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Abstract

In this work we answer a conjecture posed by Wegner in 1977 [28] for the vertex colouring problem on the *square* of any planar graph, proving a stronger result for this problem. This problem is motivated by the *Frequency Assignment Problem* in radio networks.

Let $G = (V, E)$ be a graph. A *vertex colouring* of G is a function $c : V \to \{1, 2, ..., k\}$, such that $c(v) \neq c(u)$ if $(v, u) \in E$. The *chromatic number*, $\chi(G)$, of G is the smallest integer k for which there exists a vertex colouring of G. The *clique number*, $\omega(G)$, of G is the size of its maximum clique. The *square*, G^2, of G is a graph defined on the same vertex set as G which has an edge between any pair of vertices of distance at most two in G.

In this work we define the class of *almost perfect* graphs. A graph G is *almost perfect* iff G and each of its induced subgraphs have the property that their chromatic number are bounded above by their clique number plus a constant, independent of the graph size.

In this work we prove the following for any planar graph G with maximum degree Δ.

1. The clique number of G^2 is at most $\lfloor 3\Delta/2 \rfloor + 1$, if $\Delta \geq 6$. Otherwise, $\omega(G^2) \leq \Delta + 4$.
2. The chromatic number of G^2 is at most $\omega(G^2) + 3$, if $\omega(G^2) \leq 8$. Otherwise $\chi(G^2) \leq \omega(G^2) + 1$. We also prove that G^2 is an *almost perfect graph.* A consequence of our main result is that:
3. Any efficient approximation algorithm for the chromatic number of G^2 is also an efficient approximation algorithm for its clique number and vice-versa.

1 Introduction

The *Frequency Assignment Problem* (FAP) in radio networks is the problem of assigning frequencies (nonnegative integers) to transmitters exploiting frequency reuse while keeping signal interference to acceptable levels. Meaning that interfering transmitters get frequencies whose separation is not in a set of disallowed separations. This problem was first formulated as a variation of the *vertex coloring problem* on graphs by Hale [17], called *T-colouring.* Here, the transmitters are represented by vertices, the potential interference is represented by edges, the frequencies by colours, while T is the set of disallowed separations. The resulting graph called *interference graph* (IG). Some recent works about the T-colouring and the frequency assignment problem are [7, 8, 13, 14, 17, 22, 23, 26].

Roberts [15] proposed a variation of FAP, where "close" transmitters must receive different frequencies and "very close" transmitters must receive frequencies at least two apart. Griggs and Yeh [15] and Yeh [27] considered the following generalization. Given a real number $d > 0$ try to find an $L_d(2,1)_-$*labeling* for a given graph G. This labeling is an assignment f of nonnegative real numbers (labels) to the vertices of G such that, if x and y are two adjacent vertices, then $|f(x) - f(y)| \geq 2d$. If the distance between x and y (i.e. the length of the shortest path joining x and y) is two, then $|f(x) - f(y)| \geq d$. The difference between the largest and the smallest frequency used is called *span* and is denoted by $\lambda(G, d)$. The number of discrete frequencies used is called *order* and is denoted by $\lambda_{order}(G, d)$. For $d = 1$ the problem is called *radiocolouring problem.*

The *square*, G^2, of a graph G is a graph denoted on the same vertex set as G which has an edge between any pair of vertices if their distance in G is at most two. The graph G is called a *square root* of G^2. A *vertex colouring* of a graph G is a function $c : V \to \{1, 2, ..., k\}$, such that $c(v) \neq c(u)$ if $(v, u) \in E$. For simplicity reasons in the rest of the paper we use the term *colouring* when we refer to a *vertex colouring.* The *chromatic number* of G is the smallest positive integer k for which there exists a *colouring* of G and is denoted by $\chi(G)$. The *clique number* of G is the size of its maximum clique and is denoted by $\omega(G)$.

For some special classes of graphs, exact values or tight bounds are known and can be computed efficiently (i.e. in polynomial time) for the radiocolouring problem. These include paths, cycles, wheels and complete k-partite graphs [15], trees [6, 15], cographs [6], k-almost trees [10], cacti, unicycles and bicycles [20], and grids, hexagonal grids and cellular grids [3]. On the other hand, for other types of graphs it was proved that the problem is NP-complete on them, so approximation algorithms and upper bounds were provided. These graphs are chordal and unit interval [25, 4], interval graphs [6], bipartite graphs [27, 4], outerplanar and planar graphs [20, 12], graphs of diameter two [11, 16], general graphs [15], split graph [4], and cographs [6].

Fotakis et al. proved in [11] that $\lambda_{order}(G) = \chi(G^2)$. They also proved in that work that $\chi(G^2) \leq \lambda(G) \leq 2\chi(G^2)$. In this work we study $\lambda_{order}(G)$ for the case where G is a planar graph, thus the problem of *colouring the square of a planar graph*. The first reference on this problem was by Wegner [28] in 1977, who posed the following conjecture.

Wegner's conjecture: Let G be a planar graph.

(i) $\chi(G^2) \leq \Delta(G) + 5$, for $4 \leq \Delta(G) < 8$, **(ii)** $\chi(G^2) \leq \lfloor 3\Delta(G)/2 \rfloor + 1$, for $\Delta(G) \geq 8$,

where $\Delta(G)$ denotes the maximum degree of G. When no confusion arises we omit G.

Ramanathan and Lloynd showed in [21] that the colouring problem on the square of any planar graphs is an NP-Complete problem. There are three efficient approximation algorithm for this problem with asymptotic ratios: 1.8 for $\Delta \geq 749$ due to [1], and 2 for any Δ due to [18, 12].

Observe that for each vertex v of a graph G, v and its neighbours form a clique in G^2. Thus, the minimum number of colours needed for colouring G^2 is at least $\Delta(G) + 1$. From the best of our knowledge, all known colouring algorithms for squares of graphs use a number of colours that is a function of Δ [1, 18, 12, 15, 4]. In this work we first express $\chi(G^2)$ as a function of another parameter, the clique number of G^2, when G is a planar graph.

A graph G belongs to the class of *perfect* graphs iff G and each of its induced subgraphs, have the property that their chromatic number is equal to their clique number. Now, we define an extension of this class of graphs.

Definition 1.1 *A graph G belongs to the class of the* almost perfect *graphs iff there exists a constant c, independent of the size of G, such that G and each of its induced subgraphs, have the property that their chromatic number is at most equal to their clique number plus c.*

In summary our work derives the following results for any planar graph G of maximum degree Δ.

1. The size of the maximum clique in G^2 is at most $\lfloor 3\Delta/2 \rfloor + 1$, if $\Delta \geq 6$. Otherwise $\omega(G^2) \leq \Delta + 4$.

2. The chromatic number of G^2 is at most $\omega(G^2) + 3$, if $\omega(G^2) \leq 8$. Otherwise $\chi(G^2)) \leq \omega(G^2) + 1$. Note that this result closes the above Wegner's conjecture. Since $\omega(G^2) \leq \lfloor 3/2\Delta \rfloor + 1$ for $\Delta \geq 8$. We also prove that G^2 is an almost perfect graph, and we conclude that,
3. Any efficient approximation algorithm for $\chi(G^2)$ is also an efficient approximation algorithm for $\omega(G^2)$ and vice-versa, meaning that they have the same asymptotic approximation ratio. Thus,
4. $\omega(G^2)$ can be efficiently approximated too within a ratio of 2 (or 1.8 if $\Delta(G) \geq 749$). This is the first result for the clique number on the square of planar graphs.

The rest of the paper is organized as follows: Let G be a planar graph. In Section 2 we bound from above the clique number of G^2. In Section 3 we present some definitions and properties on colour critical graphs which are used form this work. In Section 4 we provide a tight upper bound of the chromatic number of G^2 as a linear function of $\omega(G^2)$. Finally, in Section 5 we conclude our work and we present some points for future work.

2 The Maximum Clique in the Square of any Planar Graph

In this Section we bound from above the size of the maximum clique in the square of any planar graph G, in terms of its maximum degree Δ.

Theorem 2.1 *Let $G = (V, E)$ be a planar graph of maximum degree Δ. The size of the maximum clique in the square of G ($\omega(G^2)$) is bounded from above as follows:*

(1.) $\omega(G^2) \leq \lfloor 3/2\Delta \rfloor + 1$, *if* $\Delta \geq 6$, **(2.)** $\omega(G^2) \leq \Delta + 4$, *if* $\Delta \leq 5$.

Sketch of proof: To prove the first assertion of this Theorem we propose a procedure which, for a given Δ, produces an extremal planar graph S. The square of S is a clique of size the maximum between the sizes of the maximum cliques in the squares of all planar graphs of maximum degree Δ. We denote the maximum clique in G^2 by C and we define $S = (V_S, E_S)$ to be the induced subgraph of G on the vertices of C.

We first provide a constructive procedure which defines the structure of S. Then we show that the structure of S is unique and maximum (i.e. with the maximum number of vertices).

By the definition of S it can be easily seen that any two vertices in S, let them $\{a, b\}$, are at distance at most two apart each other in S. Thus, $\{a, b\}$ satisfies at least one of the following conditions (we call them *distance constrains*): i)The vertices a and b have at least one common neighbour in S, ii) $(a, b) \in E_S$.

Having in mind the distance constrains we designed the following constructive procedure:
Procedure RootOfMaxClique
Input: An integer Δ.
Output: A planar graph S whose square is a clique of size equal to the maximum among the sizes of all maximum cliques in the squares of all planar graphs with maximum degree Δ.

