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# A Polyhedral Approach to Surface Reconstruction from Planar Contours*

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**Abstract.** We investigate the problem of reconstruction a surface given its contours on parallel slices. We present a branch-and-cut algorithm which computes the surface with the minimal area. This surface is assumed to be the best reconstruction since a long time. Nevertheless there were no algorithms to compute this surface. Our experiments show that the running time of our algorithm is very reasonable and that the computed surfaces are highly similar to the original surfaces.

## 1 Introduction

We investigate the problem of reconstruction a surface given its contours on parallel slices. This problem has its applications mainly in medical imaging such as computer tomography (CT) and magnetic resonance imaging (MRI). The result of those scans is a series of slices of the object. These slices are converted by edge-detection methods to sets of simple, non-intersecting polygons, called the contours. These are the input of our algorithm. The goal is to visualize the original surface in a computer. Typical visualization software requires triangulated surfaces as input. Thus, we have to solve the surface reconstruction from planar contours problem, i.e. we have to convert the polygonal slices to a triangulated surface that looks like the surface of the original object. The problem is illustrated in Fig. 1.

It is assumed since a long time [7, 10] that the triangulated surface with minimal area is the best reconstruction. As we will see, the problem to compute this surface is NP-complete. Up to now there where only algorithms that solve the area minimization problem either for special cases [10, 8] or only approximatively [7, 8]. We present a branch-and-cut algorithm which computes the surface with the minimal area exactly. Our experiments show that the running time of the algorithm is very reasonable and that the computed surfaces are highly similar to the original surfaces.

We assume that the reader is familiar with basic polyhedral theory and branch-and-cut algorithms. For an introduction, we refer to [14]. In Sect. 2, we relate our work to extant work. In Sect. 3, we introduce some basic notations

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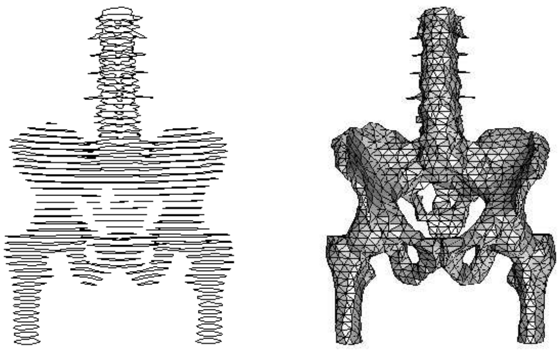

**Fig. 1.** The input is a series of parallel slices. We compute a triangulated surface.

and definitions. In particular, we give two possible characterizations of triangulated surfaces. These characterizations lead to integer linear programming (ILP) formulations formulations, which are given in Sect. 4. Our investigations on the corresponding polyhedra follow in Sect. 5 and 6, respectively. In Sect. 7, we show the NP-completeness of both problems. In Sect. 8 and 9 we describe the resulting branch-and-cut algorithms and the experiments we made with the algorithms. Finally we give some conclusion.

## 2 Comparison to Extant Work

Our algorithm as well as most previous algorithms compute the triangulated surface by iteratively computing the part of the surface that lies between two neighbored slices. Thus we assume in the following, that we want to compute a surface between two parallel slices. Most previous algorithms split the surface reconstruction into three subproblems.

1. the correspondence problem, i.e. the problem to decide which contours on the one slice are connected to which contours on the other slice.
2. the branching problem, i.e. if there is no one-to-one correspondence between the contours on the two slices, the contours must be split (see Fig. 2d) or connected (see Fig. 2c) by adding one or two additional segments to the contour. These segments are called the *bridge*.
3. the tiling problem, i.e. given one contour in each slice which triangles should be chosen to get an appropriate surface

Keppel [10] proposes to solve the tiling problem by area minimization, i.e. given two contours, the triangulated surface with the minimal surface area should be chosen as reconstruction. Sederberg, Hong, Kaneda, and Klimaszewski [8] go one step beyond and propose to solve the branching problem by area minimization, too. They describe an algorithm that, given two contours in one slice and

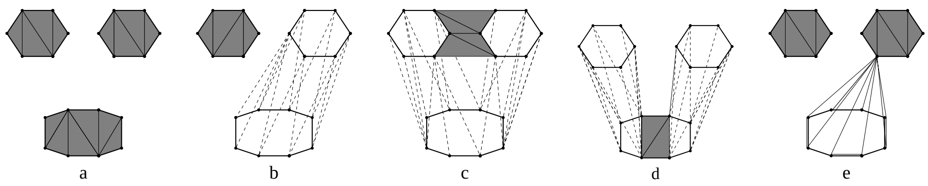


**Fig. 2.** Different solutions for the branching and correspondence problem.

one contour in the other slice, computes the best possible bridge, i.e. the two segments, so that the area of the minimal-area triangulated surface is minimal.

We go even a step further and solve the correspondence problem by area minimization, too. Thus we have to deal with the following problem: Given a set of contours in two parallel slices, we compute the triangulated surface with the minimal area.

