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Performance of TCP in the presence of correlated packet loss

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Abstract

We study in this paper the performance of a single persistent TCP connection, which experiences packet loss occurring in clumps. More precisely, the transmission of packets is altered by several packet losses during some time periods, which can span over several round trip time intervals. In spite of multiple packet loss, we assume that TCP remains in the congestion avoidance regime. We perform, by properly rescaling time, an exact analysis of the behavior of TCP when the loss rate becomes arbitrarily small. In particular, we investigate by means of elementary q-calculus the stationary distribution of the congestion window size embedded at the finishing times of loss periods. This allows us to exactly compute the mean throughput of the TCP connection. Then, by using stochastic ordering arguments, we investigate the impact of correlations on the mean throughput. Surprisingly, for a same loss rate, the mean throughput in the case of correlated losses is greater than that obtained under independent loss conditions. We conclude the paper by discussing how the results could be extended to account for additional features of TCP. In particular, we show that the slow start algorithm and variable round trip times have no impact on the asymptotic results of this paper. Finally, timeouts could easily be introduced in the model.

Keywords: TCP, throughput, q-calculus, stochastic ordering

1 Introduction

With the emergence of the Internet as the versatile network for computer communications, TCP (Transmission Control Protocol) has become the ubiquitous transport protocol for data transmission. Over the past ten years, a huge amount of research has been devoted to analysing the performance of a single TCP connection under various conditions, in particular with regard to the loss process affecting the transmission of packets. The simplest situation is when packets experience independent loss. In that case, the probability that a packet is lost is constant for every packet and loss events are independent one from each other.

Since the initial work by Floyd [1], who derived via simulation an asymptotic estimate for the throughput of a TCP connection experiencing a constant loss rate p (the celebrated $c/\sqrt{p}$ formula), several studies have refined the results obtained by Floyd. Under the independent loss assumption, Padhye *et al* [2] derived an asymptotic estimate for the throughput, which accounts for the main features of TCP (window size reduction, slow start, timeouts, etc.) and which has become central in TCP modeling. Still along this line of investigations, Dumas *et al* [3] provides rigorous convergence results and explicit expressions of the stationary distributions for the congestion avoidance regime, when packet losses are independent and the loss rate tends to 0.

However, the constant loss model rapidly proves not sufficient to reflect the behavior of a TCP connection through a real network. Indeed, recent traffic measurements carried out by Paxson [4] on the loss process affecting TCP connections in the Internet (see also Bolot [5] and Yajnik *et al* [6]) have shown that packet losses occur in clumps and that the distance between two clumps is large. This phenomenon can be explained by the fact that packet loss is mainly due to buffer overflow in the network.

Since buffer capacities in routers available today on the market are very large, buffer overflow in a router becomes a rare event, but when such an event occurs, then a large number of packets, possibly belonging to different flows, are lost. When considering a single TCP connection, this entails that once a packet is lost, a certain number of subsequent (but not necessarily consecutive) packets are lost. Note that such a loss period can span over several round trip time intervals. This corresponds to the well-known heuristic developed in probability theory by Aldous [7], which is referred to as *Poisson clumping heuristic* and which can safely be used to estimate rare event proba-

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bilities.

From the above discussion, the fact that loss events occur in clumps is essential to study the behavior of a real TCP connection. In this context, Misra *et al* [8] analyzed, in a setting similar to Ott *et al* [9], a representation of the sequence of the congestion window sizes as an $M/G/1$ queue. This $M/G/1$ representation is also used in Altman *et al* [10] to study grouped packet losses. It is worth noting that in those papers, the loss process is independent of the current congestion window size.

In the present paper, in order to study the performance of a TCP connection experiencing packet loss, which occurs in clumps, we introduce a model in which the occurrence of a clump of packet losses *depends* on the current congestion window size. Thus, the more packets are sent into the network, the more likely a packet loss occurs. This is the main difference with earlier studies published on the same topic. This allows us to capture qualitative phenomena, which cannot be seen by using the above cited models. In particular, we show by means of stochastic ordering arguments that for a fixed loss rate, constant loss is the worst case and hence, correlations in the loss process is "beneficial" for the mean throughput.

The organisation of this paper is as follows: The stochastic model for analysing a TCP connection experiencing correlated packet loss is introduced in Section 2, where some basic convergence results, when the loss rate becomes arbitrarily small, are rapidly established. The stationary distribution of the congestion window size embedded at the end of loss periods is computed in Section 3. These results are then used in Section 4 to establish a closed formula for the mean throughput of the TCP connection. Stochastic arguments are in particular used to study the impact of correlations in the packet loss process on the mean throughput. Some concluding remarks are presented in Section 5, where it is shown that the slow start mechanism and variable round trip times have no impact on the results established in this paper and how timeouts could be taken into account in the model.

2 Modeling TCP in the presence of correlated packet loss

Throughout this paper, we consider a single persistent TCP connection traversing a network and experiencing packet loss. We assume that the round trip time (RTT) along the connection is constant and sufficiently large so that a complete congestion window can be acknowledged over an RTT interval. Moreover, we assume that packet loss is lightweight (with intensity $\alpha \ll 1$) and that TCP always remains in the congestion avoidance regime. In the following, to simplify the notation, we set $\delta = 1/2$ and the MTU is taken as unity.

Let W_n^α denote the congestion window size over the nth RTT (i.e., the total number of packets, which can be sent during this time interval). The evolution of the process (W_n^α) is governed by the following recursion:

$$W_{n+1}^\alpha = \begin{cases} W_n^\alpha + 1, & \text{if no packets are lost} \\ \max\left(\lfloor \delta W_n^\alpha \rfloor, 1\right), & \text{otherwise.} \end{cases} \tag{1}$$

In the independent loss model (see Dumas *et al* [3] for instance), each packet has a probability $1 - \exp(-\alpha)$ of being lost. The loss process is given by a non-decreasing sequence (t_n^α), t_n^α being the number of the RTT interval where the nth loss occurs. Using the notation of Dumas *et al* [3], if $W_{t_n^\alpha}^\alpha = x$, then

$$t_{n+1}^\alpha - t_n^\alpha = G_x^\alpha$$

where the random variable G_x^α is given by

$$\mathbb{P}\left(G_x^\alpha > m\right) = e^{-\alpha(mx + m(m-1)/2)}, \quad m \geq 1. \tag{2}$$

As pointed out by Paxson [4], this non-correlated loss process proves not sufficient to reflect what happens in real networks. In fact, when packet loss is due to buffer overflow, if one packet is loss, it is very likely that some of the subsequent packets are also lost. This results in clumps of lost packets. Moreover, since the state of the network changes quite rapidly, if t_n^α denotes the number of the RTT interval over which the nth clump of lost packet begins, it is quite reasonable to assume that the sequence $(t_{n+1}^\alpha - t_n^\alpha)$ is independent and identically distributed (i.i.d.), provided that these quantities are not too small. Note that a clump of lost packets may span over several RTT intervals.