1. Initially S is an empty graph.
2. Add three vertices, x, y, z, in S. For each of them, add Δ' new vertices in S and connect them to their respective vertex.
   The neighbours of each of the vertices x, y, z are called its *initial neighbours*.
3. Join the vertices x, y, z in such a way that they form a K_3 (see figure 2f).
4. Join each initial neighbour of y with x, analogously each initial neighbour of x with z and each initial neighbour of z with y.
5. Optionally, join any pair of vertices keeping S planar.
6. Remove the edge (x, z) and join vertices x and z with a new vertex, call it w (see figure 2g).

End

The following facts are necessary for the correctness of the Theorem and are proved in the Appendix:

1. Only three vertices can be concerned (have initial neighbours) in Step 1 of the procedure.
2. The vertices x, y, z of Step 1 have the same number of initial neighbours.
3. S is maximum and unique.
4. The clique in the square of any planar graph of maximum degree Δ other than S has size at most equal to the size of S.

■

3 A survey of the theory of colour critical graphs

In this section we present a set of definitions and known propositions for colour critical graphs that are used later for the proof of the basic Theorem of this work. Most of the material can be found in an excellent book of Béla Bollobás [5] and in the [24]. The colour critical graphs are coloured with fewer colours when any of their edges is removed. Formally,

Definition 3.1 *If G is a (colour) critical graph with $\chi(G) = k$ (G is also called k_critical), then $\chi(G - e) = k - 1$ for each edge e of G (the graph $G - e$ obtained from G removing the edge e from it).*

Definition 3.2 *The complete join $G = G_1 \nabla G_2$ of the disjoint graphs G_1, G_2 is defined by* $V(G) = V(G_1) \bigcup V(G_2)$, $E(G) = E(G_1) \bigcup E(G_2) \bigcup \{\{x, y\} | x \in V(G_1), y \in V(G_2)\}$.

Definition 3.3 *The partial join of two disjoint graphs G_1 and G_2 is a graph G defined as follows:* $V(G) = V(G_1) \bigcup V(G_2)$, $E(G) = E(G_1) \bigcup E(G_2) \bigcup \{\textit{some edges } \{x, y\} \textit{ where } x \in V(G_1) \textit{ and } y \in V(G_2)\}$.

Definition 3.4 *The class of the k_critical graphs is denoted by Cri(k). Analogously, the class of the k_chromatic graphs (i.e. have chromatic number equal to k) is denoted by Chr(k). Finally, the class of the k_colourable graphs (i.e. have chromatic number bounded above by k) is denoted by Col(k).*

Proposition 3.1 *(Dirac's construction). Let $G_1 \in Chr(k_1)$ and $G_2 \in Chr(k_2)$ be disjoint graphs and $G = G_1 \nabla G_2$. Then, (a) $G \in Chr(k_1 + k_2)$, (b) $G \in Cri(k_1 + k_2)$ iff $G_1 \in Cri(k_1)$ and $G_2 \in Cri(k_2)$.*

Proposition 3.2 *For $k > 1$, $Chr(k) = Col(k) - Col(k - 1)$, $Cri(k) \subseteq Chr(k) \subseteq Col(k) \subseteq Col(k + 1)$*

Proposition 3.3 *Every k_chromatic graph G contains a non empty set of k_critical subgraph. These subgraphs of G are its minimal subgraphs which have chromatic number equal to the chromatic number of the whole graph.*

Proposition 3.4 *If $G = (V, E)$ is a k_critical graph and Q is a clique where $Q \neq V$ then $G - Q$ is connected. In particular, this means that G is connected and has no cut vertex*

Proposition 3.5 *Every k_critical graph, G, that is not isomorphic to the K_k, has a cut set of size at least two and there exist at least two vertices in this cut set that are not neighbours. So, G has a cycle.*

Proposition 3.6 *If $G \in Cri(k)$ then $\delta(G) \geq k - 1$.*

4 The Chromatic Number of the Square of any Planar Graph

In this Section we prove our main result which is that the chromatic number of the square of any planar graph is at most equal to its clique number plus three. We prove the above assertion based in the theory of *colour critical graphs*.

Let G be a graph with chromatic number k. It is well-known (see Prop. 3.3) that there exists at least one subgraph of G, let it be Cr, which is k_*critical*. We exploit this property in order to produce a $k+1$-chromatic graph, G', from G. If the graphs G and G' have to satisfy some restrictions (i.e belong to a class of graph), then the chromatic number of G can be increased only by proper insertion of new edges and/or vertices in it producing the graph G', in such a way that a subgraph of G' to be a critical graph with chromatic number greater than $\chi(G)$, while the restrictions of G are still hold on G'.

Our aim in this Section is to provide a tight upper bound for the difference between the chromatic number and the clique number of the square of any planar graph. We succeed it via a constructive procedure, we call it M, which generates planar graphs whose squares have chromatic number as far away as possible from their clique number. It is easily understood that the following restrictions have to be satisfied by any graph G obtained from M: 1) does not have neither K_5 nor $K_{3,3}$ as induced subgraphs, and 2) $\Delta(G) \leq \omega(G^2) - 1$ (since $\omega(G^2) \geq \Delta(G)$); we call this restriction *degree constraint*).

M implements in a high level the following design idea:
Given the size of the maximum clique, let it be q, in the square of a planar graph G, grow G properly increasing as much as possible the chromatic number of G^2 keeping (a) $\omega(G^2) = q$, (b) $\Delta(G) \leq q - 1$ and (c) G planar.

In the following we prove that at least one of the restrictions (a), (b) and (c) is violated while trying to increase the difference between the chromatic number and the clique number of the critical subgraphs of the square of any planar graph by more than three. We now state our main Theorem, which is proved in Section 4.5.

Theorem 4.1 *Let G be a planar graph. Then the chromatic number of G^2, $\chi(G^2)$, is bounded above by a linear function of the clique number of G^2, $\omega(G^2)$. More precisely:*

1. $\chi(G^2) \leq \omega(G^2) + 3$, *if* $\omega(G^2) \leq 8$.
2. $\chi(G^2) \leq \omega(G^2) + 1$, *otherwise.*

4.1 Preliminaries

We first take a look at the main theory (theory of colour critical graphs) we use to derive a tight upper bound for the difference between the chromatic number and the clique number of any critical subgraph of the square of any planar graph. In the literature there are several models for the construction of critical graphs (a very good survey paper on this subject is [24]). Let Cr be a k_-critical graph obtained by such a model. The design of any of these models guarantees the existence of graphs in Cr, denote them graphs by SCr, which force a vertex of Cr (not always the same) to be adjacent to a set of vertices with $k-1$ different colours in every *minimum colouring* of Cr (i.e. a colouring using $\chi(Cr)$ colours). These constructive models differentiate in the SCr. Unfortunately, we do not known of an efficient model, which generates every critical graph. This fat beyond others shows the hardness of the colouring problem. Every k_-critical graph is k_-constructible using the Dirac-Hajós constructive model ([24]). However this model is not of polynomial time.

Our aim in this work is to bound the difference between the chromatic number and the clique number of any critical subgraph of the square of any planar graph. We manage to derive our result showing how can grow up a critical subgraph of such graphs, in order to produce an other critical graph with greater difference beyond the above numbers. For any of these graphs, we begin with its maximum clique, which is a critical graph and always exists. its growth achieved using the following complementary approaches for the growing of a critical graph Cr, resulting another critical graph, adding to it vertices.

A1. *Complete join* Cr with the another critical $Cr2$, resulting the critical graph Cr'.

A2. *Partial join* Cr with the non critical graph $Chr2$, resulting the critical graph Cr''.

It is known by Proposition 3.1 that the result of the complete join of two graphs is a critical graph iff the initial graphs are critical graphs. Otherwise, if at least one of them is not critical, then the resulting graph is critical iff these graphs are joined via partial join.

In this work is also considered, the other hand where Cr grow up via:

A3. *Edges insertion.* Add edges Cr, resulting the critical graph Cr''.

We proceed by considering two cases on the structure of any planar graph, which has as subgraph in its square, a graph obtained via: *Case 1* complete join, and *Case 2* partial join. Due to lack of space the proofs of almost all of the following Lemmas are presented in the Appendix.

4.2 Complete Join

Firstly, we study the Case 1. The following Lemma is crucial for the proof of our main Theorem.

Lemma 4.2 *Let $G1$ and $G2$ be two graphs such that $\chi(G1) > \omega(G1)$ and $\chi(G2) > \omega(G2)$. Let $G' = G1 \nabla G2$. There not exist any planar graph G which has G' as induced subgraph in its square.*

In order to prove of this Lemma we first prove that, any graph G with $\chi(G) = \omega(G) + 1$ has a cycle of size at least five. In the sequel when we refer to cycles we assume cycles without chords.

4.3 Partial Join

Secondly, we work out the case of the $(k + c)_{-}$critical graphs which are obtained from k_{-}critical graphs via partial join with other graphs and belong in the square of any planar graph, where c is a constant.