## 3 Definitions and Notation

We introduce some notations. We assume that we are given the set of contours on the two slices as a graph $G = (V, S)$, whose nodes $V$ correspond to the corners of the contours and where two points are connected by an edge in $S$, if the points are connected by a segment in a contour. Let $n = |V|$ and $k$ be the number of polygons in $G$. Let furthermore $E$ be the set of edges of the complete graph over $V$ and $T$ all possible triangles over $V$. A triangle is *adjacent* to a point $p$ or an edge $pq$ if it contains the point or the edge, respectively. We call edges in $S$ segment edges and edges in $E \setminus S$ non-segment edges. Let $F$ be a set of triangles over $V$. For two points $p, q \in V$ let $F_p = \{t \in F \mid p \in t\}$, $F_{pq} = \{t \in F_p \mid q \in t\}$ and $\mathcal{E}^p(F) = \{rs \mid prs \in F\}$. Furthermore we assume that the contours are directed in some way. For a point $p$ let $p^-$ its predecessor on the contour and $p^+$ its successor. We also assume that the contours are numbered from 0 to $k-1$ and that $\oplus$ is the addition modulo $k$. For a point $p$ let $c_p$ the index of the contour of $p$. For an set of edges $E' \subseteq E$ let $E'^i = \{pq \in E' \mid c_p = c_q = i\}$. For a subset $V' \subseteq V$, let $V'^i = \{p \in V' \mid c_p = i\}$.

First we have to characterize those sets of triangles that correspond to triangulated surfaces for the input. We define a triangulated surface between two slices as a set of triangles with the following two properties.

1. For each segment-edge of an input contour, exactly one triangle adjacent to it is in the surface.
2. Each non-segment-edge is either in the surface or not. In the first case, exactly two adjacent triangles are in the surface, in the second case no adjacent triangle is in the surface.

One could further restrict the set of surfaces by insisting to the following property.

3. Let $F$ be the set of the triangles of the surface. Then $\mathcal{E}^p(F)$ is a path betweeen $p^-$ and $p^+$ for every $p \in V$.

The last property forbids configurations as shown in Fig. 2e to be surfaces. We call surfaces with property 3 *normal* surfaces. If we do not restrict the surfaces to have property 3 we talk about *general* surfaces. We present ILP-formulations and derive branch-and-cut algorithms for both cases. In all our experiments, the general surface with the minimal area was a normal surface.

Let $C = (p_0, \ldots, p_l, p_0)$ be a cycle. A set of triangles $F$ *closes* the cycle, if $|F_{pq}| = 1$ for all edges $pq$ of the cycle and $|F_{pq}|$ is 0 or 2 for other pairs of points in $\{p_0, \ldots, p_l\}$. Furthermore $\mathcal{E}^{p_i}(F)$ is a path between the two adjacent points of $p_i$. A cycle can be closed e.g. by the triangles $p_i p_{i+1} p_l$, $0 \le i \le l-2$. One possible normal surface is obtained by closing all contours. It is an easy observation that all segment and non-segment conditions are satisfied for this surface.

## 4 ILP-formulations and Cutting Planes

Both ILP-formulations use binary decision variables for the triangles, indicating whether a triangle belongs to a surface or not. For all triangles $t \in T$, let $a_t$ be the surface area of the triangle $t$. The first two conditions above are easily formulated as linear constraints.

$$\begin{aligned}
\min \sum_{t \in T} & a_t x_t & & \\
s.t. \sum_{t \in T_{pq}} & x_t = 1 & \text{for all } pq \in S & \qquad (1)\\
\sum_{t \in T_{pq}} & x_t \le 2 & \text{for all } pq \in E \setminus S & \qquad (2)\\
x_{pqr} & \le \sum_{s \in V^p \setminus \{q\}} x_{pqs} & \text{for all } pq \in E \setminus S, r \in V \setminus \{p, q\} & \qquad (3)\\
x_t & \in \{0, 1\} & \text{for all } t \in T &
\end{aligned}$$

The Equalities (1) enforce for each segment that exactly one adjacent triangle in the surface, thus property 1 is satisfied. To guarantee property 2, we need two classes of inequalities. The Inequalities (2) enforce that at most two triangles adjacent to a non-segment-edge are selected and Inequalities (3) enforce that either no triangle or at least two triangles adjacent to a non-segment are selected.

The following lemma is crucial for the formulation for normal surfaces and for the validity of the cutting planes we use.

**Lemma 1.** *Let $F$ be a surface and $p \in V$ be fixed. The edges $\mathcal{E}^p(F)$ form a unique path $P$ between $p^-$ and $p^+$ and a, possibly empty, set of cycles $C_1, \ldots, C_l$. Furthermore $P, C_1, \ldots, C_l$ are pairwise node-disjoint.*

*Proof.* From property 1 we conclude that there is exactly one edge adjacent to $p^-$ and $p^+$. From property 2 we conclude that for all other points there are either 0 or 2 adjacent edges. □

We now introduce some additional notation. For a point $p$ let $G^p = (V^p, E^p)$ be the complete graph whose nodes correspond to $V \setminus \{p\}$. We call a set of edges

$E' \subseteq E^p$ a $p$-path configuration, if $E'$ is the union of a path $P$ between $p^-$ and $p^+$ and cycles $C_1, \dots C_l$, so that $P, C_1, \dots, C_l$ are pairwise node-disjoint. This name is justified in Lemma 2, where we show that for every $p$-path configuration $P$ there is a surface $F$, so that $\mathcal{E}^p(F) = P$.

**Observation 1.** *If $\sum_{q,r \in V^p} a_{qr} x_{qr} \leq a_0$ is a valid inequality of the p-path configuration problem in $G^p$, the inequality $\sum_{pqr \in T_p} a_{qr} x_{pqr} \leq a_0$ is valid for the general surface problem. Analogously, if $\sum_{q,r \in V^p} a_{qr} x_{qr} \leq a_0$ is a valid inequality of the $(p^-, p^+)$-path problem in $G^p$, the inequality $\sum_{pqr \in T_p} a_{qr} x_{pqr} \leq a_0$ is valid for the normal surface problem.*

This transformation of an inequality for $G^p$ to an inequality for the surface problem is called *lifting*. The reverse transformation is called *projection*.