Paxson [4] and Zhang *et al* [11] showed through measurements that the process associated with lost packets can be described as follows:

- over the t_n^αth RTT interval, a packet loss period begins ((t_n^α) is the sequence considered in the non-correlated case);
- a certain number of packets, say X_n are lost during the loss period;
- the loss period may span over several RTT intervals; let l_n denote the number of RTT intervals affected by the loss period; l_n is assumed to be "sufficiently small".

The independence assumption implies that the random variable (X_n) are i.i.d. Moreover, for the sake of simplicity, we assume that the distribution of X_n does not depend on α but it is possible to let X_n depend on α.

The TCP connection is affected by multiple packet losses during a loss period and after such a loss period, there is large interval (of size $t_{n+1}^\alpha - t_n^\alpha \sim 1/\sqrt{\alpha}$) without any loss. Therefore, the distribution of the random variable X_n is a measure of the correlation between losses in a clump. The evolution of the congestion window size

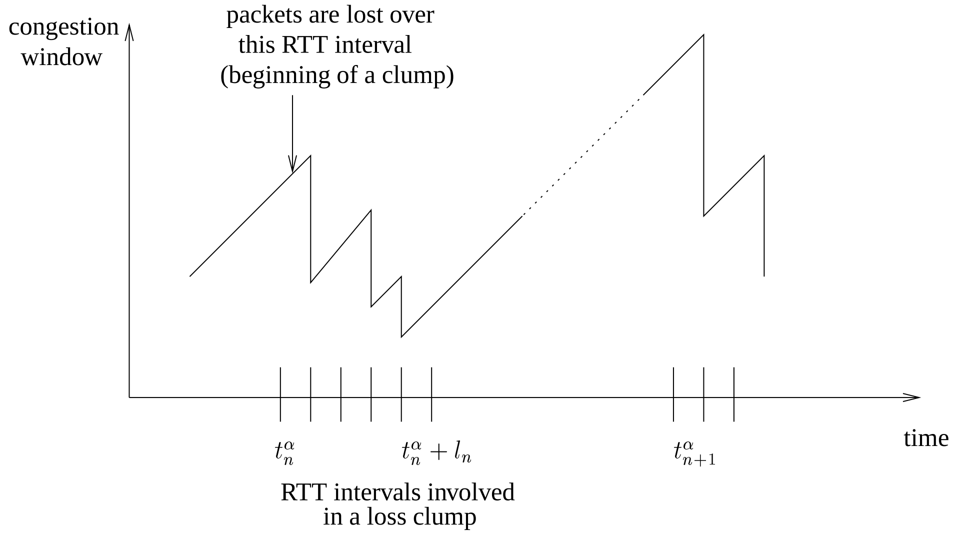

Figure 1: Evolution of the congestion window over time.

is illustrated in Figure 1. At the end of the nth loss period, the congestion window is approximately reduced by a factor δ^{X_n}.

If N_n^α is the number of packets transmitted between t_n^α and t_{n+1}^α, then asymptotically, when α tends to 0, the random variable αN_n^α is exponentially distributed with parameter 1. At the packet level, the loss process can thus be described as a Poisson process with clumps. As mentioned in the Introduction, this representation of the occurrence of rare events is quite universal in probability theory. Aldous' book [7] illustrates, through a large collection of examples, the generality of this description.

Before proceeding with convergence results, let us make the following points:

1. The case $X_1 \equiv \{1\}$ (and then $l_n \equiv 1$) corresponds to the uncorrelated case.

2. The i.i.d assumption for the sequence (X_n) is a consequence of the fact that the network rapidly forgets the past: at "time" t_{n+1}^α the nth loss period has been forgotten, in particular X_{n+1} is independent of X_n

3. The random variable X_n could depend on α, provided that the length l_n of the loss period is negligible compared to $1/\alpha$. For simplicity, the independence with respect to α is assumed; it is easily checked that this is not really restrictive.

With such a loss process the sequence (W_n^α) does not necessarily have the Markov property. The i.i.d. property of the sequence (X_n) nevertheless shows that the embedded chain $(V_n^\alpha) = (W^\alpha_{t_n^\alpha + l_n})$ (the sequence of congestion window sizes at the end of loss clumps) enjoys the Markov property. The next proposition shows that when time is properly renormalized, the transitions of this Markov chain converge. The proof is omitted (see [12] for details).

Proposition 1 *For $x > 0$, as α goes to 0, the random variable $\sqrt{\alpha}\, G^\alpha_{\lfloor x/\sqrt{\alpha}\rfloor}$ converges in distribution to a non negative random variable $\overline{G}_x$ such that for $y \geq 0$,*

$$\mathbb{P}\left(\overline{G}_x \geq y\right) = \exp\left(-x^2 - xy\right), \tag{3}$$

If $V_0^\alpha = \lfloor x/\sqrt{\alpha}\rfloor$, then as α tends to 0, the random variable $\sqrt{\alpha} V_1^\alpha$ converges in distribution to $\overline{V}_1$ with

$$\overline{V}_1 = \delta^{X_1}\left(x + \overline{G}_x\right) \tag{4}$$

where X_1 and $\overline{G}_x$ are independent random variables.

The model considered here does not exclude that consecutive losses may occur. In the present implementations of TCP, if a packet loss results into a timeout, then the congestion window is set equal to one segment and the TCP connection enters a slow start period. (See Stevens [13] for details.) For the moment, timeouts are not taken into account; we shall see in Section 5.2 that timeouts can be accounted for without any major difficulty.

The above proposition shows that if $V_0^\alpha = \lfloor x/\sqrt{\alpha}\rfloor$, then the Markov chain $(\sqrt{\alpha} V_n^\alpha)$ converges to the continuous state space Markov chain $(\overline{V}_n)$ with $\overline{V}_0 = x$ and

$$\overline{V}_{n+1} = \delta^{X_n}\left(\overline{V}_n + \overline{G}_{\overline{V}_n}\right), \tag{5}$$

for $n \in \mathbb{N}$. The above result can now be used to show the main convergence results, which are given below without proof. Arguments to show these results are exactly the same as in the uncorrelated case investigated in Dumas *et al* [3]. The main difference, which is precisely the main focus of the present paper, is in that closed form expressions of the limiting distributions are much more difficult to derive as it will be seen in the following.

Theorem 2 *When α tends to 0, the invariant distribution of the Markov chain $(\sqrt{\alpha}V_n^\alpha)$ converges in distribution to the invariant distribution $\overline{V}_\infty$ of the Markov chain $(\overline{V}_n)$.*

With a slight abuse of notation, the expression "the invariant distribution $\overline{V}_\infty$" means "a random variable $\overline{V}_\infty$ whose distribution is invariant for the Markov chain".

Proposition 3 *The invariant distribution $\overline{V}_\infty$ of the continuous state space Markov chain $(\overline{V}_n)$ satisfies the following identity:*

$$\overline{V}_\infty^2 \stackrel{dist.}{=} \delta^{2X_1}\left(\overline{V}_\infty^2 + 2E_1\right) \tag{6}$$

where X_1, E_1 and $\overline{V}_\infty$ are independent random variables, E_1 is exponentially distributed with parameter 1.