Lemma 4.3 *Let G be a critical graph. Then every one of its vertices can take the maximum colour (according to at least one optimal colouring of G).*

Note 4.4 *Let G' be a $(k + 1)_{-}$critical graph obtained by a k_{-}critical graph G via partial join. This implies that added vertices in G in such a way that: for each of them, let v', there exists a vertex in G which corresponds to it, let it be v. The vertex v' is joined to appropriate vertices of the augmented G, in order to have the same colour as v according to any minimum colouring of G. In this case the vertex of maximum colour (i.e $k+1$) can be joined either to the vertex v or v'. As a generalization the vertex v' can be replaced by a subset, let it be S', of the added vertices of G. The vertices of S' will receive the colours of a set of vertices, let it be S, of the initial G. In this case, is possible the neighbours of any vertex of S' to not its fix colour. The colour of such a vertice is determined after the choice of the colour of some of its neighbours in S' (this will be clear in the following and with the use of examples).*

4.4 The Constructive procedure M

As we said in the beginning of this Section, the constructive procedure M designed in order to produce planar graphs, whose squares have chromatic number as far away as from their clique number. Let P be any planar graph obtained by M. By Lemma 4.2 we conclude that $\chi(P^2) - \omega(P^2) \leq 1$ if the $\chi(P^2)_{-}$critical subgraph of P^2 comes via complete join. By Proposition 3.1 we also conclude that, if a graph G is the result of a join between two other graphs, then its chromatic number can't be greater than the sum of the chromatic numbers of these graphs. Having in mind the above statements the need for the following Lemma comes directly.

Lemma 4.5 *Let $G1$ and $G2$ be two graphs. Let $G1'$ and $G2'$ be two graphs obtained by graphs $G1$, $G2$, respectively, having the same chromatic number as their corresponding graphs. The graph G' obtained by any join (complete or partial) of graphs $G1'$ and $G2'$ has a chromatic number at most equal to the chromatic number of G, where $G = G_1 \nabla G_2$.*

Having in mind the above Lemmas and Notes about $(k+1)_{-}$critical graphs we designed the following procedure, which grows the K_k (which is a k_{-}critical graph) in order to be a $(k+2)_{-}$colourable graph. For a given graph and a given colouring of it We define the *allowed colours* of a vertex to be the colours not assigned to its neighbours in this colouring.

Procedure 1 ()

Input: an integer k

Output: a graph G

1. $G \cong K_k$ and find an optimal colouring of G.
2. Insert a set N of new vertices in G so that in any minimum colouring of G the followings hold:

   (i.) The colour of each new vertex, v, to be fixed (i.e there is only one colour available for it from the set of used colours) and among the colours used for K_k. If the colour of v is not fixed by the colours of its neighbours in K_k, then it becomes fixed from the colours of its neighbours in N. This is achieved by choosing randomly the colours for some of the vertices of N, so that for any such vertex, all of its allowed colours are assigned to its neighbours.

   (ii.) The vertices of G (which are the vertices of K_k and the vertices of N) to be able to be split in x different sets, let them $X = \{X_1, X_2, ...X_x\}$ as follows: each X_i has vertices with at least k different colours, for $1 \leq i \leq x$.
3. Let $IS = \{is_1, is2, ...\}$ be a maximum independent set which has the following properties: **(i.)** it has at most one vertex from each X_i, **(ii.)** its vertices have different colours, **(iii.)** its vertices are ordered based on the indexes of the sets of X where they belong to. We define $MaxIS$ to be the size of IS.
4. Insert a new clique, let it be $CL = \{cl_1, cl_2, ..., cl_{MaxIS+1}\}$, in G as follows:

   (i.) Join the vertex cl_i with each vertex of X_j, where j is the index of the set of X where is_i belongs to, for $1 \leq i \leq MaxIS$. Denote by Y_i the subgraph of G induced by the vertices of the set X_j plus the vertex cl_i.

   (ii.) Join $cl_{MaxIS+1}$ with each vertex of X_j, where j is the index of a set of X, where none of the vertices of IS belongs to X_j.

End

Lemma 4.6 *Procedure 1 (k) produces $(k+2)_{-}$colourable graphs.*

Proof: Colour the vertices of the independent set IS with colour $k+1$. This implies that the vertices $\{cl_1, cl_2, cl_{MaxIS}\}$ of the clique CL can receive the $MaxIS$ different colours of the vertices of IS. Finally, the vertex $cl_{MaxIS+1}$ receives colour $k+2$. ■

Remark. Let G be a graph obtained by Procedure 1. If $|CL| < k$ and there is a set of vertices with $k+1$ different colours in G, which do not form a clique of size k and at least one of them, let it be v, isn't joined to a vertex with colour $k+2$, then the clique CL (of Procedure 1) can be increased, without increase of the chromatic and clique numbers of G, by inserting a new vertex in it and by joining it to the above set of vertices.

The chromatic number of G remains the same, because the currently last vertex of CL can be coloured with the colour of vertex v, and v can be coloured with colour $k+2$.

In a high level description our constructive Procedure M generates planar graphs such that the vertices of their squares are able to be split in sets with the same properties as set X of Procedure 1. Moreover, the vertices of every one of these sets are at distance at most two from a new vertex, which plays the role of one of the vertices of the clique CL.

In the following we prove that the value of the variable x of the above procedure is at most 7 in the square of any planar graph. In terms of Lemma 4.5 if the graph G' is a planar graph and $G2$ is the K_x, then it can be proved that the difference between the chromatic number and the clique number of the square of any planar graph is at most 7. Furthermore we improve this bound to 3 by proving that there is always a re-colouring of the square of any planar graph G that uses at most $\omega(G^2)+3$ colours.

Before the formal presentation of the constructive procedure M, we prove the following important Lemmas.

Lemma 4.7 *Let P be any planar square root of the K_k, where $k \geq 4$. Let u, v, w be three vertices on the exterior face of P. Then at most two vertices, we call them x and y, can be added in this face so that each of them to be adjacent to each of the vertices u, v, w.*

Remark. Let an arbitrary embedding of P where the vertices u, v, w appear in the exterior face of this graph in this order. Each vertex, on the exterior face of P, which appear between the vertices u and w can also be accessible only from the vertices x and/or y and no other vertex of the exterior face of P.

Remark. Let I be the induced subgraph of P on the vertices u, v, w, x. If I isn't the star with root one of the vertices u, v, w, then the vertices u, v, w can be joined to only one vertex of the exterior face of P.

Lemma 4.8 *Let P be a planar graph and let $q = \omega(P^2) \geq 7$. If there are at least seven vertices of the maximum clique of P^2 in the same face of P, then the subgraph of P which forms this clique in P^2, call it S, is the $K_{(1,(q-1))}$ plus optional edges between consecutive vertices of S (given any arbitrary planar embedding of P).*

Now we are ready to present our constructive Procedure M.

Procedure M

Input: an integer q.

Output: a planar graph P.

1. Set P to be a planar graph, such that K_q is a subgraph of P^2. We call the graph P of this Step as *initial* P.
2. Insert new vertices in P keeping it planar with $\Delta(P) \leq q$, so that its vertices to be able to be grouped in sets, let them $X'_1, X'_2, ..., X'_x$. Each of these sets, i.e. X'_i for $1 \leq i \leq x$, satisfies the following conditions:

(a) X_i' has vertices of q different colours in every minimum colouring of P^2.

(b) Each vertex of X_i' is at distance at most two from a new vertex (which is added in P according to Step 3).

(c) At least two of the vertices of X_i' are not adjacent.

3. Insert in P a set of new vertices which form a clique in P^2, in such a way that each of them has as neighbours at distance at most two each vertex of one of the sets $X_1', X_2', ..., X_x'$.

End M

Lemma 4.9 *Let P be a planar graph obtained by $M(q)$. Let P' be the initial P and let Em to be an arbitrary planar embedding of P' in the plane. Then the value of the variable x is at most equal to the number of vertices of P which are in the exterior face of Em.*

Proof: Let c be a vertex inserted at Step 3 of M. Assume by contradiction that, the set X_i', where $1 \leq i \leq x$, of its distance at most two neighbours has vertices which are joined to only one vertex of P'. Let p be a vertex on the exterior face of Em and B' its neighbours. If c is joined only to vertices of B', then its distance at most two neighbours are coloured in P^2 with less than q colours. This is because none of its neighbours of distance at most two, uses the colour of vertex p. Hence, vertex c have to be at distance at most one from other vertices of P' (i.e joined to vertex p) or at most two from neighbours of vertices of P' not joined to p, so these vertices could be coloured with the colour of p.

Until now we proved that any two vertices of the exterior face of Em, let them p, p', and their neighbours can define a set X_i'. If these vertices are not consecutive in Em, then using Lemma 4.7 we conclude that all other vertices (and their neighbours) of P' located between p and p' can't be accessible from other vertices inserted at Step 3, except in the case where the vertices of Step 3 are nested and each of them has two neighbours. If these vertices are nested then each other vertex inserted at Step 3 have to be joined to at least one of the two common neighbours of the nested vertices. Because of the degree constraint, the nested vertices can not be more than $q - 2$. In this case we could have a set of more than one cliques in P^2 of size q inserted at Step 3. However, the full proof of Theorem 4.10 shows that this is not possible, because if P' isn't the star graph then it has a cycle (chords are allowed). So, each vertex of K_q has at least two neighbours in P'. This implies that each vertex inserted at Step 3 should have three neighbours of distance one in P' or neighbours from the set of neighbours of three of the vertices of P'. So the vertices inserted at Step 3 can not be nested.

We remark here that the graph P' may have vertices which do not belong to K_q, call them auxiliary vertices. Let p be a vertex of K_q which has neighbours auxiliary vertices. Then, even if it has $q - 1$ degree, its neighbours out of P' can't have fixed colours, in any minimum colouring of the graph resulted by graph P minus the vertices inserted at Step 3. If this is the case, then each vertex inserted at Step 3 have to be at distance at most two from as many vertices so that to have neighbours of q different colours. Thus for these cases, the difference between the chromatic number and the clique number of the square of the graph obtained is more difficult to be increased, than for the cases where the graph P' is a square root of K_k. So, we examine only such cases.