Look at any $(p^-, p^+)$-cut in $G^p$. At least one edge of any path between $p^-$ and $p^+$ crosses this cut. Lifting the corresponding inequalities leads to the following (exponential) class of valid inequalities for the general and normal surface polytope.

$$\sum_{q \in V', r \in V^p \setminus V'} x_{pqr} \geq 1 \text{ for all } V', \{p^-\} \subseteq V' \subseteq V^p \setminus \{p^+\} \tag{4}$$

We use this inequalities as cutting planes. We show a separation algorithm in Sect. 8.

To restrict the set of feasible solutions of the ILP above to normal surfaces, we have to eliminate those surfaces, where at least one set $\mathcal{E}^p(F)$ is not only a path from $p^-$ to $p^+$ but contains also a cycle $C$. The following (exponential) set of inequalities can be used.

$$\sum_{r \in V^p \setminus \{q\}} x_{pqr} \leq \sum_{r \in V', s \in V^p \setminus V'} x_{prs} \text{ for all } p, q \in V, \{q\} \subseteq V' \subseteq V^p \setminus \{p^-, p^+\} \tag{5}$$

We first argue the validity of the inequality. Look at a node $p$ and the path between $p^-$ and $p^+$ of the edges $\mathcal{E}^p(F)$. Let $q$ be any node. If $q$ is not on the path, the left hand side of the inequality is 0 and thus the inequality is satisfied for all $V'$. If $q$ is on the path, the left hand side of the inequality is 2. The right hand side of the inequality is at least two, since there is a path from $q$ to $p^-$ and a path from $q$ to $p^+$.

We now argue that the inequalities restrict the set of feasible solutions as desired. Assume that for one point $p$ the edges $\mathcal{E}^p(F)$ form a path $P$ and a cycle $C$. Choosing $q$ as any point in $C$ and $V'$ as the set of nodes in $C$, the left hand side of the inequality is 2 whereas the right hand side is 0. Thus this inequality is violated.

We handle these inequalities also by separation. The separation algorithm is given in Sect. 8.

## 5 Polyhedral Investigations for General Surfaces

We want to show that all described inequalities of the general surface ILP are facet defining.

If one takes a closer look to the inequalities, one makes the following observation: for every described inequality $ax \leq b$ for the general surface polytope, there is at least one point $p$, so that $a_t = 0$ for all $t \in T \setminus T_p$. Thus we can project these inequalities to an inequality in $G^p$. Looking at the resulting inequalities one observes that they are valid for the $p$-path configuration problem.

We first proof, that all $p$-path configurations can indeed be extended to general surfaces (Lemma 2). Furthermore we will see that for all facets of the $p$-path configuration polytope with a special property, the lifted inequality defines a facet of the general surface polytope. We finally show that the projected inequalities of all described inequalities for the general surface polytope are facets of the $p$-path configuration polytope and that they have the special property.

We will use the following observation several times.

**Observation 2.** *Let $P = (q_i, \ldots, q_j)$ be a path of segment edges, so that the length of the path is smaller than the size of the contour. For any (normal) surface $F$ of the modified problem, where the path $P$ is replaced by an edge $(q_i, q_j)$, the set of triangles $F \cup \{q_l q_{l+1} q_j \mid i \leq l \leq j-2\}$ is a (normal) surface for the original input (see Fig. 3).*

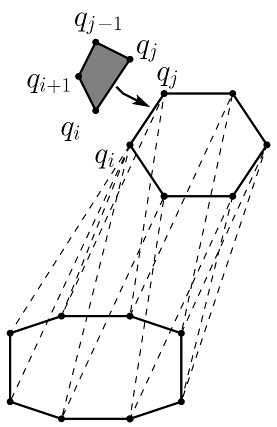


**Fig. 3.** The nodes $q_{i+1}, \ldots q_{j-1}$ were removed and a surface is constructed. Then the triangles $\{q_l q_{l+1} q_j \mid i \leq l \leq j-2\}$ are added to the surface.

We now show the crucial lemma of our proof. Every $p$-path configuration can be extended to a general surface.

**Lemma 2.** *Let $p \in V$ and $P$ be a set of edges forming a $p$-path configuration in $G^p$. There is a general surface with $T_p = \{pqr \mid qr \in P\}$.*

*Proof.* Let $F$ be the set of triangles induced by $P$. First notice that no segment edge has more than one adjacent triangle in $F$ and no non-segment edge has more than two adjacent triangles in $F$. Furthermore only non-segment edges adjacent to $p$ can have two adjacent triangles. Let $E'$ be the set of edges that need a further triangle, i.e. the set of segment edges without adjacent triangle in $F$ and the set of non-segment edges which have exactly one adjacent triangle in $F$. We claim that $E'$ is a union of cycles, not containing the point $p$. Thus closing all those cycles will lead to a correct surface.