This proposition is a simple consequence of the elementary identity in distribution: for $x \geq 0$, $(x+\overline{G}_x)^2 \stackrel{\text{dist}}{=} x^2 + 2E_1$. (See Dumas *et al* [3] for details.)

Similarly convergence results can also be obtained for the original sequence (W_n^α).

Proposition 4 *If $\lim_{\alpha\to 0}\sqrt{\alpha}W_0^\alpha = \overline{w}$, then*

$$(W^\alpha(t)) = \left(\sqrt{\alpha}W^\alpha_{\lfloor t/\sqrt{\alpha}\rfloor}\right)$$

converges in distribution to the Markov process $\left(\overline{W}(t)\right)$ such that $\overline{W}(0) = \overline{w}$ and with the infinitesimal generator given by

$$\Omega(f)(x) = f'(x) + x\int_{\mathbb{R}_+}\left(f(\delta^u x) - f(x)\right)X_1(du),$$

for any C^1-function f on $\mathbb{R}_+$, where $X_1(dx)$ denotes the distribution of X_1 on $\mathbb{N}$.

3 Invariant distributions

By setting $\beta = \delta^2 = 1/4$, it is easily checked from equation (6) that $\overline{V}_\infty^2 = 2\delta^{2X_1} I$ with I being the solution to the equation

$$I \stackrel{\text{dist.}}{=} \beta^X I + E_0, \tag{7}$$

where E_0, I and X are independent one of each other, E_0 is an exponentially distributed random variable with parameter 1, and X is some non-negative random variable such that $\mathbb{P}(X > 0) = 1$. In this section, we do not assume that X takes integer values. The goal of this section is to study the random variable I solution of the above stochastic equation.

3.1 Laplace transform and moments

First note that by iterating Relation (7), the variable I can be represented as

$$I = \sum_{n=0}^{+\infty}\beta^{S_n}E_n, \tag{8}$$

where (E_n) is an i.i.d. sequence of exponentially distributed random variables with parameter 1 and $(S_n) = (X_1+\cdots+X_n)$ is the random walk associated with the i.i.d. sequence (X_n). If $(N(t))$ is a Poisson process with intensity 1, let $(\xi(t))$ be the compound Poisson process

$$\xi(t) = \log(1/\beta)\sum_{k=1}^{N(t)} X_k.$$

From representation (8), I can be written as

$$I = \int_0^{+\infty} e^{-\xi(t)}\,dt. \tag{9}$$

The variable I is thus the *exponential functional* associated with the Lévy process $(\xi(t))$. Such exponential functionals naturally occur in mathematical finance (Asian options) and in many other fields, when the Lévy process ξ is a Brownian motion with drift. See Yor [14] and Yor [15, Chapter 8], and Yor [16, Section 15.4] for a more theoretical point of view.

In a first step, we determine the integral and fractional moments of the random variable I. The determination of the fractional moment is clearly useful since the variable $\overline{V}_\infty$ is the square root of stationary window size $2\beta^{X_1}I$. Section 4 uses a fractional moment of I to derive an explicit expression of the throughput of the TCP connection.

It is precisely possible to establish the following result concerning the fractional moments of the random variable I, which extends the result obtained by Carmona *et al* [17].

Proposition 5 *For any $s \in \mathbb{R}$, the moments of the random variable I satisfy the following recursion:*

$$\mathbb{E}\left(I^{s-1}\right) = \frac{1-\mathbb{E}\left(\beta^{sX_1}\right)}{s}\mathbb{E}\left(I^s\right), \tag{10}$$

the moment of order s of I is finite if $\mathbb{E}\left(\beta^{(s+1)X_1}\right) < +\infty$.

Proof. For $\lambda \geq 0$, denote by $\psi(\lambda)$ the Laplace transform of I at λ. Equation (7) gives

$$\psi(\lambda) = \frac{1}{1+\lambda}\mathbb{E}\left(\psi(\lambda\beta^{X_1})\right). \tag{11}$$

The Mellin transform of ψ is

$$\psi^*(s) = \int_0^{+\infty}\psi(\lambda)\lambda^{s-1}\,d\lambda.$$

Since ψ is bounded, ψ^* is defined for $\Re(s) > 0$. (See Flajolet and Sedgewick [18] for a survey on Mellin transform methods.)

By using the definition of ψ,

$$\begin{aligned}
\psi^*(s) &= \mathbb{E}\left(\int_0^{+\infty} e^{-\lambda I}\lambda^{s-1}\,d\lambda\right) \\
&= \mathbb{E}\left(\frac{1}{I^s}\right)\int_0^{+\infty} e^{-\lambda}\lambda^{s-1}\,d\lambda \\
&= \mathbb{E}\left(\frac{1}{I^s}\right)\Gamma(s). \qquad (12)
\end{aligned}$$

On the other hand, relation (11) rewritten as

$$(1+\lambda)\psi(\lambda) = \mathbb{E}\left(\psi(\lambda\beta^{X_1})\right)$$

becomes via Mellin transform

$$\psi^*(s) + \psi^*(s+1) = \mathbb{E}\left(\beta^{-sX_1}\right)\psi^*(s),$$

and by relation (12),

$$\mathbb{E}\left(\frac{1}{I^{s+1}}\right) = \frac{\mathbb{E}\left(\beta^{-sX_1}\right)-1}{s}\mathbb{E}\left(\frac{1}{I^s}\right). \qquad (13)$$

This relation extends to $\mathbb{R}_-$ formula (10) obtained by Carmona *et al.* [17] for $s \in \mathbb{R}_+$. If I has a finite moment of order $-s$ and $\mathbb{E}\left(\beta^{-sX_1}\right)$ is finite, then I has a finite moment of order $-s-1$. Since all the positive moments of I are finite (see Carmona *et al.* [17] for example), by induction one gets that $\mathbb{E}(1/I^s)$ is finite when $\mathbb{E}\left(\beta^{-(s-1)X_1}\right)$ is finite. The proposition is proved. $\square$

As a consequence of the above result, we can obtain a closed formula for the fractional moment of I. The proof is omitted; see [12] for details.

Proposition 6 *For any $s \in \mathbb{R}$, $-s \notin \mathbb{N} - \{0\}$, provided that $\mathbb{E}\left(\beta^{(s+1)X_1}\right) < +\infty$ and*

$$\mathbb{E}\left(\frac{1}{1-\beta^{X_1}}\right) < +\infty,$$

the moment of order s of the random variable I can be expressed as

$$\mathbb{E}\left(I^s\right) = \Gamma(s+1)\prod_{k=1}^{+\infty}\frac{\phi(s+k)}{\phi(k)}, \qquad (14)$$

where $\phi(u) = 1 - \mathbb{E}\left(\beta^{uX_1}\right)$ for $u \geq \min(s,0)$.

The integer moments of the variable I can naturally be used to get a representation of the Laplace transform of I. We eventually get the following representation.