Now, the Lemma can be easily proved using above arguments. ■

The next Theorem is the base of the proof of our main Theorem 4.1. There we prove that if the critical subgraph of maximum chromatic number in the square of any planar graph P is obtained via a partial join of other graphs and the difference between its chromatic number and clique number is the maximum possible, then P can be obtained by M.

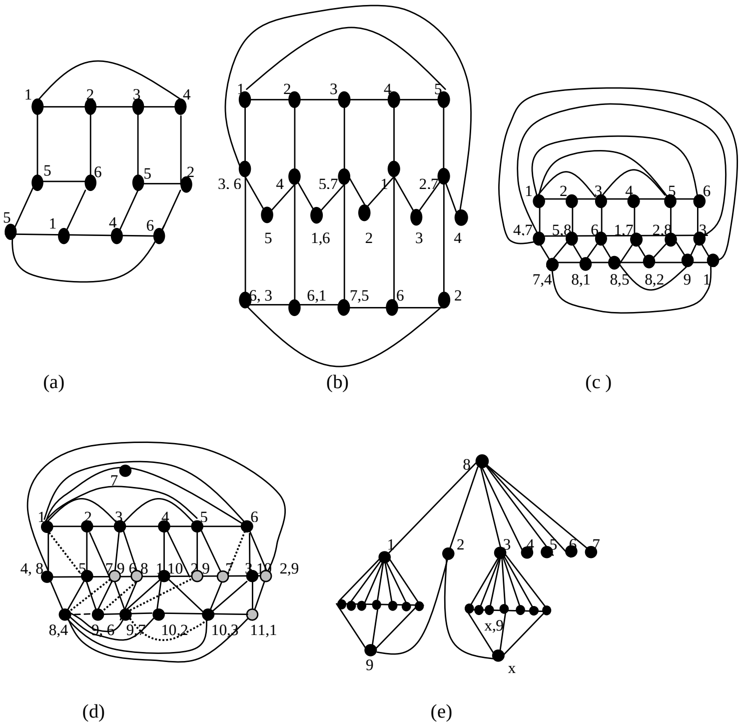

Figure 1: The graphs depicted in this figure are representable planar graphs obtained by Procedure M for the cases where $q = 4, 5, 6, 7, 8$. The integer near to each of the vertices of this graph represents its colour in the square of these graphs.

Theorem 4.10 *Let P denote any planar graph obtained by procedure $M(q)$. Then* $\chi(P^2) \leq \omega(P^2) + 3$.

Sketch of proof: Due to lack of space we present here a sketch of the proof of this Theorem. The full proof is presented in the Appendix. By the definition of Procedure M, the maximum clique in P^2 is of size q. So, we examine all the possible values of q ($q \geq 2$). We manage to handle all these cases based on Lemmas 4.8 and 4.9.

By the first of these two Lemmas, we get that if $q \geq 8$ and more than 7 vertices of the K_q are in the exterior face of the initial P, then graph initial P must be the star $K_{1,q-1}$. Because of this and by Lemma 4.9, the vertices inserted at Step 3 can't be more than 7. This implies that, even if they form a clique in P^2, the difference between the chromatic number and the clique number of this graph is bounded by 7. Actually, this difference can't be more than one in this case because of the graph depicted in Figure 1f. The integer near to each of the vertices of the graphs defected in this Figure a represent its colour in the square of these graphs.

On the other hand, if the initial P isn't the star graph, as above, then at most 6 vertices can be in the exterior face of the initial P. In this case, Figure 1d shows representable what happens. The basic observation is that there are vertices in the inside faces of the initial P. So, we have to add properly vertices in P in such a way that some of them to have the same colours as the colours of these vertices.

At the very opposite, if q is very small, i.e. 2 or 3, by the degree constraint it can by easily seen that the Theorem holds. If $q = 3$ then the worst case is when the graph P is an odd cycle.

Now consider all the possible cases left, i.e. $q = 4,\ 5,\ 6,\ 7$. The graphs of Figures 1a,b, and c are representable for these cases. Observe that for these cases, the degree constraint plays again the main role. ■

4.5 Proof of Theorem 4.1

Sketch of proof: The proof of this Theorem is split in the following three parts, according to the approaches given in the beginning of this Section, of how a critical graph Cr can be grown, in order to give another critical graph Cr' with larger chromatic number. The possible ways to get the Cr' are the followings:

From **A1**: by the complete join of Cr with the critical graph $Cr'' = Cr' - Cr$.

From **A2**: by a partial join of Cr with the graph $Cr'' = Cr' - Cr$.

From **A3**: by insertion of edges in Cr.

For the first case, it is obvious that the difference between the chromatic number and the clique number of Cr' is increased analogously to the corresponding difference on each of the graphs Cr and Cr''. The least meaningful value of this difference is one. So, by Lemma 4.2, we get that there is no planar graph that has in its square a sugbgraph which is the result of the complete joint of graphs Cr and Cr'.

The second case includes the planar graphs obtained by procedure M, concerning the partial join of two graphs. By Lemma 4.3, we get that for any critical graph, there is a colouring that gives the maximum colour to any vertex of this graph. So, for the cases where $q \geq 8$, if there are vertices in the inside faces of the initial P, then it is possible some of them to receive the maximum colour. As we can see in Figure 1d the vertices of the inside faces of the initial P must have two common neighbours, from the vertices of K_q stated in the exterior face of initial P. Thus, by the degree constraint is proved the chromatic number can't be far away from the clique number. In order to increase the difference between the chromatic number and the clique number in the square of any planar graph, we assume one fixed colouring of the initial P, and we produce as much as possible sets with vertices of q different colours in P^2. Observe that, the planarity and the degree constrains do not allow the production of many such sets. As it is proved in Lemma 4.9. Then, we join each of them with one of the vertices of a clique. So, the maximum possible difference is obtained when $x - MaxIS$ is maximized. Observe that $MaxIS$ can be smaller than two and this case was proved in Theorem 4.1 about the graphs obtained by procedure M.

The last case comes from the observation that even if we make the graphs obtained by M maximal planar graphs then the results still hold.

■

4.6 For a planar G, G^2 is almost perfect

Theorem 4.11 *The square of any planar G is an almost perfect graph.*

Proof: This is straight forward by Theorem 4.1. ■

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APPENDIX

5 Proof of Theorem 2.1

Theorem 2.1 *Let $G = (V, E)$ be a planar graph of maximum degree Δ. The size of the maximum clique in the square of G ($\omega(G^2)$) is bounded from above as follows:*

(1.) $\omega(G^2) \leq \lfloor 3/2\Delta \rfloor + 1$, *if* $\Delta \geq 6$, **(2.)** $\omega(G^2) \leq \Delta + 4$, *if* $\Delta \leq 5$.

Proof: At this point we explain the way of reasoning on the design of the constructive procedure *RootOfMaxClique* and the unavoidable constrains which lead to the restricted number of vertices in the Step 1 of this procedure and equality on the size of the sets of their initial neighbours.

Let RC be a graph that contains the maximum clique, C, in RC^2. It is interesting to note here that the graph RC may be contains vertices which are not members of the C.

We consider the structure of the graph RC on two vertices of C, let them x, y, plus their neighbours. We assume that the degree of the vertices x,y in RC is greater than six. Let CN be the set of the common neighbours of the vertices x, y in RC, and let N_x and N_y the sets of the rest of the neighbours of the vertices x and y, respectively. We proceed with a detailed study of the structure of the graph RC based on the sizes of the above sets.

Initially we assume that the sets N_x and N_y are empty. In this case the size of the set $CN > 5$. This implies that any other vertex of the C, let it be w, has to be joined to at least one of the vertices x and y. This is because the middle common neighbour of the vertices x and y is impossible to be at distance at most two from w via any other join of w with the vertices x, y and their neighbours. It is easily seen that similar arguments prove that w has to be joined in the same way as explained above, even in the case that it belongs in one of the faces which are formed from two consecutive vertices of the CN. At this point we have to consider what happens with the vertices which belong to the sets N_x and N_y. Every one of these vertices must be at distance at most two from each of the others.

If $|N_x| \leq 2$ and $|N_y| \leq 2$ then their exist two cases. In the first one each of the vertices of any of the above sets has to be joined to at least one of the vertices of the other set. It is interesting to mention here that at most one of the above vertices can be joined to two of the vertices of the other set. So, the vertices of the second set have to be neighbours. In the second case each of the vertices of the first set is joined to exactly one of the vertices of the second set. This implies that the vertices of at least one of the above sets are neighbours.

Figure 2b and c show what happens if $|N_x| = 3$ and $|N_y| = 3$. For the case where the above sets have size 4 see figures 2d and e. The basic observation on the above figures is that one of the vertices of the second set, which is called z, is adjacent to each vertex of the sets N_x and N_y. It is easily seen that the generalization for bigger sets N_x and N_y is straight forward.