We show that every point $q$ has an even number of adjacent edges in $E'$. If $q$ is not in $P$, the two adjacent segment edges are in $E'$, namely $qq^-$ and $qq^+$. If $q$ is on the path and $q^-$ and $q^+$ are on $P$, too, $E'$ contains no edge adjacent to $q$, since both segment edges have an adjacent triangle. If exactly one of $q^-, q^+$ is also on the path, say $q^-$, the segment $q^-q^+$ and $qr$ are in $E'$, where $r$ is the adjacent point of $q$ in $P$ different from $q^-$. Finally, if neither $q^-$ nor $q^+$ is on the path, there are 4 adjacent edges in $E'$, namely $q^-q$, $q^+q$, $rq$ and $sq$, where $r$ and $s$ are the two adjacent points of $q$ in $P$. The point $p$ is not on a cycle, since there are triangles adjacent to $pp^-$ and to $pp^+$ in $F$. □

Note that, if the path is different from $(p^-, p^+)$, the surface can be chosen, so that it does not use the segment $p^-p^+$. This is obvious, if $p^-$ and $p^+$ are on different cycles. If $p^-$ and $p^+$ are on the same cycle, we have to chose the triangles that close the cycle carefully (Note that the edge $p^-p^+$ is not on a cycle). If the path contains no further point on $c_p$, $p^-$ and $p^+$ lie on the same cycle, since all edges on $c_p$ different from $p^-p$ and $pp^+$ are in this cycle (all those edges are in $E'$ and the nodes have degree 2 in $E'$). In this case there is also an surface that contains the edge $p^-p^+$, since we can close the cycle so that the edge $p^-p^+$ is chosen.

Before we can make a facet proof, we need to know the dimension of the polytope.

**Lemma 3.** *The dimension of the general surface polytope is $\binom{n}{3} - n$, where $n$ is the number of nodes in $G$.*

*Proof.* Let $r_i \in V$ be any point with $c_{r_i} = i$. We have $n$ equalities of type 1 in our description, one for each segment. Let $pq \in S^i$. The coefficient of the triangle $pqr_{i\oplus 1}$ is 1 for the equality for $pq$ and 0 for all other equalities. Thus all equalities are linearly independent. Hence the dimension of the surface polytope is at most $\binom{n}{3} - n$.

We now look at the special case of $k$ contours each containing exactly 3 points. For $k = 2$ and $k = 3$ the lemma can be easily checked by enumerating all surfaces. Let $k \geq 4$ and assume that we have proven the statement for all $l \leq k - 1$. It is known that the dimension of the polytope is $\binom{n}{3} - n$, if and only if all valid equalities are linearly dependent to the known equalities.

We can use the fact that $a_{pqr_{i\oplus 1}}$ is 1 for one equality and 0 for all other equalities to define a normal form of equalities. An equality is in *normal form*, if $a_{pqr_{i\oplus 1}} = 0$ for all segments $pq$. Every equality that is linearly independent to the equalities in our description, has an equivalent equality in normal form. There exists a valid equality linearly independent to the known equalities if and only if there exists a equality in normal form different from $0 = 0$.

Assume there is a valid inequality $\sum_{t\in T} a_t x_t = a_0$ in normal form and $a_{pqr} \neq 0$ for three points $p, q, r$. Let $i$ be any contour with $i \neq c_p, i \neq c_q$ and $i \neq c_r$. We look at all surfaces closing polygon $i$ and choosing no other triangle adjacent to any of the three points. The equality is valid for all those surfaces, which is equivalent to $\sum_{t\in T\setminus(T_p\cup T_q\cup T_r)} a_t x_t = a_0 - a_{pqr}$. This inequality must be a valid inequality for $k-1$ contours, thus we conclude that $a_t = 0$ for all $t \in T \setminus (T_p \cup T_q \cup T_r)$. This contradicts the assumption that $a_{pqr} \neq 0$.

Let's turn to the general case. In the facet proof, we will show that there are $\binom{n}{3} - n$ affinely independent points that are active for a specific facet, e.g. for the facet $x_{p^-pp^+} \geq 0$. This proof is by induction and uses as basis the case above. Observing that a facet is not active for all surfaces (in the example, notice that there is a surface that uses the triangle $p^-pp^+$), we conclude that there are at least $\binom{n}{3} - n + 1$ affinely independent surfaces. This proves that the dimension is $\binom{n}{3} - n$. □

We resign to proof the following lemma.

**Lemma 4.** *The dimension of the p-path configuration polytope is* $\binom{n-1}{2} - 2$.

We are now turning to our main theorem.

**Theorem 1.** *Let $p \in V$ be fixed so that the size of the contour of $p$ is at least 4. Let $\sum_{q,r\in V^p} a_{qr}x_{qr} \leq a_0$ be a facet of the p-path configuration polytope. If (C1) $a_{p^-p^+} = a_0$ and (C2) there is a $p^-p^+$-path $P$ with $P \cap V^{c_p} = \{p^-, p^+\}$ and $\sum_{e\in P} a_e = a_0$ then $\sum_{pqr\in T_p} a_{qr}x_{pqr} \leq a_0$ defines a facet for the general surface polytope.*

*Proof.* Let $k$ be the number of contours. For every $k$ we make an inductive proof over the total number of points in the contours to show that the inequalities that have the properties of the lemma are active for at least $\binom{n}{3} - n$ affinely independent surfaces. For $n = 3k$ there is nothing to show (since there is no contour with size at least 4). Thus let $n > 3k$.

