))$, one gets

$$\mathbb{P}\left(\sup_{0\le t\le T}\sqrt{N}\,\overline{Z}_N(t)\ge a\right)\le \varepsilon+\mathbb{P}_{Z_N(0)}\left(H\left(\left\lfloor a\sqrt{N}\right\rfloor\right)\le NT\right)$$

$$\le \varepsilon+\mathbb{P}_{Z_N(0)}\left(H\left(\left\lfloor a\sqrt{N}\right\rfloor\right)\le H(0)\right)+\mathbb{P}_0\left(H\left(\left\lfloor a\sqrt{N}\right\rfloor\right)\le NT\right). \tag{19}$$

The classical ruin probability formula (or the fact that $\left(1/\kappa^{t\wedge H(0)\wedge H(\lfloor a\sqrt{N}\rfloor)}\right)$ is a martingale) gives that

$$\mathbb{P}_{Z_N(0)}\left(H\left(\left\lfloor a\sqrt{N}\right\rfloor\right)\le H\left(0\right)\right)=\frac{1/\kappa^{Z_N(0)}-1}{1/\kappa^{\lfloor a\sqrt{N}\rfloor}-1}$$

this term converges to 0 as $N$ goes to infinity since $Z_N(0)/\sqrt{N}\to 0$ (see Condition (11)).

The convergence in distribution of $(\kappa^K H\left(K\right))$ implies that

$$\mathbb{P}_0\left(H\left(\left\lfloor a\sqrt{N}\right\rfloor\right)\le NT\right)=\mathbb{P}_0\left(\kappa^{\lfloor a\sqrt{N}\rfloor}H\left(\lfloor a\sqrt{N}\rfloor\right)\le N\kappa^{\lfloor a\sqrt{N}\rfloor}T\right)$$

converges to 0 as $N$ gets large. Inequality (19) shows that the variable

$$\sup\left\{\sqrt{N}\,\overline{Z}_N(t):0\le t\le T\right\}$$

converges in distribution to 0 as $N$ goes to infinity. The proposition is proved. $\square$

Define

$$\widehat{L}_{N,r}(t)=\sqrt{N}(\overline{L}_{N,r}(t)-\overline{l}_r(t))=\frac{L_{N,r}(t)-N\overline{l}_r(t)}{\sqrt{N}}.$$

According to Relation (7) the renormalized process satisfies the following identity, for $r=1,\dots,R$ and $t\ge 0$,

$$\begin{aligned}\overline{L}_{N,r}(t)=\overline{L}_{N,r}(0)+\overline{M}^N_{1,r}(t)+\overline{M}^N_{2,r}(t)\\+\lambda_r\int_0^t 1_{\left\{\sum_{k=1}^R \overline{L}_{N,k}(s)<1\right\}}\,ds-\mu_r\int_0^t\overline{L}_{N,r}(s)\,ds.\end{aligned}\tag{20}$$

Equation (8) for the fluid limits and the assumptions on the initial state give that, for $t<T$,

$$\begin{aligned}\widehat{L}_{N,r}(t)=\widehat{L}_{N,r}(0)+\sqrt{N}\overline{M}^N_{1,r}(t)+\sqrt{N}\overline{M}^N_{2,r}(t)\\+\lambda_r\sqrt{N}\int_0^t 1_{\left\{\sum_{k=1}^R \overline{L}_{N,k}(s)<1\right\}}\,ds-\lambda_r\sqrt{N}\int_0^t\frac{\left\langle\mu,\overline{l}(s)\right\rangle}{\Lambda}\,ds-\mu_r\int_0^t\widehat{L}_{N,r}(s)\,ds,\end{aligned}\tag{21}$$

using Relation (15), one gets

$$\begin{aligned}\widehat{L}_{N,r}(t)=\widehat{L}_{N,r}(0)+M^N_r(t)\\-\frac{\lambda_r}{\Lambda}\sqrt{N}(\overline{Z}_N(t)-\overline{Z}_N(0))+\int_0^t\left(\frac{\lambda_r\left\langle\mu,\widehat{L}_N(s)\right\rangle}{\Lambda}-\mu_r\widehat{L}_{N,r}(s)\right)\,ds,\end{aligned}\tag{22}$$