Our aim in the sequel is to prove the maximality and the uniqueness of the planar graph obtained be the procedure *RootOfMaxClique*. We make frequently use of the symbols defined in this procedure. Initially, we assume that S has a set of vertices, we denote it by I, and every one of them has a distinct set of initial neighbours (not necessarily of the same size). Our aim is to modify the current graph S in such a way that the resulting S becomes a graph of diameter (i.e. the maximum distance between any pair of vertices of a given graph) 2. Thus, we join the vertices of I in a clique, thus making the distance between any pair of vertices of I to be one and the distance of each vertex of I from the initial

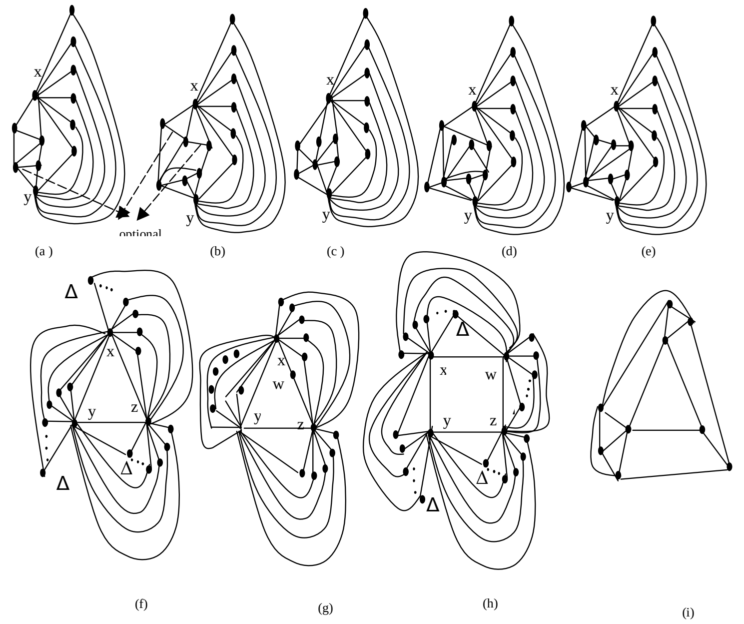

Figure 2: a) The vertices x and y have at least five common neighbours. b) and c) concern the case where $|Nx| = |Ny| = 3$. d) and e) concern the case where $|Nx| = |Ny| = 4$ f) The basic structure of the graph S which forms the maximum possible clique in the squares of planar graphs. g) The graph $S5$ for $\Delta = 12$. h) This figure represents a graph which can't be extended in such a way that its square to a clique. i) The graph S which corresponds to the tight upper bound for the maximum clique in the square of a planar graph of maximum degree 5.

neighbours of the rest of the vertices of I to be two. The above modification on S implies that $|I| \leq 4$, because it is a planar graph.

Until now we manage to decrease the diameter of S to three. There vertices of distance three are the initial neighbours of each vertex of I from the initial neighbours of the rest of the vertices of this set. The distance between these vertices can be decreased with uniform insertion of the least number of edges (the increase of the degree of each vertex of I to be almost the same), keeping the planarity of S (see Step 3 of the procedure). Observe that Step 3 can't be extended in order to concern and the fourth vertex of I, because of the planarity constraint of P. So $|I| \leq 3$ and for simplicity we call the vertices of I x, y, z.

It is easily understood that the degrees of the vertices x, y, z will define the size of the maximum clique in S^2. Therefore, the maximum possible clique in S^2 comes when the sizes of the sets of the initial neighbours of each of the vertices x, y, z are the same.

Step 4 in the above constructive procedure $RootOfMaxClique$ allows the insertion of edges between the initial neighbours of any one of the vertices x, y, z, optionally. This step is necessary because we will prove that the structure of S is unique. Finally, the last step is used for increasing the size of C by one without an increase of the maximum degree of S.

According to the constructive procedure *RootOfMaxClique* the graph S obtained from it has degree $\Delta = 2\Delta' + 2$ and its square is a clique of size $3\Delta' + 4 = 3\Delta/2 + 1$. figure 2f shows the graph S when $\Delta = 12$.

The uniqueness of the structure of S comes from the next two points:

1. The uniform assignment of the neighbours/edges to the vertices x, y, z. Any other case would leads to a smaller maximum clique in S^2. This is because the maximum degree of S would be

increased without an analogous growth of the maximum clique in S^2 given that the size of S remains the same.

2. The distance constrains on its vertices. These do not allow S's structure to looks like the graph of figure 2h. It is easily seen from this figure that it is impossible to have enough edges between the vertices of this graph in such a way that the maximum distance between each pair of these vertices to be at most two, while keeping S planar.

Finally we have to show that S has the maximum size. This holds because it is not possible to insert a set X of new vertices in S (while its distance and planarity constrains hold) without increasing its maximum degree at least by $\lceil 2|X|/3 \rceil$. The new vertices can be inserted in one of the following three positions: i) inside the cycle x, y, z, w ii) outside of the cycle x, y, z, w iii) at least one of these vertices inside and at least one outside of the above cycle.

Let a be one of the new vertices. Then, a must be at distance at most two from each vertex of S, according to the definition of S. If a is only joined with one of the vertices x, y, z, then it would violated the distance constrains in S. Hence, the insertion of any new vertex in S, implies the insertion of at least two edges in it. In order to keep the final degree of S as small as possible we should join every one of the vertices x, y, z to at least $\lfloor 2|X|/3 \rfloor$ of the new vertices (and to at most $\lceil 2|X|/3 \rceil$ of them). It is obvious that $\lfloor 3/2(\Delta + \lceil 2|X|/3 \rceil) \rfloor \geq \lfloor 3/2\Delta \rfloor + |X|$. It is easily seen that there is no difference between the ways the new vertices are placed with respect to the cycle x, y, z, w (because each new vertex is joined to two of the vertices x, y, z 'uniformly'. This means that there is no need to have edges between the new vertices).

Note that, the distance constrains in S will not hold if a is joined to two of the initial neighbours of any one of the vertices x, y, z. It is easily checked that it is impossible for the vertex a to be joined to more than two of these vertices (initial neighbours) keeping S planar.

For the proof of the second part of this Theorem, where considered planar graphs with maximum degree Δ less than five are considered, we examine the following cases according to the value of Δ.

(i) $\Delta = 1$ that is trivial.

(ii) $\Delta = 2$. In this case are included beyond others the cycles (without chords) and the trees. As it is easily checked the maximum cycle such that its square is a clique is the cycle on five vertices (i.e. C_5). Hence, $\omega(C_5^2) = 5 \leq \Delta + 4 = 6$.

(iii) $\Delta = 3$, $\Delta = 4$ and $\Delta = 5$. The first two of these cases follows immediately from the third. Therefore, to prove it is sufficient to show that $\omega(S^2) \leq \Delta + 4$, when $\Delta = 5$. Figure 2i shows a planar graph with maximum degree 5, which has a clique of size 9 in its square. In this case we add the optional edges between the initial neighbours of every one of the vertices x, y, z (according to Step 4 of the above construction). After careful study of this case we join properly the initial neighbours of the above vertices in such a way that the distance constraints in S are satisfied.

It is easily checked that, this is not possible when $\Delta \geq 6$. This is because is not possible to form a clique of size more than three in the initial neighbours of each of the vertices x, y, z.

Observe that when Δ is an odd number then the maximum clique has size at most equal to the size of the maximum clique for $\Delta - 1$ plus one. ■

6 Proof of Lemma 6.1

Lemma 6.1 *Let G be a graph such that, $\chi(G) = \omega(G) + 1$. Then, there is a cycle of size at least five in G.*

Proof: If G is not a critical graph then we prove the Lemma for its critical subgraph that has the same chromatic number as G. So, w.l.o.g we assume that G is a critical graph. The difference between the chromatic number and the clique number of G implies that there is a vertex (in each minimum colouring of G not always the same), we call it w, which is adjacent to $\chi(G)-1$ vertices coloured by different colours with at least two of them not adjacent, we call them u and v. We remark here that in the sequel of this proof we use the facts that G doesn't have a *cut vertex* and it has a cycle, according to Propositions 3.4 and 3.5.

The difference between the colours of the vertices u and v implies that there is a neighbour of u (v), let it be u' (v'), which is not neighbour of v (u). Otherwise, if the vertices u and v have the same neighbours, then they could receive the same colour (as in the case where the neighbours of one of them is a subset of the neighbours of the other). In this case we concern an other vertex w having the desired property, or we choose an other pair of non adjacent neighbours of w which have different colours. We are sure that always exist such a vertex. Otherwise, the graph G would be possible to be coloured with fewer colours. Since G is a critical graph, then the vertex w isn't joined to the vertices u' and v'. Otherwise the removal of the edge $\{u, w\}$ would not influence the chromatic number of G. In the place of the above edge could be the edges $\{v, w\}$, $\{v', w\}$ or $\{u', w\}$. In order to exist a cycle of size five as induced subgraph of G the edge $\{u', v'\}$ must belongs to G. Otherwise either the vertices $\{u', v'\}$ have at least one common neighbour, or we recursively apply the same way of reasoning given that the vertices u' and v' are non adjacent vertices with distinct colours. We can guarantee that this recursive procedure ends with a cycle of size greater than five, because G does not have a cut vertex (i.e. the vertex w). ■

7 Proof of Lemma 4.2

Lemma 4.2 *Let $G1$ and $G2$ be two graphs such that $\chi(G1) > \omega(G1)$ and $\chi(G2) > \omega(G2)$. Let $G' = G1 \nabla G2$. There not exist any planar graph G which has G' as induced subgraph in its square.*

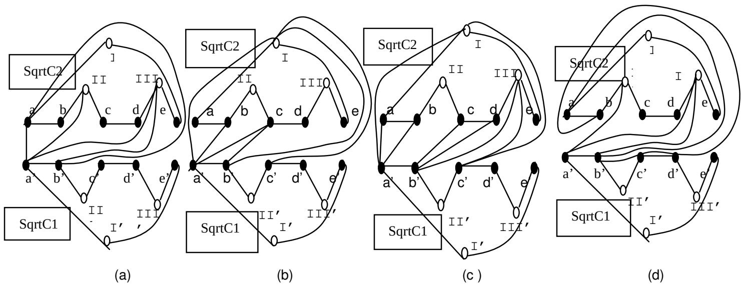

Figure 3: The graphs of this Figure concern the case of Complete Join of two graphs of chromatic number greater than their graphs, in their squares.