We define the *p-reduced surface polytope* as the surface polytope where the path $p^-pp^+$ is replaced by the edge $p^-p^+$. If $n - 1 > 3k$, there is at least one inequality of the $p$-reduced surface polytope that has the properties of the lemma. Since this inequality is not active for all surfaces, we can conclude that the dimension of the $p$-reduced surface polytope is $\binom{n-1}{3} - n + 1$. If $n - 1 = 3k$, we have already shown that the dimension of the $p$-reduced surface polytope is $\binom{n-1}{3} - n + 1$.

There are $\binom{n-1}{3} - n + 2$ affinely independent surfaces which have the triangle $pp^-p^+$, since fixing the triangle $p^-pp^+$ is equivalent to replacing the path $p^-pp^+$

by the edge $p^-p^+$ and thus we have the same number of affinely independent surfaces as in the $p$-reduced surface polytope. From condition (C1) we conclude that all of them satisfy the facet with equality. Furthermore there are $\binom{n-1}{2} - 3$ affinely independent $p$-path configurations different from the path $(p^-, p^+)$ in $G^p$ which satisfy the projected facet with equality. These path can be extended to surfaces which do not use the edge $p^-p^+$. These surfaces also satisfy the facet with equality. We resign to give the rather simple proof that all those surfaces are affinely independent. Thus we have $\binom{n-1}{3} - n + 2 + \binom{n-1}{2} - 3 = \binom{n}{3} - n - 1$ affinely independent surfaces. Let $\mathcal{F}$ be the set of these surfaces.

We need one further surface to finish the proof. All surfaces in $\mathcal{F}$ satisfy the equality

$$x_{pp^-p^+} = \sum_{r \in V^p \setminus \{p^-, p^+\}} x_{p^-p^+r} \quad . \tag{6}$$

Due to (C2), there is a path $P$ with $P \cap V^{c_p} = \{p^-, p^+\}$ and $\sum_{e \in P} a_e = a_0$. A surface that does not satisfy this equality can be obtained by extending the path $P$ so that the non-segment edge $p^-p^+$ is realized. The resulting surface is active for the facet. Since it does not satisfy the Equality (6) it is affinely independent from the surfaces $\mathcal{F}$. □

We resign to proof the following lemma which summarizes our polyhedral investigations on the $p$-path configuration polytope.

**Lemma 5.** *The following valid inequalities for the p-path configuration polytope are facet defining.*

1. $x_{qr} \geq 0$ *for all* $qr \in E^p$.
2. $x_{qr} \leq \sum_{s \in V^p, s \neq r} x_{qs}$ *for all* $q, r \in V^p$, $q \neq r, p^-, p^+$.
3. $\sum_{q \in V', r \in V^p \setminus V'} x_{qr} \geq 1$ *for all* $\{p^-\} \subseteq V' \subseteq V^p \setminus \{p^+\}$ *with* $2 \leq |S| \leq |V^p| - 2$.

Now we can formulate the following lemma.

**Lemma 6.** *Let* $p \in V$ *so that the size of the contour of* $p$ *is at least* 4. *The following inequalities are facet defining for the general surface polytope.*

1. $x_{pqr} \geq 0$ *for all* $q, r \in V$, $p, q, r$ *pairwise disjoint.*
2. $x_{pqr} \leq \sum_{s \in V \setminus \{p,q,r\}} x_{pqs}$ *for all* $q, r \in V^p$, $q \neq r$.
3. $\sum_{q \in V', r \in V^p \setminus V'} x_{pqr} \geq 1$ *for all* $\{p^-\} \subseteq V' \subseteq V^p \setminus \{p^+\}$ *with* $2 \leq |S| \leq |V^p| - 2$.

*Proof.* We start with the first statement. If $qr \neq p^-p^+$, $x_{qr} \geq 0$ is facet defining for the $p$-path configuration polytope. Furthermore $a_{p^-p^+} = 0 = a_0$ and $a_{p^-r} + a_{rp^+} = 0 = a_0$ for any point $r \neq q$. If $qr = p^-p^+$, the inequality $x_{pr} \geq 0$ is facet defining for the $q$-path configuration polytope and the conditions (C1) and (C2) hold.

For the other two items notice that the corresponding path inequality is facet defining for the $p$-path configuration polytope of $G^p$ and that the condition (C1) holds. It is easy to find a path $P$ so that (C2) holds. □

## 6 Polyhedral Investigations for Normal Surfaces

The polyhedral investigations for normal surfaces follow almost the same path as for general surfaces. There is one additional complication. We were not able to show that all $(p^-, p^+)$-paths are extensible to normal surfaces. If the degree of a node $q$ in $E'$ is 4, the adjacent edges of $q$ in the surface we described needn't form only a path, but can have additional cycles. If every node has degree at most 2, we can show that the constructed surface is normal.

From the following set of path we can proof that we can extend them to normal surfaces.

**Definition 1.** *A $p$-*ordered path *is a path $P$ in $G^p$ between $p^-$ and $p^+$ with the following properties.*

- *For any point $q$ in $P$ different from $p^-$ and $p^+$, at least one of its adjacent points in $P$ lies in the same contour as $q$.*
- *For every contour $i$ let $s$ be the first point of $P$ in contour $i$ and $t$ be the last point. The subpath of $P$ consisting of all nodes in contour $i$ is a subpath of one of the two $s, t$ paths in the contour $i$.*

We were able to show that the dimension of the $p$-ordered path polytope is $\binom{n-1}{2} - 2$, where $n-1$ is the number of nodes of $G^p$. Thus we can exactly argue as in the case of general surfaces and obtain the following theorem.