Proposition 7 *The Laplace transform of I is given for $\lambda \in [0,1[$ by*

$$\mathbb{E}\left(e^{-\lambda I}\right) = \sum_{n=0}^{+\infty}\frac{(-\lambda)^n}{\prod_{k=1}^n\left(1-\mathbb{E}(\beta^{kX})\right)}. \qquad (15)$$

3.2 Special cases

If the random variable X_1 has a rational generating function, i.e. there exist two polynomials P and Q such that $\mathbb{E}\left(z^{X_1}\right) = P(z)/Q(z)$, then, for some $a_1,\ldots a_M$, $b_1,\ldots$, $b_N \in \mathbb{C}$

$$1 - \mathbb{E}\left(z^{X_1}\right) = \frac{(1-z)\prod_{j=1}^M(1-b_jz)}{\prod_{i=1}^N(1-a_iz)}$$

A direct consequence of equation (15) is the following representation for the Laplace transform of random variable I:

$$\mathbb{E}\left(e^{-\lambda I}\right) = \sum_{n=0}^{+\infty}\frac{(a_1\beta;\beta)_n\ldots(a_M\beta;\beta)_n}{(b_1\beta;\beta)_n\ldots(b_N\beta;\beta)_n}\,\frac{(-\lambda)^n}{(\beta;\beta)_n} \qquad (16)$$

where, for $x \in \mathbb{C}$, $q \in [0,1[$, $(x;q)_n$ is defined by

$$(x;q)_n = (1-x)(1-xq)\ldots\left(1-xq^{n-1}\right)$$

for $k \geq 1$ and $(a;q)_0 = 1$. Expression (16) for the Laplace transform can be transformed so that it can be expressed as a q-hypergeometric function (see [19] for a tutorial of q-special functions). This suggests that q-calculus is the natural setting to study the density of exponential integral functionals for discontinuous Lévy processes. See also Bertoin *et al.* [20] for some developments in this domain, when the L—'evy process $(\xi(t))$ is purely Poisson.

Under some special assumptions, the probability density function of the random variable I can be explicitly computed. Let us first consider the case when X_1 has a shifted geometric distribution, that is, for some $a < 1$ and $n \geq 1$,

$$\mathbb{P}(X_1 = n) = a^{n-1}(1-a).$$

We assume in a first step that a is not a power of β. In that case, we can state the following result.

Proposition 8 *If X_1 has a shifted geometric distribution with parameter a, the probability density function (pdf) of random variable I is given by*

$$h(x) = \frac{1}{(\beta;\beta)_\infty}\sum_{n=0}^{+\infty}\frac{(a\beta^{-n+1};\beta)_\infty}{(1/\beta;1/\beta)_n}\,\beta^{-n}e^{-\beta^{-n}x} \qquad (17)$$

Proof. For $|z| \leq 1$,

$$1 - \mathbb{E}\left(z^X\right) = \frac{1-z}{1-az},$$

and from Equation (16), one gets the relation

$$\mathbb{E}\left(e^{-\lambda I}\right) = \sum_{n=0}^{+\infty}\frac{(a\beta;\beta)_n}{(\beta;\beta)_n}(-\lambda)^n.$$

Applying the q-Binomial Theorem:

$$\sum_{n=0}^{+\infty}(a;q)_k\frac{x^k}{(q;q)_k} = \frac{(ax;q)_\infty}{(x;q)_\infty}, \qquad (18)$$

yields

$$\mathbb{E}\left(e^{-\lambda I}\right) = \frac{(-\lambda a\beta;\beta)_\infty}{(-\lambda;\beta)_\infty}.$$

Therefore, the Laplace transform I has simple poles at the points $-\beta^{-n}$, $n \geq 0$, and the residue at point $-\beta^{-n}$ is

$$\frac{\beta^{-n}(a\beta^{-n+1};\beta)_\infty}{(\beta;\beta)_\infty \prod_{k=1}^n (1-\beta^{-k})} = \frac{\beta^{-n}(a\beta^{-n+1};\beta)_\infty}{(\beta;\beta)_\infty (1/\beta;1/\beta)_n}.$$

The knowledge of the poles and the residues of the Laplace transform yields the pdf of the random variable I. This completes the proof. □

Note also that if a is a power of β, say, $a = \beta^p$ for some $p \geq 1$, the Laplace transform of I is a rational function given by

$$\mathbb{E}\left(e^{-\lambda I}\right) = \frac{1}{(-\lambda;\beta)_{p+1}}$$

and thus,

$$h(x) = \sum_{n=0}^{p} \frac{1}{(1/\beta;1/\beta)_n(\beta,\beta)_{p-n}} \, \beta^{-n} e^{-\beta^{-n}x}.$$

To conclude this section, let us consider the case when X_1 can take two values, namely $\mathbb{P}(X_1 = 1) = p$ and $\mathbb{P}(X_1 = 2) = 1-p$ so that $1 - \mathbb{E}(z^{X_1}) = (1-z)(1+(1-p)z)$.

Proposition 9 *When X_1 takes the values 1 and 2 with probability p and $1-p$, respectively, the pdf of the random variable I is given for $x \geq 0$ by*

$$h(x) = \sum_{n=0}^{+\infty} r_n \, \beta^{-n} e^{-\beta^{-n}x},$$

where

$$r_n = \frac{1}{(-(1-p)\beta;\beta)_\infty(\beta;\beta)_\infty} \times \sum_{m=0}^{n} \frac{(-1)^m(1-p)^m}{(1/\beta;1/\beta)_m(1/\beta;1/\beta)_{n-m}}. \quad (19)$$

Proof. From relation (15), the Laplace transform of the random variable I is is given by

$$\mathbb{E}\left(e^{-\lambda I}\right) = \sum_{n=0}^{+\infty} \frac{(-\lambda)^n}{(\beta;\beta)_n(-\beta(1-p);\beta)_n}.$$

Using the elementary relation

$$(z;q)_n = \frac{(z;q)_\infty}{(q^n z;q)_\infty}$$

yields

$$\mathbb{E}\left(e^{-\lambda I}\right) = \frac{1}{(-(1-p)\beta;\beta)_\infty} A(\lambda),$$

where

$$A(\lambda) = \sum_{n=0}^{+\infty} \frac{(-(1-p)\beta^{n+1};\beta)_\infty}{(\beta;\beta)_n} (-\lambda)^n.$$

From Euler's identity:

$$\sum_{n=0}^{+\infty} (-1)^n q^{n(n-1)/2} \frac{x^n}{(q;q)_n} = (x;q)_\infty,$$

one gets

$$A(\lambda) = \sum_{m=0}^{+\infty} \beta^{m(m-1)/2} \frac{(1-p)^m \beta^m B_m(\lambda)}{(\beta;\beta)_m}$$

with

$$B_m(\lambda) = \sum_{n=0}^{\infty} \beta^{nm} \frac{(-\lambda)^n}{(\beta;\beta)_n} = \frac{1}{(-\lambda\beta^m;\beta)_\infty},$$

where we have used the second Euler's identity:

$$\sum_{n=0}^{+\infty} \frac{x^n}{(q;q)_n} = \frac{1}{(x;q)_\infty}$$

in the last step. Hence,

$$A(\lambda) = \sum_{m=0}^{+\infty} \beta^{m(m-1)/2} \frac{(1-p)^m \beta^m}{(\beta;\beta)_m(-\lambda\beta^m;\beta)_\infty},$$

the above term being equal to

$$\frac{1}{(-\lambda;\beta)_\infty} \sum_{m=0}^{+\infty} \beta^{m(m+1)/2} \frac{(1-p)^m(-\lambda;\beta)_m}{(\beta;\beta)_m}.$$