where $(M^N(t)) = (M_r^N(t))$ is the martingale defined by

$$M_r^N(t) = \sqrt{N}\left(\overline{M}_{1,r}^N(t) + \overline{M}_{2,r}^N(t)\right) - \frac{\lambda_r}{\Lambda}\sum_1^R \sqrt{N}\left(\overline{M}_{1,k}^N(t) + \overline{M}_{2,k}^N(t)\right). \tag{23}$$

**Lemma 3.** *The sequence of martingales $(M^N(t))$ converges in distribution to a degenerate $R$-dimensional Gaussian process $(G(t))$ of rank $R-1$ such that*

$$G(t) = \Gamma \cdot \left(B_r\left(\gamma_{1,r}(t) + \gamma_{2,r}(t)\right);\, 1 \leq r \leq R\right), \tag{24}$$

*where $(\gamma_{1,\cdot}(t))$ and $(\gamma_{2,\cdot}(t))$ are given in Proposition 1, $(B_r(t))$ is a standard $R$-dimensional Brownian motion and $\Gamma$ is a $R \times R$ matrix of rank $R-1$, with*

$$\Gamma_{rr} = 1 - \frac{\lambda_r}{\Lambda}, \quad \Gamma_{rq} = -\frac{\lambda_r}{\Lambda},$$

*for $q \neq r \in \{1, \dots, R\}$. The range of $\Gamma$ is $\{y \in \mathbb{R}^R : y_1 + \cdots + y_R = 0\}$.*

*Proof.* The convergence is a straightforward application of Proposition 1 and the Continuous Mapping Theorem. □

**Proposition 4.** *Under the conditions*

$$\lim_{N\to+\infty} L_{N,r}(0)/N = \bar{l}_r, \quad \lim_{N\to+\infty} (L_{N,r}(0) - N\bar{l}_r)/\sqrt{N} = v_r,$$

*where the vectors $\bar{l} = (\bar{l}_r)$ and $v = (v_r)$ are such that $\bar{l}(0) = (\bar{l}_r) \in \Delta^+$ and $v_1 + \cdots + v_R = 0$, if $T > 0$ is such that the fluid limit $(\bar{l}_r(t))$ belongs to $\Delta^+$ for $0 \leq t \leq T$, then the vector $(\widehat{L}_{N,r}(t),\, r = 1, \dots, R;\, 0 \leq t \leq T)$ converges in distribution to the process $(\widehat{L}(t))$, solution of the following stochastic differential equation*

$$d\widehat{L}(t) = dG(t) + A \cdot \widehat{L}(t)\, dt, \tag{25}$$

*or, equivalently, defined by*

$$\widehat{L}(t) = \int_0^t e^{(t-s)A} dG(s) + e^{tA} v, \tag{26}$$

*where $(G(t))$ is the Gaussian process defined by Equation (24), $v = (v_r)$ and $A$ is the $R \times R$ matrix*

$$A_{rr} = \left(\frac{\lambda_r}{\Lambda} - 1\right)\mu_r \text{ and } A_{rq} = \frac{\lambda_r}{\Lambda}\mu_q,$$

*for $q \neq r$.*

*Proof.* For a process $X$ with values in $\mathbb{R}^R$ and $\delta > 0$, $w_X(\delta)$ denotes the modulus of continuity of $(X_r(t))$ on the interval $[0, T]$,

$$w_X(\delta) = \sup_{1\leq r\leq R} \sup\left\{|X_r(t) - X_r(s)| : 0 \leq s \leq t \leq T,\, t - s \leq \delta\right\}.$$

The sequence of process

$$\left(\widehat{L}_N(t)\right) = \left(\widehat{L}_{N,r}(t),\, r = 1, \dots, R;\, 0 \leq t \leq T\right)$$

is tight and any of its limiting points is continuous. To prove that assertion, it is sufficient to show that for any $\varepsilon, \eta > 0$ there exist $\delta$ and $N_0$ such that the inequality

$$\mathbb{P}\left(w_{\widehat{L}_N}(\delta) \geq \varepsilon\right) < \eta$$

holds for $N\geq N_0$ (see Ethier and Kurtz [3]).