Proof: **Note 7.1** *Each square root of a cycle C_x, call it SQ, for $x \geq 5$ has as subgraph a cycle (chords are allowed) of size at least 8.*

Proof: Let SQ be a square root of C_x. Observe that for each three consecutive vertices, let them a, b, c, of the cycle C_x at most one of the edges, joining them, can belongs to SQ (otherwise the vertices a, b, c would form a triangle in SQ^2). W.l.o.g. we assume that there is the edge $\{a, b\}$ in SQ. Then the vertices b and c have to become distance two neighbours. So, there is a new vertex (auxiliary vertex), we call it I, joined to the vertices b and c.

Some one can obviously conclude that applying the above way any square root of C_x is constructed. It is clear that are allowed edges (chords) only between auxiliary vertices. ■

According to Lemma 6.1 there is at least one cycle of size at least five in each of the graphs $G1$ and $G2$. We call them $C1$ and $C2$, respectively. The existence of these cycles in G^2 implies the existence of bigger cycles in G. According to Lemma 7.1 each square root of the cycles $C1$ and $C2$ has size at least eight. We prove in details the case where both of the graphs $G1$ and $G2$ have cycles ($C1$, $C2$) of size five and their square roots, let them $SqrtC1$ and $SqrtC2$ respectively, have size 8 and are unchord cycles.

Let a' be one of the vertices of the cycle $C1$. figure 3 depicts three graphs according to which a' is at distance at most two from each of the vertices of the cycle $C2$. We note here that the vertices of the cycles $C1$ and $C2$ are represented by the black vertices in the graphs of this figure. Studding these graphs we conclude that no more than three of the vertices of the cycle $C1$ can be at distance at most two from each vertex of the cycle $C2$ while the graph G is keeping planar and the above cycles without chords in G^2. Before the detailed explanation of the way of construction the graphs of figure 3 we note that we omit unnecessary edges because of the above reason. The existence of such edges may be exclude the existence of edges which would enable more vertices of the cycle $C1$ to be at most distance two neighbours from more vertices of the cycle $C2$.

Now lets go back to the graphs of this figure. More precisely, in figure 3 a the vertex a' is joined to the vertex a of $C2$. In order to be at distance at most two from each vertex of $C2$, it could be joined to the auxiliary vertices (are the white vertices in the figure) and/or to at most one of the vertices b and e and no one else (because the cycles $C1$ and $C2$ have to remain unchordal in the square). Let the first case where a' is joined to the auxiliary vertices II and III (its join to the vertex I is unnecessary). In this case at least the vertex b' can't be at distance at most two from the vertex c. Due to the vertex b' has the neighbours shown in the above figure the rest of the vertices of cycle $C1$ can't be distance two neighbours of the vertex c. In the graph of figure 3b the vertex a' is joined to the vertices b, c and I. This forces the vertex b' to be joined to one of the vertices c, d or III and to the vertex I or a, inhibiting the vertices c', d', e' to be at distance at most two from the vertices b, c and d. figure 3c consider the case where a' is joined to the vertices a and b of $C2$. In order to be at distance two from each other vertex of this cycle with the most efficient way, (increasing the possibility the other vertices of $C1$ to be at distance at most two from each vertex of $C2$) a' is joined to the auxiliary vertex I and the vertex b' to the vertices c, d and III. In this case the vertices c', d', e' can't be at distance at most two from the vertices b and c.

The graphs of figure 3 cover with the best way all the cases given that the vertex a' is at distance at most two from each vertex of $C2$ (the other cases are covered by the figure 3d where the first neighbour of vertex a' is neither the vertex a nor the vertex b. This implies that the vertex b have to have as neighbour the vertex a or b. Hence, the rest of the vertices of $C1$ can't be distance two neighbours of the vertices c and d). So, is clear that Lemma follows.

The rest of the cases were the cycles $C1$ and/or $C2$ have size different of five and where their square roots aren't the C_8 are proved analogously. ■

8 Proof of Lemma 4.3

Lemma 4.3 *Let G be a critical graph. Then every one of its vertices can take the maximum colour (according to at least one optimal colouring of G).*

Proof: The proof of this Lemma is based on Proposition 3.1. We assume that the assertion is false and we prove it by contradiction. Thus, according to our assumption there are vertices which can't be coloured with the maximum colour according to any minimum colouring of G. We call this set of vertices S. This implies that it is possible to be added a new vertex, we call it v, in G in such a way that the resulting graph to be a $(\chi(G)+1)_{-}$critical graph via partial join. According to such a partial join, the vertex v can be not joined with the vertices of G which can't receive the maximum colour, if there is at least one vertex in $G-S$ coloured with the colour of every one of the vertices of S. We prove the last condition by contradiction. We assume that it doesn't hold. Hence, there is a vertex, let it be u, which can't receive the maximum colour and has unique colour. In this case u can receive the maximum colour be suitable interchange of colours and this is a contradiction. The contradiction to the first assumption comes from the Proposition 3.1 according to which two critical graphs, in our case are the graphs G and K_1, give a new critical graph only via complete join. ■

9 Proof of Lemma 4.5

Lemma 4.5 *Let $G1$ and $G2$ be two graphs. Let $G1'$ and $G2'$ be two graphs obtained by graphs $G1$, $G2$, respectively, having the same chromatic number as their corresponding graphs. The graph G' obtained by any join (complete or partial) of graphs $G1'$ and $G2'$ has a chromatic number at most equal to the chromatic number of G, where $G = G_1 \nabla G_2$.*

Proof: The proof of this Lemma is based on Proposition 3.1. From Proposition 3.1 we know that even if we increment the graph $G1$ in any way keeping its chromatic number unchanged its complete join with $G2$ will results a graph with chromatic number equal to the sum of the chromatic numbers of $G1$ and $G2$. So, if the graph $G1$ is augmented in such a way that, following one colouring of the initial $G1$, it has sets of vertices with $\chi(G1)$ different colours each, then it is easily understood that a graph G' witch is constructed joining each vertex of $G2$ to a set of vertices of $G1$ having the above property is $\chi(G1)+\chi(G2)$-colourable.

In fact the chromatic number of G' is may be smaller than $\chi(G1)+\chi(G2)$. This is true, because if we colour the vertices of $G2$ after a colouring of the vertices of $G1$ in G' one by one and we recolour the vertices of $G1$ and the coloured part of $G2$ each time that the next vertex of $G2$ is coloured then in each such step we can find a colouring of G' where the maximum colour used is not greater than the one used before the recolouring. ■

10 Proof of Lemma 4.7

Lemma 4.7 *Let P be any planar square root of the K_k, where $k \geq 4$. Let u, v, w be three vertices on the exterior face of P. Then at most two vertices, we call them x and y, can be added in this face so that each of them to be adjacent to each of the vertices u, v, w.*

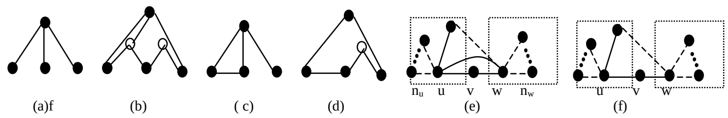

Figure 4: This figure shows parts of planar square roots of the graph K_k, where at least three of the vertices of the above clique are in the exterior face of these graphs.

Proof: In order to prove this Lemma we have to consider the structure of each square root of K_k on the vertices u, v, w and at least one more, we call it x, of this clique. We are sure that the fourth vertex exists, because $k \geq 4$. For simplicity reasons we call P each of the square roots of K_k provided that no confusion can rise.

We call I the induced subgraph of P on the vertices u, v, w, x. Thus, we have to consider the following five cases, which cover all the possible graph instances that can be assigned to I.

Case 1: I is an independent set of size four. **Case 2:** I has an edge plus two vertices. **Case 3:** I has two edges and a vertex. **Case 4:** I has three edges and a vertex. **Case 5:** I has four edges.

Of course for the last four cases we have to consider all the possible combinations for one, two, three and four edges between the vertices u, v, w, x.