It follows that the Laplace transform of the random variable I has simple poles at the points $-\beta^{-n}$, $n \geq 0$. The residue of the Laplace transform at $-\beta^{-n}$ is equal to $r_n\beta^{-n}$, where r_n is given by equation (19). This completes the proof. □

Similar derivations can be carried out when the generating function of X_1 is of the form

$$1 - \mathbb{E}\left(z^{X_1}\right) = \frac{(1-z)\prod_{j=1}^N (1-b_j z)}{(1-az)},$$

but calculations rapidly become intricate and it is then very difficult to derive qualitative results on the random variable I (see [12] for further examples). Instead, it is possible to establish a closed formula for the mean throughput of the TCP connection and to get stochastic ordering results. This point is addressed in the next section.

4 Throughput of the connection

4.1 Computation of the throughput

In this section, the random variable X_1 takes integer values and is greater than or equal to 1. The loss rate of packets is of order $\alpha\mathbb{E}(X_1)$ and the throughput of the TCP connection is by definition

$$\rho^\alpha = \lim_{n\to+\infty} \frac{1}{nRTT}\sum_{i=1}^{n} W_k^\alpha = \frac{1}{RTT}\mathbb{E}\left(W_\infty^\alpha\right). \quad (20)$$

In this definition, the round trip time RTT is assumed to be a constant. In reality, RTT is affected by random fluctuations due, for instance, to queueing in routers. Thus, RTT should be considered as a random variable, varying around a mean value. As it will be seen in Section 5, assuming that a constant RTT is by no means restrictive and random fluctuations could easily be accounted for. In the following, we take RTT constant and equal to unity.

Using the embedded Markov chain (V_n^α), the throughput can also be written as

$$\begin{aligned}\rho^\alpha &= \frac{\mathbb{E}\left(\displaystyle\sum_{k=0}^{G^\alpha_{V^\alpha_\infty}-1}\left(V_\infty^\alpha + k\right)\right)}{\mathbb{E}\left(G^\alpha_{V^\alpha_\infty}\right)} \\ &= \frac{\mathbb{E}\left(2G^\alpha_{V^\alpha_\infty}V^\alpha_\infty + \left(G^\alpha_{V^\alpha_\infty}\right)^2\right)}{2\mathbb{E}\left(G^\alpha_{V^\alpha_\infty}\right)} - \frac{1}{2}.\end{aligned}$$

By multiplying this identity by the square root of the loss rate of packets, i.e. by $\sqrt{\alpha\mathbb{E}(X_1)}$, Theorem 2 shows that the convergence

$$\begin{aligned}\overline{\rho}_{X_1} &\stackrel{\text{def.}}{=} \lim_{\alpha\to 0}\sqrt{\alpha\mathbb{E}(X_1)}\rho^\alpha \\ &= \sqrt{\mathbb{E}(X_1)}\,\frac{\mathbb{E}\left(2\overline{G}_{\overline{V}_\infty}\overline{V}_\infty + \overline{G}^2_{\overline{V}_\infty}\right)}{2\mathbb{E}\left(\overline{G}_{\overline{V}_\infty}\right)} \quad (21)\end{aligned}$$

holds.

By using repeatedly equation (6), we get on the one hand, by using the fact that

$$2\overline{G}_{\overline{V}_\infty}\overline{V}_\infty + \overline{G}^2_{\overline{V}_\infty} = \left(\overline{V}_\infty + \overline{G}_{\overline{V}_\infty}\right)^2 - \overline{V}^2_\infty,$$

the relation

$$\mathbb{E}\left(2\overline{G}_{\overline{V}_\infty}\overline{V}_\infty + \overline{G}^2_{\overline{V}_\infty}\right) = \mathbb{E}\left(\overline{V}^2_\infty\right)\frac{1-\mathbb{E}\left(\delta^{2X_1}\right)}{\mathbb{E}\left(\delta^{2X_1}\right)}$$

and on the other hand,

$$\begin{aligned}\mathbb{E}\left(\overline{V}^2_\infty\right) &= 2\frac{\mathbb{E}\left(\delta^{2X_1}\right)}{1-\mathbb{E}\left(\delta^{2X_1}\right)}, \\ \mathbb{E}\left(\overline{G}_{\overline{V}_\infty}\right) &= \frac{1-\mathbb{E}\left(\delta^{X_1}\right)}{\mathbb{E}\left(\delta^{X_1}\right)}\mathbb{E}\left(\overline{V}_\infty\right).\end{aligned}$$

These last three identities show that relation (21) can be rewritten as

$$\overline{\rho}_{X_1} = \frac{\sqrt{\mathbb{E}(X_1)}\mathbb{E}\left(\delta^{X_1}\right)}{\left(1-\mathbb{E}\left(\delta^{X_1}\right)\right)\mathbb{E}\left(\overline{V}_\infty\right)}. \quad (22)$$

The next proposition gives an explicit formula for the asymptotic throughput.

Theorem 10 *The asymptotic throughput of the TCP connection in a correlated loss model by means of the random variable X_1 is given by*

$$\begin{aligned}\overline{\rho}_{X_1} &= \lim_{\alpha\to 0}\sqrt{\alpha\mathbb{E}(X_1)}\rho^\alpha \\ &= \sqrt{\frac{2\mathbb{E}(X_1)}{\pi}}\prod_{n=1}^{+\infty}\frac{1-\mathbb{E}\left(\delta^{2nX_1}\right)}{1-\mathbb{E}\left(\delta^{(2n-1)X_1}\right)}. \quad (23)\end{aligned}$$

Proof. Since $\overline{V}^2_\infty = 2\delta^{2X_1}I$, where I is solution to equation (7), we have $\mathbb{E}\left(\overline{V}_\infty\right) = \sqrt{2}\mathbb{E}\left(\delta^{X_1}\right)E\left(\sqrt{I}\right)$. Formula (14) yields

$$E\left(\sqrt{I}\right) = \Gamma(3/2)\prod_{n=1}^{+\infty}\frac{1-\mathbb{E}\left(\delta^{(1+2n)X_1}\right)}{1-\mathbb{E}\left(\delta^{2nX_1}\right)},$$

since $\Gamma(3/2) = \sqrt{\pi}/2$. Equation (22) gives the desired formula. This completes the proof. □

4.2 Impact of correlations

In this section, we investigate the impact of correlations in the loss process on the throughput $\overline{\rho}_X$. In particular, we pay special attention to the variance of X, which is to some extent a measure reflecting how loss events are correlated. To carried out comparisons, we assume that the mean loss rate ($\alpha\mathbb{E}(X_1)$ is fixed.