**Theorem 2.** *Let $p$ be fixed so that the size of the contour of $p$ is at least 4. Let $\sum_{q,r \in V^p} a_{qr} x_{qr} \leq a_0$ be a facet of the $p$-ordered path polytope. Let furthermore $\sum_{pqr \in T_p} a_{qr} x_{pqr} \leq a_0$ be a valid inequality for the surface polytope. If (C1) $a_{p^- p^+} = a_0$ and (C2') there is a $p$-ordered path $P$ with $P \cap V^{c_p} = \{p^-, p^+\}$ and $\sum_{e \in P} a_e = a_0$, then $\sum_{pqr \in T_p} a_{qr} x_{pqr}$ defines a facet for the normal surface polytope.*

Note that in the theorem there is the additional precondition that the facet of the $p$-ordered path polytope is valid for the normal surface problem.

**Lemma 7.** *Let $p \in V$ and $P$ be a set of edges forming an $p$-ordered path in $G^p$. There is a normal surface with $T_p = \{pqr \mid qr \in P\}$.*

*Proof.* Assume first that every edge $e$ in $P$ that lies completely in one contour is a segment edge. We conclude that for every node of the path, at least one adjacent edge is a segment edge.

We show that the surface we constructed in Lemma 2 is a normal surface. Let $q$ be any point. Assume first that $q$ lies on the path $P$. If $q^-$ and $q^+$ are on the path $P$, the triangles $qq^-p$ and $qq^+p$ are the triangles that are adjacent to $q$. Assume now that exactly one of $q^-$ and $q^+$ are on the path, say $q^-$. Let $r$ be the other adjacent edge of $q$ on the path. The path contains the triangles $qq^-p$ and $qrp$ that are adjacent to $q$. The cycle containing $q$ contains the path $rqq^+$ and thus the triangles that close the cycle form a path between $q^+$ and $r$. Thus all triangles adjacent to $q$ form a path between $q^-$ and $q^+$. If $q$ is not on

$P$, the cycle containing $q$ contains the path $q^- q q^+$. Thus closing this cycle will guarantee that the triangles adjacent to $q$ form a path between $q^-$ and $q^+$.

Assume now that there is a non-segment edge of $P$ that lies in one contour, say $i$. Let $q_0$ be the first node of $P$ on contour $i$ and $q_1, \dots q_l$ be the remaining points of the contour going along the segment edges, so that the indices of the nodes in contour $i$ in $P$ are increasing. Let $q_a q_b$, $b > a$ be a non-segment edge of $P$ in this contour. We can close the cycle $q_a, q_{a+1}, \dots, q_b$ and handle $q_a q_b$ as a segment edge. This is possible since no point between $q_a$ and $q_b$ is on the path $P$. □

We now summarize our investigations on the $p$-ordered path polytope.

**Lemma 8.** *The following inequalities define facets for the p-ordered path polytope.*

1. $x_{qr} \geq 0$, *for all* $qr \in E^p$ *with* $qr \neq p^- p^{--}$ *and* $qr \neq p^+ p^{++}$. *If* $q$ *and* $r$ *lie in the same contour different from* $c_p$, *we assume furthermore that the size of the contour is at least* 4.
2. $x_{qr} \leq \sum_{s \in V^p, s \neq r} x_{qs}$ *for all* $q, r \in V^p$ *with* $q, r \neq p^-, p^+$ *and* $c_q = c_r \neq c_p$.
3. $\sum_{q \in V', r \in V^p \setminus V'} x_{qr} \geq 1$ *for all* $\{p^-\} \subseteq V' \subseteq V^p \setminus \{p^+\}$ *with* $|V' \cap V^i| \neq 1$ *and* $|(V^p \setminus V') \cap V^i| \neq 1$ *for all* $i \neq c_p$. *The nodes of the path* $(p^-, p^{--}, \dots, p^{++}, p^+)$ *have an initial part of nodes in* $V'$ *followed by a path of nodes that are not in* $V'$.
4. $\sum_{r \in V^p \setminus \{q\}} x_{qr} \leq \sum_{r \in V', s \in V^p \setminus V'} x_{rs}$ *for all* $\{q\} \subseteq V' \subseteq V^p \setminus \{p^-, p^+\}$ *if either*
   (a) $c_q \neq c_p$, $|V' \cap V^i| \neq 1$, $|(V^p \setminus V') \cap V^i| \neq 1$ *for all* $i$, $|V' \cap V^{c_q}| \geq 3$, *and* $|V^{c_q}| \geq 4$, *or*
   (b) $c_q = c_p$, $|V' \cap V^i| \neq 1$, $|(V^p \setminus V') \cap V^i| \neq 1$ *for all* $i$, $V' \cap V^{c_p} = \{q\}$

We finally state the following lemma.