We begin with the case where it is allowed to inserted two vertices, x, y, in the exterior face of P, which can be joined to each of the vertices u, v, w. If P has only one face (so, only the exterior face), then the above vertices remain in the exterior face after the insertion of the vertex x in P iff the graph I is the star (i.e. $K_{1,3}$) with one of the vertices u, v, w in the root. In any other case at least one of the vertices u, v, w isn't any more in the exterior face. This is the case even if the graph P has more than one faces from the beginning, before the insertion of the vertex x. The proof of this follows directly from the graphs of figure 4 which correspond to the Cases 1-5. ∎

11 Proof of Lemma 4.8

Lemma 4.8 *Let P be a planar graph and let $q = \omega(P^2) \geq 7$. If there are at least seven vertices of the maximum clique of P^2 in the same face of P, then the subgraph of P which forms this clique in P^2, call it S, is the $K_{(1,(q-1))}$ plus optional edges between consecutive vertices of S (given any arbitrary planar embedding of P).*

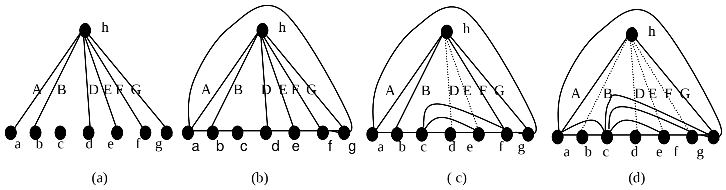

Figure 5:

Proof: We prove the assertion inductively by contradiction. The basic case is when $q = 8$. So, let S be the $K_{(1,7)}$ minus an edge. In order to be unambiguous our proof we use the figure 5. The vertices of the graph, which is represented in this Figure named be small letters and the edge that join each of these vertices with the root named be the corresponding capital latter. According figure 5 the edge that is missed is the C. It is clear that all the edges are equivalent (have the same possibility to be the missed edge). In this case the vertex c have to be neighbour of each of the other seven vertices in S^2. According to the assertion seven of the vertices of S are in the same face. Hence c can't be joined to any vertex of S, except of the vertices b and d which are next to it. So, each of the other vertices have to be distance two neighbour of c. In order to achieve this the least we have to do is to replace S with the graph of figure 5b. In this case the vertices f and g are at distance three from c. This implies that we have to insert the edges $\{c, e\}$ and $\{c, f\}$ in S and delete the edges D and E because S has to remain planar. The vertex c would be at distance at most two from g and f if it is only joined to f. In fact the avoidance of the edge $\{c, g\}$ leads to the need of an edge that will join the vertex d with the vertex f or g. If holds the first case, then this implies that the edge $\{c, e\}$ must be removed (because S must be planar). But the removal of the last edge influences the distance between the vertices b and e. So, the edges $\{c, e\}$ and $\{c, g\}$ must belong to S. Following the same way of reasoning we conclude that the edge $\{a, c\}$ have to be inserted in S. The final graph is presented in the figure 5d. It is easily seen that there is a $K_{1,q-1}$ in this graph. Actually this graph has as root any more the vertex c. The Lemma has been proved, because if q is greater than 8 then for the same reasons each of the added vertices have to be joined finally to vertex c.

By the same arguments is proved the case where $q = 7$. ■

12 Proof of Theorem 4.10

Theorem 4.10 *Let P denote any planar graph obtained by procedure $M(q)$. Then $\chi(P^2) \leq \omega(P^2) + 3$.*

Proof: We prove this Theorem considering six cases, based on the size q of the maximum clique in P^2. More precisely these cases are:
Case 1: $q \leq 4$, **Case 2:** $q = 5$, **Case 3:** $q = 6$, **Case 4:** $q = 7$, **Case 5:** $q = 8, 9$, and **Case 6:** $q \geq 10$

In order to increase the difference between the chromatic number and the clique number in P^2, we have to maximise the difference $x - MaxIS$. So, the split of the proof in the above cases comes from the hardness to get big values the variable x.

The Figures 5-11 play a great role in the study of the above cases, because it is easier to explain our arguments using them. In these figures we show planar graphs generated by procedure M for $3 \leq q \leq 9$. The numbers that correspond to the vertices of these graphs are their colours in the corresponding square of each of these graphs. If the colour of a vertex changed during the colouring of the vertices inserted in P along to the Step 3 of M, then its colours are written near to it, chronologically from left to right. The grey vertices in the following figures indicate vertices witch can't be coloured with unique colour with respect to their neighbours in the initial P. We choose such colour for them, which leads to as much as possible difference between the chromatic and the clique number on the squares of these planar graphs. Most of the times, in order to limit their choices, we join some of them to not only vertices of the initial P but and/or to vertices inserted at Step 2. The importance of unique colours to the vertices, which are inserted during Step 2, is explained in the following remarks. The edges in the following figures, which are represented by dots mention that if they were part of these graphs then would exist greater clique in their squares.

During the description of the graphs of these figures we name the vertices as follows:

i) The vertices of the initial P are named from A1 to Aq from left to right. If there is a cycle C_x in this graph, then we named the vertices of the cycle first and then the vertices inside the cycle from Ax+1 to Aq up to down.**ii)** The vertices inserted at Step 2 and joined to at least one of the vertices of the initial P are named B1, B2, ... from left to right. **iii)** The rest of the vertices inserted at Step 2 are named D1, D2, ... from left to right. **iv)** The vertices inserted at Step 3 are named C1, C2, ... from left to right.

Before the proof of the above cases we mention some important remarks which are crucial for the understanding of the following figures.

Remark. Let P' be a planar graph obtained by the procedure $M(q)$ during the Steps 1 and 2. Let u be a vertex of the clique which is inserted in P' during the Step 3. One of our aims is to find how to join each of the vertices of this clique in P' in such a way that the size of the maximum clique in P^2 to remains q and the difference $\chi(P^2) - \omega(P^2)$ to be as big as possible. Hence, because of Lemma 4.7 if the vertex u wants to be at most distance two neighbour of a vertex of colour a, then we achieve this joining it to propriety vertices which are most close to it.

Remark. Let P be a graph obtained by M. Let UD be a subset of consecutive vertices, from the vertices which were inserted in P during the execution of Step 2, without unique colours (most of the times have common neighbours in the initial P). If a vertex of the clique inserted at Step 3 wants to be at distance at most two from the vertex of colour x of UD, then it must be at distance at most two from each of the vertices of UB. This is because each of the vertices of this set can receive the colour x.

At this point we start the proof of each of the above cases separately.

Case 1: $q \leq 4$

Proving this case we have to consider the following subcases: $q = 2$, $q = 3$ and $q = 4$. We refer to each of them independently, because their characteristics are not common.

We start with the case where $q = 2$. It is well known that the odd cycles are the only 3_critical graphs. Thus, we want to find a planar graph P such that $\chi(P^2) \geq 3$ and $\omega(P^2) = 2$. There isn't a graph with this property, because even a path on three vertices leads to a clique of size three in its square.

At the case where $q = 3$ we have to consider what happens for each square root of the K_3 (i.e. the values of the graph initial P). The initial P can't be the C_3, because each vertex in the cycle has degree two and P's maximum degree has to be at most two, otherwise would created bigger maximum clique in its square. This graph can't also be the C_i, where $i \geq 6$, because of the maximum degree constraint as above. We note here that the squares of the cycles C_4 and C_5 are the K_4 and K_5, respectively. Thus this cycles can't be the initial P for this case. The only valid value for this graph is a path on three vertices. We denote the vertices of this path by a, b, c and its edges by $\{a, b\}$ and $\{b, c\}$. The middle vertex of this path (i.e. b) has degree two. Thus, no one of the vertices inserted at Step 2 can be joined to b. Also, because of the degree constraint at most one more vertex can be joined to the vertices a and c. Hence, at most one set (i.e.X'_1) with vertices with three different colours in every minimum colouring of P^2 with the properties of the sets of the Step 2 of M is possible to be created. Until now X'_1 has vertices coloured with the same colours as the vertices a and c but non with the colour of vertex b. This implies that another vertex is inserted. Obviously it can be adjacent only to a vertex which doesn't belong to the initial P_3, and we are sure that its colour is the colour of vertex b (is in a path of three vertices and the other two coloured without use of the colour of the vertex b). The final graph comes from the insertion of the vertex of Step 3. So, for this case our procedure generates odd cycles.

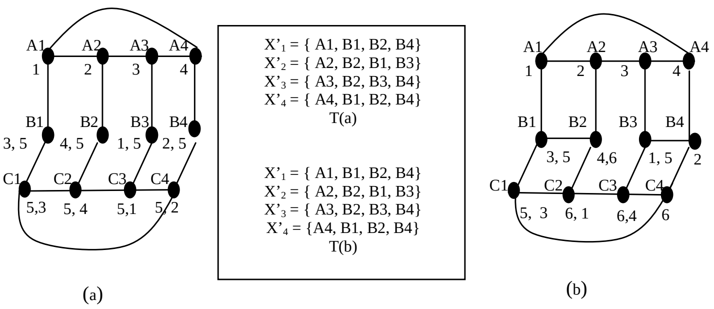

Figure 6: These figures correspond to the case $q = 4$ of the procedure M

At this point we prove the Theorem for $q = 4$. We start by considering in details the case where the initial P is the C_4. The other cases are proved in the Appendix B. We observe that, if we add a new neighbour to each of the vertices of the initial P, then each of them has distinct colour according to any minimum colouring of P^2. Based on the above remarks someone expects that the set X'_1 (and analogously the other sets of X') would be the $\{A1, A2, B1, B2\}$. This is not the case because we observe that if we join a vertex of the clique of Step 3 to any two consecutive vertices of the set B, then a cycle of size five is created ($C_5^2 = K_5$). This is the reason that the final graph P is the graph of figure 6a. In order to minimise the value of the variable $MaxIS$ we add two edges (we can't insert more edges because of the degree constrains) according to figure 6b.

The rest of the cases of the square root of K_4.

We conclude saying that the maximum difference between the chromatic and the clique number of squares of planar graphs with maximum clique of size 4 in their squares is two and no more because each vertex of the graph of figure 6b has degree 3 (which is the largest allowable).