Before proceeding with the main results of this section, let us recall the definitions of stochastic and concave orders. (See Stoyan [21] for the basic definitions and results on stochastic orderings.)

Definition 11 *The order relations $\leq_{st}$ and $\leq_{cv}$ are defined as follows: for two random variables X and Y taking values in $\mathbb{R}$,*

1. *the inequality $X \leq_{st} Y$ holds when*

   $$\mathbb{E}(f(X)) \leq \mathbb{E}(f(Y))$$

   is true for any non-decreasing function on $\mathbb{R}$. Equivalently, $X \leq_{st} Y$ if and only if the inequality $\mathbb{P}(X \geq a) \leq \mathbb{P}(Y \geq a)$ holds for any $a \in \mathbb{R}$.

2. *the inequality $X \leq_{cv} Y$ holds when*

   $$\mathbb{E}(f(X)) \leq \mathbb{E}(f(Y))$$

   is true for any non-decreasing concave function on $\mathbb{R}$. Equivalently, $X \leq_{cv} Y$ if and only if the inequality $\mathbb{E}((a-Y)^+) \leq \mathbb{E}((a-X)^+)$ holds for any $a \in \mathbb{R}$.

If X and X' [resp. Y and Y'] are independent real random variables such that $X \leq_{cv} Y$ and $X' \leq_{cv} Y'$ then $X+X' \leq_{cv} Y+Y'$. Indeed, for $a \in \mathbb{R}$,

$$\begin{aligned}
&\mathbb{E}\left((a-(Y+Y'))^+\right) \\
&= \int_{\mathbb{R}} \mathbb{E}\left((a-y-Y')^+\right) \mathbb{P}(Y \in dy) \\
&\qquad \leq \int_{\mathbb{R}} \mathbb{E}\left((a-y-X')^+\right) \mathbb{P}(Y \in dy).
\end{aligned}$$

The last term of the above inequality is equal to

$$\begin{aligned}
&\int_{\mathbb{R}} \mathbb{E}\left((a-x'-Y)^+\right) \mathbb{P}(X' \in dx') \\
&\leq \int_{\mathbb{R}} \mathbb{E}\left((a-x'-X)^+\right) \mathbb{P}(X' \in dx') \\
&\qquad = \mathbb{E}\left((a-(X+X'))^+\right).
\end{aligned}$$

Similarly, under the same independence assumptions, if X, X', Y and Y' are such that $X \leq_{st} Y$ and $X' \leq_{st} Y'$ then $X+X' \leq_{st} Y+Y'$.

If X and Y are random variables such that $X \leq_{st} Y$, it is possible to construct a common probability space for two random variables X' and Y' having respectively the same distribution as X and Y, and such that the inequality $X' \leq Y'$ holds almost surely. Hence, the inequality

$$\frac{\overline{\rho}_X}{\sqrt{\mathbb{E}(X)}} \geq \frac{\overline{\rho}_Y}{\sqrt{\mathbb{E}(Y)}} \tag{24}$$

should hold since the model with X experiences less losses than the model with Y. From formula (23), this property is not obvious at all. To facilitate comparisons, we rewrite the asymptotic throughput as follows:

Proposition 12 *The asymptotic throughput $\overline{\rho}_{X_1}$ can be written as*

$$\overline{\rho}_{X_1} = \sqrt{\frac{2\mathbb{E}(X_1)}{\pi}} \exp\left(\sum_{k=1}^{+\infty} \frac{1}{k}\, \mathbb{E}\left(\frac{1}{1+\delta^{-S_k}}\right)\right), \tag{25}$$

where $(S_n) = (X_1 + \cdots + X_n)$ is the random walk associated to the i.i.d. sequence (X_n).

Proof. From Formula (23), one gets (recall that $X_1 \geq 1$),

$$\begin{aligned}
&\log\left(\sqrt{\frac{\pi}{2\mathbb{E}(X_1)}}\overline{\rho}_{X_1}\right) \\
&\qquad = \sum_{n=1}^{+\infty} (-1)^n \log\left(1 - \mathbb{E}\left(\delta^{nX_1}\right)\right).
\end{aligned}$$

By expanding the logarithmic terms, the above expression is equal to

$$\begin{aligned}
&-\sum_{n=1}^{+\infty}\sum_{k=1}^{+\infty} (-1)^n \frac{1}{k} \left(\mathbb{E}\left(\delta^{nX_1}\right)\right)^k \\
&= -\sum_{k=1}^{+\infty}\sum_{n=1}^{+\infty} \frac{1}{k}(-1)^n \mathbb{E}\left(\delta^{nS_k}\right) \\
&\qquad = \sum_{k=1}^{+\infty} \frac{1}{k}\mathbb{E}\left(\frac{\delta^{S_k}}{1+\delta^{S_k}}\right),
\end{aligned}$$

and formula (25) follows. This completes the proof. □

If $X \leq_{\text{st}} Y$, the same property holds for the associated random walks (S_n^X) and (S_n^Y), i.e. for any $n \geq 1$, $S_n^X \leq_{\text{st}} S_n^Y$. Hence, equation (25) directly shows that inequality (24) holds, as expected. The following proposition gives a stronger result in this domain.

Proposition 13 *The asymptotic throughput function*

$$Z \to \overline{\rho}_Z / \sqrt{\mathbb{E}(Z)}$$

is a non-increasing function for the concave order, i.e. if X and Y are real random variables,

$$X \leq_{cv} Y \quad \Rightarrow \quad \frac{\overline{\rho}_X}{\sqrt{\mathbb{E}(X)}} \geq \frac{\overline{\rho}_Y}{\sqrt{\mathbb{E}(Y)}}. \tag{26}$$

In particular, when $\mathbb{E}(X) = \mathbb{E}(Y)$, $X \leq_{cv} Y$ implies $\overline{\rho}_X \geq \overline{\rho}_Y$.

Proof. If (S_n^X) [resp. (S_n^Y)] is the random walk associated to the variable X [resp. Y], by induction, with the help of the remark below Definition 11, it is easily seen that for $n \geq 1$ $S_n^X \leq_{\text{cv}} S_n^Y$.

The function $a \to 1/(\delta^a + 1)$ being non-decreasing and concave on $\mathbb{R}_+$, one gets that for $n \geq 1$,

$$\mathbb{E}\left(\frac{1}{\delta^{S_n^X}+1}\right) \leq \mathbb{E}\left(\frac{1}{\delta^{S_n^Y}+1}\right)$$

and hence, by subtracting 1 on both sides of the above inequality,

$$\mathbb{E}\left(\frac{1}{\delta^{-S_n^X}+1}\right) \geq \mathbb{E}\left(\frac{1}{\delta^{-S_n^Y}+1}\right).$$

Formula (25) shows that the last inequality implies that inequality (24) holds. The proof is done. □

The above proposition suggests that the greater is the variance of X the better is the asymptotic throughput. Jensen's inequality moreover gives that for any concave function f on $\mathbb{R}_+$,

$$\mathbb{E}(f(X)) \leq f(\mathbb{E}(X)),$$

hence $\mathbb{E}(X) \geq_{\text{cv}} X$. This implies in particular that

$$\overline{\rho}_X \geq \overline{\rho}_{\mathbb{E}(X)},$$

where for $t > 0$, $\overline{\rho}_t$ denotes the asymptotic throughput for the random variable constant equal to t. In other words, the asymptotic throughput with the loss process associated to X is greater than the throughput of an uncorrelated model but with a multiplicative decay $\delta^{\mathbb{E}(X)}$.