Let $t\in[0,T]$, if

$$G_N(t)\stackrel{\text{def.}}{=}\sup_{1\leq r\leq R}\sup\left\{\left|\widehat{L}_{N,r}(s)\right|:0\leq s\leq t\right\},$$

then Equation (22) gives the relation

$$G_N(t)\leq K_N+2R\mu_*\int_0^t G_N(s)\,ds$$

with

$$K_N=\sup_{1\leq r\leq R}\sup_{0\leq s\leq T}\left(\left|\widehat{L}_{N,r}(0)\right|+\sqrt{N}\,\overline{Z}_N(s)+\left|M_r^N(s)\right|\right)\text{ and }\mu_*=\max_{1\leq r\leq R}\mu_r.$$

From Gronwall's Inequality one gets the bound

$$G_N(t)\leq K_N e^{2R\mu_* t}, \tag{27}$$

for $0\leq t\leq T$. Using again Equation (22), the following inequality is derived

$$\begin{aligned}\mathbb{P}\left(w_{\widehat{L}_N}(\delta)\geq\varepsilon\right)&\leq\mathbb{P}\left(w_{M^N}(\delta)\geq\varepsilon/3\right)\\&+\mathbb{P}\left(2\sup_{0\leq t\leq T}\left(\sqrt{N}\overline{Z}_N(t)\right)\geq\varepsilon/3\right)+\mathbb{P}\left(2R\mu_*G_N(T)\delta\geq\varepsilon/3\right).\end{aligned}\tag{28}$$

Since the sequence of processes $(M^N(t))$ converges and its limit is continuous (Lemma 3), there exist some $N_0$ and $\delta_0$ such that the relation

$$\mathbb{P}\left(w_{M^N}(\delta_0)\geq\varepsilon/3\right)\leq\eta/3$$

holds for $N\geq N_0$.

From Proposition 2 one gets that there exists $N_1$ so that if $N\geq N_1$, then

$$\mathbb{P}\left(2\sup_{0\leq t\leq T}\left(\sqrt{N}\overline{Z}_N(t)\right)\geq\varepsilon/3\right)\leq\eta/3.$$

Condition (10) and Propositions 1 and 2 show that the sequence of random variables $(K_N)$ is tight, thus there exists some constant $C_0>0$ such that for $N\geq 1$ the inequality $\mathbb{P}(K_N\geq C_0)\leq\eta/3$ holds.

Now if $N_2=N_0\vee N_1$ and

$$\delta=\delta_0\wedge\frac{\varepsilon}{6C_0R\mu_*}e^{-2R\mu_*T},$$

then for $N\geq N_2$, Inequality (28) gives the relation

$$\mathbb{P}\left(w_{\widehat{L}_N}(\delta)\geq\varepsilon\right)\leq\eta.$$

Thus the sequence of processes $(\widehat{L}_N(t))$ is tight and any of its limits is continuous.

If $(\widehat{L}(t))$ is a limit of $(\widehat{L}_N(t))$, using Relation (22), one gets that $(\widehat{L}(t))$ satisfies the stochastic integral equation

$$\widehat{L}_r(t)=v_r+G_r(t)+\int_0^t\sum_{q=1}^R A_{rq}\widehat{L}_q(s)\,ds,\quad r=1,\ldots,R, \tag{29}$$