**Lemma 9.** *Let* $p \in V$ *be a point so that the size of the contour of* $p$ *be at least* 4. *The following inequalities define facets for the normal surface polytope.*

1. $x_{pqr} \geq 0$, *for all* $p, q, r \in V$, $p, q, r$ *pairwise disjoint. If* $q$ *and* $r$ *lie in the same contour, we assume furthermore that the size of the contour is at least* 4. *If* $p, q, r$ *lie in the same contour we assume that at most one of the three edges* $pq$, $qr$, $rp$ *is a segment edge.*
2. $x_{pqr} \leq \sum_{s \in V^p, s \neq r} x_{pqs}$ *for all* $p, q, r \in V^p$ *with* $c_q = c_r \neq c_p$.
3. $\sum_{q \in V', r \in V^p \setminus V'} x_{pqr} \geq 1$ *for all* $\{p^-\} \subseteq V' \subseteq V^p \setminus \{p^+\}$ *with* $|V' \cap V^i| \neq 1$ *and* $|(V^p \setminus V') \cap V^i| \neq 1$ *for all* $i \neq c_p$. *The nodes of the path* $(p^-, p^{--}, \dots, p^{++}, p^+)$ *have an initial part of nodes in* $V'$ *followed by a path of nodes that are not in* $V'$.
4. $\sum_{r \in V^p \setminus \{q\}} x_{pqr} \leq \sum_{r \in V', s \in V^p \setminus V'} x_{prs}$ *for all* $\{q\} \subseteq V' \subseteq V^p \setminus \{p^-, p^+\}$ *if either*
   (a) $c_p \neq c_q$, $|V' \cap V^i| \neq 1$, $|(V^p \setminus V') \cap V^i| \neq 1$ *for all* $i$. $|V_q^c \cap V'| \geq 3$ *and* $|V^{c_q}| \geq 4$, *or*
   (b) $c_p = c_q$, $|V' \cap V^i| \neq 1$, $|(V^p \setminus V') \cap V^i| \neq 1$ *for all* $i$ *and* $V' \cap V^{c_p} = \{q\}$.

Note that we now have to show the existence of a $p$-ordered path $P$ which satisfies condition (C2').

## 7 NP-completeness

We will now show the NP-completeness of the minimal general and normal surface problem for general cost functions. We do not analyze the Euclidean case, where any point of a contour has a coordinate in the Euclidean Space and the cost of a triangle is defined as its surface area.

**Theorem 3.** *Given a set of contours, a function $c$ that maps triples of points of the contours to integers, and a constant $k$, the question whether there exists a general surface of cost at most $k$ is NP-complete. The same holds for normal surfaces.*

*Proof.* It is easy to see that the problem is in NP.

For the completeness, we show a reduction from maximal 3-dimensional matching. The maximal 3-dimensional matching problem is defined as follows. Given three disjoint sets $X$, $Y$, and $Z$ and a subset $D \subseteq X \times Y \times Z$, find a matching of maximal cardinality. A matching is a subset $M \subseteq D$ so that no elements in $M$ agree in any coordinate. The maximal 3-dimensional matching problem is NP-complete (see [9]).

The reduction will be so that for any matching $M$ there will be a normal surface of cost $|D| - |M|$ and no (general) surface of smaller cost.

For every $v \in X \cup Y \cup Z$, we introduce a contour $v_1, v_2, v_3$. For every element $d \in D$, we introduce a contour $d_1, d_2, d_3$. The cost function $c$ is defined as follows. For all $d = (x, y, z) \in D$ (where $x \in X$, $y \in Y$, and $z \in Z$), we define $c(d_1 d_2 d_3) = 1$, $c(d_1 d_2 x_1) = 0$, $c(d_1 x_1 x_2) = 0$, $c(d_2 d_3 y_1) = 0$, $c(d_2 y_1 y_2) = 0$, $c(d_3 d_1 z_1) = 0$, $c(d_3 z_1 z_2) = 0$, $c(d_1 z_1 x_2) = 0$, $c(z_1 z_3 x_2) = 0$, $c(z_3 x_2 x_3) = 0$, $c(d_2 x_1 y_2) = 0$, $c(x_1 x_3 y_2) = 0$, $c(x_3 y_2 y_3) = 0$, $c(d_3 y_1 z_2) = 0$, $c(y_1 y_3 z_2) = 0$, $c(y_3 z_2 z_3) = 0$, and $c(x_3 y_3 z_3) = 0$. We call these triangles the triangles associated with $d$. For all $v \in X \cup Y \cup Z$, we define $c(v_1 v_2 v_3) = 0$. For the remaining triangles $t$, we define $c(t) = \infty$. (It is easy to avoid $\infty$, by defining $c(t) = |D| + 1$.) See Fig. 4 for an example.

Let $M \subseteq D$ be a matching. We can construct a surface of cost $|D| - |M|$ as follows. For all $d \in M$, choose the triangles associated with $d$ which have cost 0. For all $d \notin M$, choose the triangle $d_1 d_2 d_3$. For all $v \in X \cup Y \cup Z$ which have no adjacent triangle in $M$, choose the triangle $v_1, v_2, v_3$. This constructs a normal surface of cost $|D| - |M|$.

Assume now there is a general surface $F$ of cost less than $|D| - |M|$, where $M$ is a maximal matching. Let $M' = \{d \in D \mid d_1 d_2 d_3 \notin F\}$. Obviously $|M'| > |M|$ and thus $M'$ is not a matching. Let $d$ and $d'$ be two elements in $M'$, which share a common node, say $x$. Without loss of generality, let $x \in X$. There must be a triangle in $F$ adjacent to the edge $d_1 d_2$. The only such triangle with finite cost different from $d_1 d_2 d_3$ is $d_1 d_2 x_1$ and hence it must be in $F$. Hence there is an other triangle adjacent to $d_1 x_1$ in $F$. The only such triangle with finite cost is $d_1 x_1 x_2$ and thus it must be in $F$. In the same way, we argue that the triangle $d'_1 x_1 x_2$ must be in $F$. Thus $F$ contains at least two triangles adjacent to the segment edge $x_1 x_2$. This contradicts the property 1 of general surfaces. □