Proposition 14 *For any integer valued random variable X,*

$$\overline{\rho}_X \geq \overline{\rho}_{\mathbb{E}(X)} = \sqrt{\frac{2\mathbb{E}(X)}{\pi}} \prod_{n=1}^{+\infty} \frac{1-\delta^{2n\mathbb{E}(X)}}{1-\delta^{(2n-1)\mathbb{E}(X)}}. \quad (27)$$

The function $t \to \overline{\rho}_t$ is non-decreasing; in particular, for any random variable $X \geq 1$,

$$\overline{\rho}_X \geq \overline{\rho}_1. \quad (28)$$

Proof. Only the non-decreasing property of $t \to \overline{\rho}_t$ has to be proved. Taking $g(t) = \log \overline{\rho}_t - \log(2/\pi)/2$ and using expression (25),

$$g(x) = \frac{1}{2}\log(x) + \sum_{n=1}^{+\infty} \frac{1}{n} \frac{1}{1+\delta^{-nx}},$$

$$g'(x) = \frac{1}{2x} + \sum_{n=1}^{+\infty} \log(\delta) \frac{\delta^{nx}}{(1+\delta^{nx})^2}.$$

A glance at the right hand side of the last inequality reveals that to prove the property, it is sufficient to show that the relation

$$x \sum_{n=1}^{+\infty} \frac{e^{-nx}}{(1+e^{-nx})^2} \leq \frac{1}{2}$$

holds for $x \geq 0$. Since

$$\sum_{n=1}^{+\infty} \frac{e^{-nx}}{(1+e^{-nx})^2} \leq \int_0^{+\infty} \frac{e^{-xy}}{(1+e^{-xy})^2}\, dy = \frac{1}{2x},$$

this is clearly true and the proof is done. □

According to Equation (28), formula (23) with $X_1 \equiv 1$ proved in Dumas *et al* [3] is thus a lower bound for the real throughput, when packet losses are correlated. Equation (28) of the above proposition shows that choosing an uncorrelated loss process underestimates the real performance of the TCP connection.

A possible intuitive explanation of this phenomenon is as follows: When there are x losses in some small time interval, the congestion window size is basically reduced by a factor δ^x. If x is not too small, then, due to the exponential decay, for any $y \geq x$, the quantities δ^x or δ^y are both very small. Hence, it is better to have a very large variability in the loss process: a large number of packets are lost but this event is very rare.

Proposition 13 shows that the asymptotic throughput is a non-decreasing functional, with respect to the concave order, for random variables X with the same mean value. When the mean values are different, the comparison turns out to be more difficult. Relation (28) is an example of such a comparison for deterministic variables.

To conclude this section, let us consider the case when X is geometrically distributed. In this context, it is possible to prove a monotonicity property of the asymptotic throughput, even if mean values are not equal. This kind of distribution has also another advantage: Since the number of local losses is believed to be sharply concentrated near small values (see Paxson [4]), the geometric distribution is a good candidate to describe the loss process.

Proposition 15 *If for $p \in [0, 1[$, G_p is a shifted geometrically distributed random variable with parameter p, i.e. $\mathbb{P}(G_p = n) = p^{n-1}(1-p)$ for $n \geq 1$, the function*

$$p \to \overline{\rho}_{G_p} = \sqrt{\frac{2}{\pi(1-p)}} \prod_{n=1}^{+\infty} \frac{1-p\delta^{2n-1}}{1-p\delta^{2n}} \frac{1-\delta^{2n}}{1-\delta^{2n-1}},$$

is convex and non-decreasing.

Proof. Equation (23) gives the relation

$$\sqrt{\frac{\pi(1-p)}{2}}\overline{\rho}_{G_p} = \prod_{n=1}^{+\infty} \frac{1-p\delta^{2n-1}}{1-p\delta^{2n}} \prod_{n=1}^{+\infty} \frac{1-\delta^{2n}}{1-\delta^{2n-1}}.$$

Hence,

$$\log\left(\frac{\overline{\rho}_{G_p}}{\overline{\rho}_{G_0}}\right) = -\frac{1}{2}\log(1-p) - \sum_{n=1}^{+\infty}(-1)^n \log\left(1-p\delta^n\right).$$

By expanding the logarithmic terms, the above expression is equal to

$$\sum_{k=1}^{+\infty} \frac{1}{2k}p^k + \sum_{n=1}^{+\infty}\sum_{k=1}^{+\infty}(-1)^n \frac{p^k}{k}\delta^{nk}$$
$$= \sum_{k=1}^{+\infty} \frac{p^k}{k}\left(\frac{1}{2} - \frac{\delta^k}{1+\delta^k}\right)$$
$$= \sum_{k=1}^{+\infty} \frac{p^k}{k} \frac{1-\delta^k}{2\left(1+\delta^k\right)}.$$

The last term defines a convex function of p and the proposition is proved. □

The case $p = 0$ corresponds to the uncorrelated case considered in Dumas *et al* [3]. From the above result, this case still appears as to be the worst case for the asymptotic throughput, even if mean loss rates are not constant. (The mean value $\mathbb{E}(G_p)$ is not a constant function of p.)

5 Further considerations

In this section, several aspects of TCP, which have not been explicitly considered up to now in the stochastic model analyzed in this paper, are discussed.

5.1 Finite maximal congestion window size

So far, it has been assumed that the sequence (W_n^α) can be increased without any bound. In practice, the congestion window size is upper bounded by a maximal value w_{max}^α, referred to as maximal congestion window size. In Dumas *et al.* [3] for independent packet loss, the stationary behavior of the asymptotic sequence $(\overline{V}_n)$ is described when $w_{\text{max}}^\alpha \sim \overline{w}_{\text{max}}/\sqrt{\alpha}$.

For the present loss model, a similar analysis can also be done. The corresponding asymptotic sequence $(\overline{V}_n)$ satisfies the relation

$$\overline{V}_{n+1}^2 \stackrel{\text{dist.}}{=} \delta^{2X_n} \min\left(\overline{V}_n^2 + 2E_n, \overline{w}_{\text{max}}\right),\ n \geq 1,$$

where (X_n) and (E_n) are i.i.d. independent sequences, E_1 is exponentially distributed with parameter 1. This sequence converges in distribution to a random variable $\overline{V}_\infty$ such that

$$\overline{V}_\infty = \sqrt{\inf_{n\geq 0}\left(\delta^{2S_n}\overline{w}_{\text{max}} + 2\sum_{i=1}^{n}\delta^{2S_i}E_i\right)},$$

where (S_n) is the random walk associated to (X_n). It is possible to carry out exact computations for the density of the random variable $\overline{V}_\infty^2$ but they rapidly become intricate.