Equation (25) is therefore satisfied. The range of $A$ is

$$\mathcal{S}=\{y\in\mathbb{R}^R:y_1+\cdots+y_R=0\},$$

since $(G(t))$ is a Gaussian process in $\mathcal{S}$, the above equation can be rewritten as a classical non degenerate linear stochastic differential equation in $\mathcal{S}$. In particular, there is a unique strong solution (see Ethier and Kurtz [3] or Rogers and Williams [11]). It is easy to check that the process $(\widehat{L}(t))$ defined by Equation (26) verifies Relation (29). The proposition is proved. □

2.3. **Stationary behavior.** We conclude with some remarks on the invariant distribution of $(L_N(t))$. If $L_N(\infty)$ denotes some random variable whose distribution is stationary with respect to $(L_N(t))$ then

$$L_N(\infty) \stackrel{\text{dist.}}{=} \left((Z_{N,1},\dots,Z_{N,R}) \middle| Z_{N,1}+\cdots+Z_{N,R}\leq N\right),$$

where $Z_{N,1},\dots,Z_{N,R}$ are independent Poisson random variables whose parameters are respectively $N\rho_1,\dots,N\rho_R$. It is elementary to prove that, as $N$ tends to infinity, $L_N(\infty)/N$ converges in distribution to $\bar{l}(\infty)$, the stable point of the fluid limits, defined by Equation (9). By using the central limit theorem and the fact that the empty space in the link converges, as $N$ tends to infinity, toward a geometrically distributed random variable (see Kelly [8]), it is easily seen that $(L_N(\infty)-N\bar{l}(\infty))/\sqrt{N}$ converges in distribution to a Gaussian random variable $\widehat{L}(\infty)$ with zero mean and covariance matrix $K=(K_{ij};\,1\leq i,j\leq R)$ defined by

$$K_{ii}=\frac{\rho_i}{\rho}\left(1-\frac{\rho_i}{\rho}\right) \quad \text{and} \quad K_{ij}=-\frac{\rho_i\rho_j}{\rho^2}, \tag{30}$$

for $1\leq i,j\leq R$, $i\neq j$ and $\rho=\rho_1+\cdots+\rho_R$. For central limit theorems of the stationary distributions of loss networks, see Kelly [8], Hunt and Kelly [6], Hunt [5] and Whitt [12].

The matrix $K$ is clearly singular ($K\cdot 1=0$), with rank $R-1$. Since the matrix $K$ is symmetrical and non-negative, it is easily seen that there exists some $R\times(R-1)$ matrix $H$ such that $K=H\cdot H^t$ ($H^t$ is the adjoint of $H$) and the range of $H$ is the hyperplane $\mathcal{H}=\{x: x_1+\cdots+x_R=0\}$. The variable $\widehat{L}(\infty)$ can then be expressed as $H\cdot\mathcal{W}$ where $\mathcal{W}$ is a standard $(R-1)$-dimensional Gaussian vector.

**Proposition 5.** *The diagram*

$$\begin{array}{ccc} \left(\frac{1}{\sqrt{N}}\left(L_N(t)-N\bar{l}(t)\right)\right) & \xrightarrow{t\to+\infty} & \left(\frac{1}{\sqrt{N}}\left(L_N(\infty)-N\bar{l}(\infty)\right)\right) \\ {\scriptstyle N\to+\infty}\big\downarrow & & {\scriptstyle N\to+\infty}\big\downarrow \\ \left(\widehat{L}(t)\right) & \longrightarrow & \widehat{L}(\infty). \end{array} \tag{31}$$

*commutes, i.e. the invariant distribution of $(\widehat{L}(t))$ is a Gaussian distribution with zero mean and covariance matrix $K$ defined by Equation* (30).

*Proof.* It is not difficult to see that one can assume that the fluid limit is already at equilibrium, i.e. $\bar{l}(0)=\bar{l}(\infty)$, so that $\bar{l}(t)=\bar{l}(\infty)$ for any $t\geq 0$. In this situation, the stochastic differential Equation (25) becomes

$$d\widehat{L}(t)=\Gamma\cdot D\cdot dB(t)+A\cdot\widehat{L}(t)\,dt,$$

where $D$ is the $R\times R$ diagonal matrix such that $D_{ii}=\sqrt{2\lambda_i/\rho}$, $\Gamma$ and $A$ are the matrices defined in Lemma 3 and Proposition 4, and $(B(t))$ is a standard $R$-dimensional Brownian motion. (For $i=1,2$ and $r\in\{1,\dots,R\}$, the quantity $\gamma_{i,r}(t)$ defined in Proposition 1 is $\lambda_r t/\rho$).