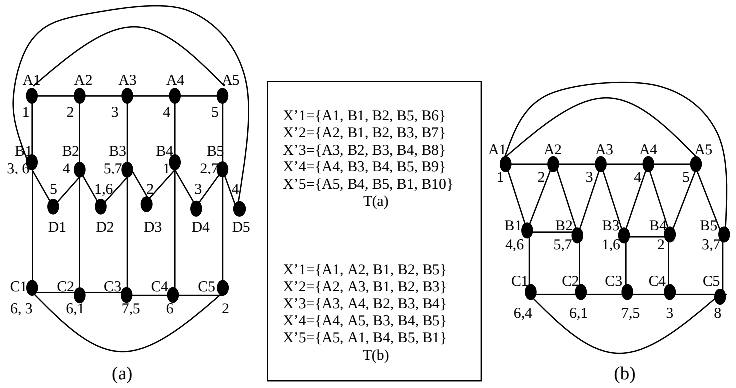

Figure 7: These figures correspond to the case $q = 5$ of the procedure M

Case 2: $q = 5$ Let C_5 to be the initial P (all the other cases proved in the Appendix B). For this case we consider in details two subcases. According to the first one we add a new neighbour to each vertex of the initial P. Studying carefully this case we observe that in every minimum colouring of P^2 each of the new vertices has two colours in its set allowed colours and that for each pair of consecutive such vertices the intersection of their sets of allowed colours has one element. Thus, if the vertices of these pairs are at most distance two neighbours, then the determination of the colour of one of them uniquely determines the colour of the others. An example of this case is shown in figure 7a where the vertex $B1$ received the color 3 (it is easily seen that there is no difference in the result if we had choose the colour 4 for the vertex $B1$). The distance two between the vertices of these pairs are achieved using auxiliary vertices. It is interesting to observe here that the auxiliary vertices have fixed colour (given the colours of the vertices of B). Having in mind the above observations and Remarks 12 and 12 we can easily understood why the sets of X' have the values depicted at the table of Figure 7. In this case $x = 5$ and $MaxIS = 2$. It is also easily seen that in the case where the nearby vertices of B are distance one neighbours the clique inserted at Step 3 could not have so big size (i.e. 5) because of the degree constraints.

Figure 7 depicts the graph P obtained by M when the colour of each vertex which is joined to vertices of the initial P defined in a fixed way. This is achieved joining each of the vertices of B to two consecutive vertices of the cycle. The size of the $MaxIS$ again is decreased by the insertion of two edges (see figure). The degree constrain does not allow more decrease of this set. It is interesting to mention here that this is one of the worst cases of the Theorem (so our bounds are tight).

In the following four cases the hardness of finding planar graphs with large value to the variable x in their squares increase rapidly. **Case 3:** $q = 6$ This case is the first one where non of the square roots of K_q is an unchorded cycle. It is clear that the above observation is true for each q greater than five. Figure 8a depicts a planar graph, P, obtained by M for this case. According to this the initial graph which is assigned to P during Step 1 of M is the C_6 plus as much as possible chords while all its

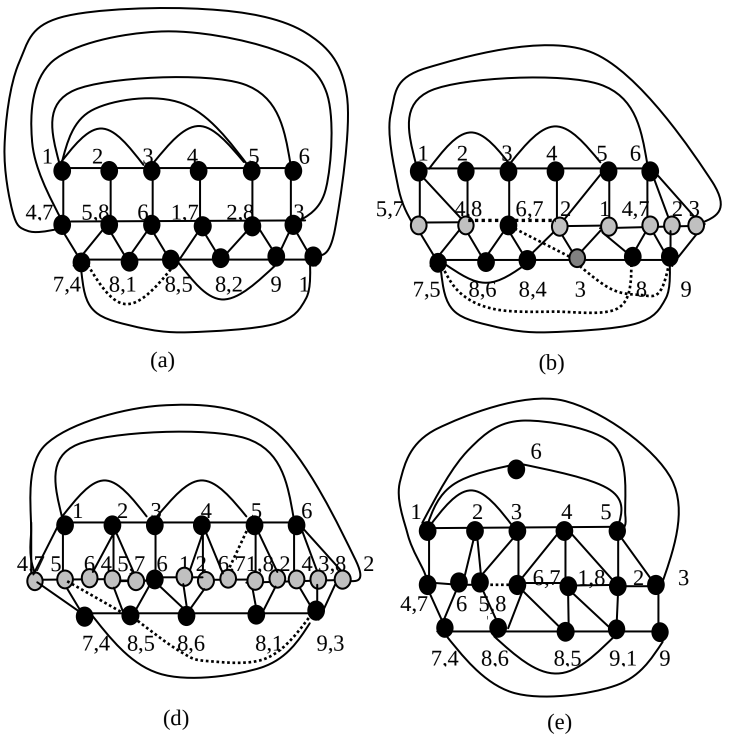

Figure 8: These figures correspond to the case $q = 6$ of the procedure M

vertices belong to the same face. The need for so much chords (one more than the necessary) comes from the observation that in this case the vertices of set B is harder to have fixed colours, than in the case where $q = 4$ and $q = 5$. In that cases the vertices of the set B had fixed colours because of their join to only one or two of the vertices of the initial P. Here, if each of the vertices of B is joined to one of the vertices of the initial P then only the vertices B1, B3, B5 have specific colours. The neighbours of the vertices $A2$ and $A4$ which belong to B receive fixed colours iff are at distance at most two from the vertices of B which are near (left and right) to them. If they are distance two neighbours, then the auxiliary vertices can't receive fixed colours except of the case where one of their neighbours in B is at distance one for the vertex of B which is near to it (except of the other neighbour of the auxiliary vertex). If this is the case then it is easily seen that the vertices of the set C are less than 6. So we consider the case where consecutive vertices of B are distance one neighbours. Now, then the auxiliary vertices play the role of the vertices of the clique inserted at Step 3. In this case $x = 6$ and $MaxIS = 2$ and the sets of X' are the expected and can easily seen from the figure.

The Figure 8d is at the very opposite. In this case each of the vertices of the initial P has as much as possible neighbours in the set B. So, these vertices has sets of allowable colours with more than one element (except of the neighbour of vertex $A3$, because is only one). In this case we join the consecutive vertices of B in order to decrease the $MaxIS$. We observe that only two vertices in the sets A and B received the colour 3. This implies that each of the vertices of the set C has to be at distance at most two from at least one of them. Because of the degree constraint only three vertices of C can be at distance at most two from the vertex of B which has the colour 3 (this is because of Lemma 4.7). So, only two of the other vertices of C can be at distance at most two from $A3$. The above discretion becomes more clear seeing the Figure. It is clear that the results of the case where the edge $\{A1, A5\}$ where part of the initial P are similar to the above case.

Now lets go to an other two case where the edge $\{A1, A5\}$ wheren't to the initial P. In these cases the vertex $A1$ can have neighbpours in B coloured with the colours 4 and 5. The vertex $A2$ can have in B neighbours coloured with the colours 1,4 and 6. Having in mind the figure 8d and for simplicity reasons we omit to present two vertices for each of the colours 4 and 5 (similar for the vertices $A4$ and $A5$). In the Figures 8b and 8c we achieve to take the set C with 6 vertices. Seeing carefully these figures we observe that only in the second one the vertices of C form a clique in P^2. But in this figure the variable x has value 5 because the vertex $C2$ actually takes the colour of one of the vertices of A. The value of the variable $MaxIS$ is 2 in both of these cases. The dotted edges in these Figures shows clearly the not allowed edges, which would increase the maximum clique in P^2.

The most important observation on the last three instances of this case is that the colours of the vertices of B are not fixed.

The last two instances of this case are presented in Figures 8e and 8f and concerns the case where the vertex $A6$ is not in the same face as the other five vertices of the initial P. In the last Figure we consider the case where the only one vertex of B has the colour 6. In this case it is clear that each vertex of C has to be at distance at most two from the above vertex. This in addition to that non of the vertices of B has the colour 3, leads to a set C with only two vertices. The Figure 8e solves the above problems with the best way.

Case 4: $q = 7$

The graph of Figure 9a is representable for this case. It is easily seen there the hardness to increase the chromatic number of the square of any planar graph which has a clique of size 7.

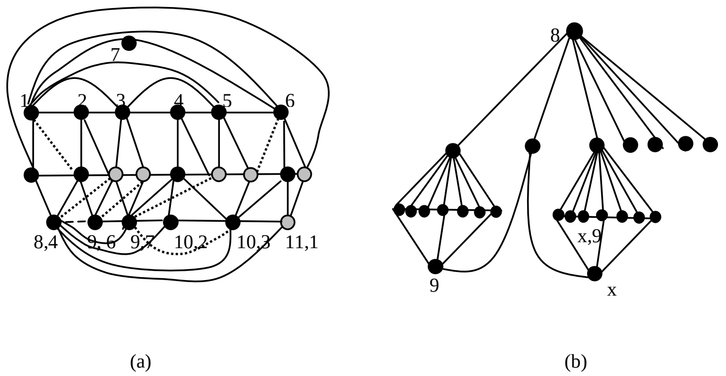

Figure 9: These figures correspond to graphs obtained by procedure M for the cases where $q = 7, 8$

Case 5: $q = 8, 9$ and **Case 6:** $q \geq 10$

The graph of Figure 9b is a graph P obtained by M when the initial P is the star. For this case is easily seen that the chromatic number of such graphs is bounded by $\omega P^2 + 1$. The graphs obtained by Procedure M for these values of q are omitted since they can be obtain in a similar way as in the cases explained.

$\blacksquare$