5.2 Timeouts

In the model considered here only losses that can be handled by the Fast Recovery algorithm have been considered (see Jacobson [22]). When there is a timeout, the congestion window size W is set equal to 1. In our limiting process this amounts to set $\overline{W}$ to 0. Thus, the evolution equation

$$\overline{V}_1^2 \stackrel{\text{dist.}}{=} \delta^{2X_1}\left(\overline{V}_0^2 + 2E_1\right)$$

is still valid provided that the value $+\infty$ is allowed for X_1. The quantity $\mathbb{P}(X_1 = +\infty)$ is then interpreted as the probability of a timeouts.

Formula (23) for the asymptotic throughput becomes

$$\overline{\rho}_{X_1} = \sqrt{\frac{2(q\mathbb{E}(X_1\,|X_1 < +\infty) + (1-q))}{\pi}} \times \prod_{n=1}^{+\infty}\frac{1 - q\mathbb{E}\left(\delta^{2nX_1}|X_1 < +\infty\right)}{1 - q\mathbb{E}\left(\delta^{(2n-1)X_1}|X_1 < +\infty\right)}, \quad (29)$$

where $q = \mathbb{P}(X_1 < +\infty)$ and the probability of a timeout is $1-q$.

In the simple uncorrelated case, $\mathbb{P}(X_1 = 1) = q = 1 - \mathbb{P}(X_1 = +\infty)$, the above formula is

$$\begin{aligned}\overline{\rho}_1 &= \sqrt{\frac{2}{\pi}}\prod_{n=1}^{+\infty}\frac{1-q\delta^{2n}}{1-q\delta^{2n-1}}\\ &= \sqrt{\frac{2}{\pi}}\exp\left(\sum_{k=1}^{+\infty}\frac{q^k}{k}\,\frac{\delta^k}{1+\delta^k}\right),\end{aligned}$$

where the last identity is proved by using the same arguments as in the proof of Proposition 15. For TCP ($\delta = 1/2$), as q varies from 1 to 0, $\overline{\rho}_1$ decreases from 1.309 to $.798$.

5.3 Slow start phase

In the current implementations of TCP, if a time out occurs when the window size is $W = w$, the congestion window is set equal to one and the *Slow Start* algorithm is then used (see Stevens [13]). This algorithm works as follows: A parameter called *Slow Start Threshold* T_{ss} is fixed to $\lfloor w/2\rfloor$ and the congestion window size is set to 1. The congestion window size is then doubled after each RTT as long as its value has not reached T_{ss}:

$$W \to \begin{cases} 2W & \text{if no loss among the } W \text{ packets,} \\ 1, & \text{otherwise.}\end{cases} \quad (30)$$

when W is greater than T_{ss}, the congestion avoidance algorithm is then used.

Ferguson [23] analyses a related stochastic model of the slow start algorithm. In the probabilistic model investigated here (and also in Dumas *et al.* [3]), this feature of the TCP is taken into account.

Indeed, in the setting of this paper, the slow start algorithm can be included without changing the results obtained so far. Recall that by Proposition 4, the chain (W_n^α) is properly renormalized as

$$\left(\sqrt{\alpha}W^\alpha_{\lfloor t/\sqrt{\alpha}\rfloor}\right) \quad (31)$$

to get the asymptotic Markov process $(\overline{W}(t))$. To show that the slow start algorithm can be neglected, it is sufficient to show that if $W_0^\alpha = 1$ and the transitions (30) are used, the mean time T_x to reach the level $x/\sqrt{\alpha}$ is $o(1/\sqrt{\alpha})$. In other words, the time necessary to reach a slow start threshold of the order $x/\sqrt{\alpha}$ is negligible in the time scale defined by (31), and hence, the slow start period vanishes because of the time scale.

Proposition 16 *If $W_0^\alpha = \lfloor x/\sqrt{\alpha}\rfloor$, (W_n^α) is a TCP session starting after a loss and T^α is the first index n when the congestion avoidance algorithm is used, then*

$$\lim_{\alpha\to 0}\mathbb{P}\left(\frac{T^\alpha}{-\log_2\sqrt{\alpha}} \leq 1\right) = 1$$

The variable T^α is of the order $-\log_2\sqrt{\alpha}$, hence the interval $\{0,1,\ldots,T^\alpha\}$ (where the slow start algorithm is used) vanishes under the scaling (31) defined by: $t \to \lfloor t/\sqrt{\alpha}\rfloor$. Consequently, the slows start algorithm does not appear in the rescaled stochastic model.

Proof. Since the distribution of the size of a group of losses is independent of α, it can be assumed that the initial loss is the last loss of a group. (Recall that time is shrunk by $1/\sqrt{\alpha}$). The next loss will thus occur as in the independent loss model, where each packet has a probability $\exp(-\alpha)$ of being lost. If no loss occurs during the first $\log_2\lfloor x/\sqrt{\alpha}\rfloor$ steps, then necessarily $T^\alpha \leq \log_2\lfloor x/\sqrt{\alpha}\rfloor$, therefore

$$\mathbb{P}\left(T^\alpha \leq \log_2\lfloor x/\sqrt{\alpha}\rfloor\right) \geq \prod_{i=1}^{\log_2\lfloor x/\sqrt{\alpha}\rfloor} \exp\left(-\alpha 2^i\right)$$
$$= \exp\left(-\alpha(\lfloor x/\sqrt{\alpha}\rfloor - 1)\right),$$

since this last expression is converging to 1 as α tends to 0, the proposition is proved. □

The slow start algorithm does not play any role in this paper because only the transfer of an infinite file is considered. The transient periods where the algorithm recover from a loss are negligible from this point of view. The problem is entirely different when "small transfers" (say, less than a few tens of packets) are considered. For these connections, the reverse situation prevails; they are actually finished before the congestion avoidance algorithm starts.

5.4 Variable RTT's

In Section 4 devoted to the asymptotic throughput of the TCP connection, the round trip times have been assumed constant. In practice, this is not the case since packets experience variable delays in buffers of the various routers along their paths.

If, for $n \in \mathbb{N}$, R_n is the delay experienced by the packets of the nth round trip, the random variables W_n^α and R_n are correlated random variables. The average throughput after the nth round trip is

$$\sum_{i=1}^{n} W_i^\alpha \Big/ \sum_{i=1}^{n} R_i^\alpha.$$

If we assume that, asymptotically, the sequence (R_n) is stationary and ergodic, the pointwise ergodic theorem shows that, almost surely,

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{i=1}^{n} W_i^\alpha = \mathbb{E}(W_\infty^\alpha)$$

and

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{i=1}^{n} R_i = \mathbb{E}(R_\infty),$$

the asymptotic throughput is therefore $\mathbb{E}(W_\infty)/\mathbb{E}(R_\infty)$. Notice that the dependence between the two sequences (W_n^α) and (R_n) does not play any role for this result. Hence, up to the constant $1/\mathbb{E}(R_\infty)$, and under a mild assumption on the stationarity of (R_n), even when the RTT are variable, Theorem 10 gives the correct expression for the asymptotic throughput of a persistent TCP connection.

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