The infinitesimal generator of the Markov process $(\widehat{L}(t))$ is given by the second order differential operator

$$\Omega(f)(x)=\frac{1}{2}\sum_{1\leq i,j\leq R}\Sigma_{ij}\frac{\partial^2 f}{\partial x_i\partial x_j}(x)+\langle A\cdot x,\nabla f(x)\rangle\,, \tag{32}$$

for a twice differentiable function $f$ on $\mathbb{R}^R$ with compact support, the diffusion coefficient $\Sigma=(\Sigma_{ij};\,1\leq i,j\leq R)=\Gamma\cdot D^2\cdot\Gamma^t$ is given by

$$\Sigma_{ii}=\frac{2\lambda_i}{\rho}\left(1-\frac{\lambda_i}{\Lambda}\right)\quad\text{ and }\quad\Sigma_{ij}=-\frac{2}{\rho}\frac{\lambda_i\lambda_j}{\Lambda},$$

for $1\leq i\neq j\leq R$. To prove that the distribution of $\widehat{L}(\infty)$ is indeed the invariant distribution of $(\widehat{L}(t))$, it is sufficient to prove the identity $\mathbb{E}(\Omega(f)(\widehat{L}(\infty))=0$ holds for any twice differentiable function $f$ with compact support, or that

$$\mathbb{E}\left(\Omega(f)(H\cdot\mathcal{W})\right)=0 \tag{33}$$

holds. (See Ethier and Kurtz [3] page 290). For $w\in\mathbb{R}^{R-1}$, define $\phi(w)=f\,(H\cdot w)$, the following identities are easily checked

$$\left(\frac{\partial\phi}{\partial w_i}(w);\,1\leq i\leq R-1\right)=\left(\frac{\partial f}{\partial x_\ell}(H\cdot w);\,1\leq\ell\leq R\right)\cdot H, \tag{34}$$

$$\begin{aligned}\left(\frac{\partial^2\phi}{\partial w_i\partial w_j}(w);\,1\leq i,j\leq R\ -1\right)&\\ &=H^t\cdot\left(\frac{\partial^2 f}{\partial x_\ell\partial x_m}(H\cdot w);\,1\leq\ell,m\leq R\right)\cdot H.\end{aligned} \tag{35}$$

Since the range of matrix $A$ is also hyperplane $\mathcal{H}$, there exists some square matrix $\Psi$ of dimension $R-1$ such that $A\cdot H=H\cdot\Psi$. Trite calculations show the relation

$$\Sigma=-2A\cdot K=-2A\cdot H\cdot H^t=-2H\cdot\Psi\cdot H^t.$$

which implies, together with relation $A\cdot H=H\cdot\Psi$ (again) and Equations (34) and (35), that

$$\Omega(f)(H\cdot w)=-\sum_{1\leq k,\ell\leq R-1}\Psi_{k\ell}\frac{\partial^2 f}{\partial w_k\partial w_\ell}(w)+\langle\nabla\phi(w),\psi\cdot w\rangle\,.$$

Identity (33) to check becomes then

$$\begin{aligned}\sum_{1\leq k,l\leq R-1}\Psi_{kl}\int_{\mathbb{R}^{R-1}}\left(\frac{\partial^2\phi}{\partial w_k\partial w_l}(w)-w_l\frac{\partial\phi}{\partial w_k}(w)\right)&\\ \times\exp\left(-\frac{1}{2}\sum_{i=1}^{R-1}w_i^2\right)\,dw_1\ldots dw_{R-1}&=0,\end{aligned}$$

which is trivial to verify. The proposition is proved. □

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