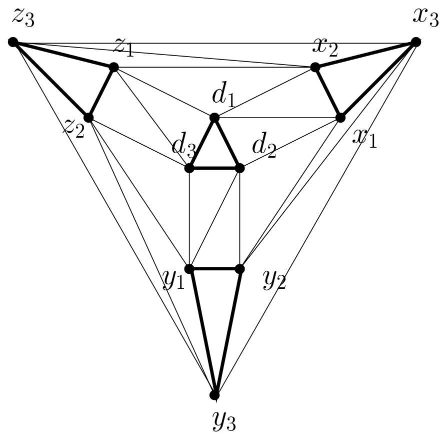


**Fig. 4.** The figure shows a tiny example: $X = \{x\}$, $Y = \{y\}$, $Z = \{z\}$ and $D = \{(x, y, z)\}$. The segment-edges are shown as bold lines. The faces of the graph (including the outer face) correspond to the triangles with finite cost. The triangle $d_1d_2d_3$ has cost 1, the remaining triangles have cost 0. There are exactly two surfaces consisting only of triangles of finite cost. We can either close all four contours, or we chose all other faces of the planar graph (including the outer face).

## 8 Branch-and-cut Algorithm

The input size of our problem instances, i.e. the total number of points in the two slices, was up to 267 points in our experiments. This would lead to over $3,000,000$ triangles. Thus we cannot afford to compute the minimal area surface with respect to all triangles. Therefore we restrict the possible triangles to the Delaunay triangles. Only considering Delaunay Triangles is wide-spread for surface reconstruction problems.

Assuming that the contours in one slice are non-intersecting, Cheng and Dey [4] show how to modify the input so that there is a surface consisting of Delaunay triangles. They showed that it is sufficient that all segments are edges of the 2D Delaunay Triangulation of the individual slices. If a segment is not a Delaunay edge, they propose to add an additional point in the middle of the segment and recompute the Delaunay triangulation. This process is iterated until all segments are Delaunay edges. We also use this approach in our experiments.

We now turn to the separation algorithms. Both classes of inequalities can be separated by computing a number of minimal cuts.

Fix a $p \in V$ and look at the inequalities $\sum_{q \in V', r \in V^p \setminus V'} x_{pqr} \geq 1$ for all $\{p^-\} \subseteq V' \subseteq V^p \setminus \{p^+\}$. The most violated inequality is achieved for the set $V', \{p^-\} \subseteq V' \subseteq V^p \setminus \{p^+\}$ that minimizes $\sum_{q \in V', r \in V^p \setminus V'} x^*_{pqr}$ where $x^*$ is the actual LP solution. Thus we have to solve a minimal $(p^-, p^+)$-cut problem in $G^p$ with edge weights $x^*$. To find the most violated inequality, we have to iterate over all $p \in V$. Thus the total running time is $O(n^4)$.

For the inequalities (5) we fix $p$ and $q$. Than we can find the most violated inequality by finding the subset $V', \{q\} \subseteq V' \subseteq V^p \setminus \{p^-, p^+\}$ that minimizes $\sum_{r \in V', s \in V^p \setminus V'} x^*_{prs}$. This set can be found by computing a minimal $(q, p^+)$-cut in $G^p$ and by setting the edge capacity of an edge $rs$ to $x^*_{prs}$ if $rs \neq p^-p^+$ and to infinity for the edge $p^-p^+$. The total running time is $O(n^5)$.

## 9 Experiments

We implemented a branch-and-cut algorithm for both ILP-formulations. We used the branch-and-cut system SCIL [12] which is based on ABACUS [1] and LEDA [11]. The LP solver can be chosen among CPLEX [5], XPRESS [15] and SOPLEX [13]. All our experiments were performed with CPLEX. SCIL provides an easy handling interface to implement branch-and-cut algorithms and LEDA provides the functionality to implement the separation algorithms. Furthermore we use CGAL [3] to compute the Delaunay triangulations.

As input for our experiments, we use a dataset given by Barequest [2]. It contains several sets of polygonal slices which originally came from CT-scans of human organs.

In all our experiments the general surface with minimal area was a normal surface.

Our experiments show that too many polygons were closed in the optimal surface. The reason is that the slices of the input are rather far away from each other compared to the size of the polygons. To reduce the number of closed polygons, we compute the area of the triangles according to slices whose distance is decreased by a constant factor. This results in surfaces that were very close to the original surfaces. All correspondence and branching problems were solved correctly.

The running time for our algorithm was very reasonable. For example the reconstruction of the 50 slices of pelvis data-set takes about 9 minutes if we compute the minimal area general surface and about 3.5 minutes if we restrict the surface to be normal. The running time to compute all Delaunay triangulations was about 25 seconds.

More experimental results can be found in [6].

## 10 Conclusion

We approached a problem arising from practical application with integer programming techniques. We studied the corresponding polyhedra and experimented with the resulting branch-and-cut algorithm. The results we obtained are promising.

It would be interesting to find further facets and corresponding separation algorithms for the polytopes. These could either be further lifted inequalities from the $p$-path configuration or the $p$-ordered path polytope, or inequalities that can not be obtained by lifting.

For the surface reconstruction problem for general point sets, there are algorithms that provably find an surface that is within a certain distance and homeomorphic to the original surface, if the point set satisfies some conditions. It would be interesting to obtain a similar result for our algorithm.

The next step seems to attack the surface reconstruction problem from points in general position with integer programming methods. We formulated this problem as ILP, but we were not able to solve practical instances with our formulation.

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