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On the Number of Maximal Bipartite Subgraphs of a Graph*

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Abstract

We show new lower and upper bounds on the number of maximal induced bipartite subgraphs of graphs with n vertices. We present an infinite family of graphs having $105^{n/10} \approx 1.5926^n$ such subgraphs, which improves an earlier lower bound by Schiermeyer (1996). We show an upper bound of $n \cdot 12^{n/4} \approx n \cdot 1.8613^n$ and give an algorithm that lists all maximal induced bipartite subgraphs in time proportional to this bound. This is used in an algorithm for checking 4-colourability of a graph running within the same time bound.

1 Notation

In this paper we look at simple, undirected graphs $G = (V, E)$ and induced subgraphs of these. The subgraph induced by a subset of the vertices $S \subseteq V$ is denoted $G[S]$. If G' is a subgraph we let $V(G')$ denote the vertices of G' and call $G[V \setminus V(G')]$ the *remaining graph*. A *maximal k-colourable subgraph* of a graph is an induced k-colourable subgraph, contained in no other induced k-colourable subgraph. If $k = 1$ or $k = 2$ we use the terminology *maximal independent set* or *maximal bipartite subgraph*, respectively. We let $n = |V|$.

The running times of all algorithms in this paper are of the form $\mathcal{O}(p(n) \cdot c^n)$ where p is a polynomial and c a constant. If we round off c to a larger value we can ignore the polynomial factor, and we say that the running time is proportional to c^n.

2 Introduction

As a natural generalisation of the use of maximal independent sets in colouring algorithms (see e.g. Lawler [5]), Schiermeyer [8] considers the use of maximal bipartite subgraphs. He devises an algorithm that checks whether a graph is 4-colourable, by generating all maximal bipartite subgraphs and checking if any of the remaining graphs are 2-colourable. It runs in time proportional to that of generating all maximal bipartite subgraphs of the graph. Schiermeyer also

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shows lower and upper bounds on the number of maximal bipartite subgraphs. He constructs an infinite family of graphs all having $10^{n/5} \approx 1.5849^n$ maximal bipartite subgraphs, which shows the lower bound. He states a matching upper bound.

In this paper we show a new lower bound of $105^{n/10} \approx 1.5926^n$ on the number of maximal bipartite subgraphs by providing a new infinite family of graphs. This invalidates the upper bound of Schiermeyer. We prove instead an upper bound of $n \cdot 12^{n/4} \approx n \cdot 1.8613^n$ and present a generating algorithm running in time proportional to this bound. The same time bound is achieved for 4-colouring.

3 Lower bound

We show a lower bound on the number of maximal bipartite subgraphs in any graph by providing and infinite family of graphs with many maximal bipartite subgraphs. The infinite family consists of disconnected copies of a single graph, the k'th one having k copies. The maximal bipartite subgraphs of a disconnected graph are exactly the union of one maximal bipartite subgraph of each connected component. Their number thus equals the product of the number of maximal bipartite subgraphs of each component. Schiermeyer [8] uses K_5 (having ten maximal bipartite subgraphs) to generate his infinite family, resulting in a lower bound of $10^{n/5} \approx 1.5849^n$.

Theorem 1. *There exists an infinite family of graphs all having $105^{n/10} \approx 1.5926^n$ maximal bipartite subgraphs.*

Proof. The generating graph for our infinite family of graphs is seen in Figure 1.[1] Let a *pair*

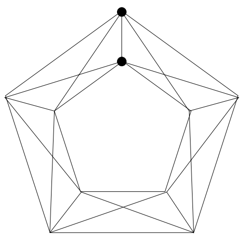
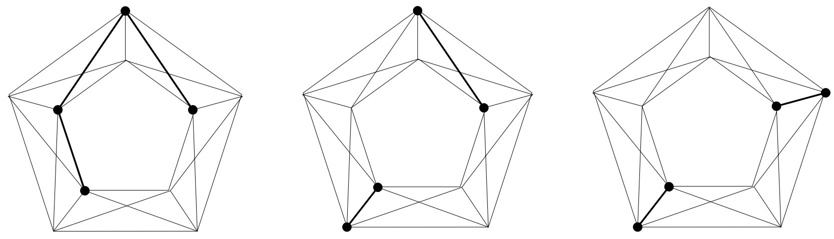


Figure 1: Generating graph with a pair marked.

denote a vertex on the outer 5-cycle and the nearest vertex on the inner 5-cycle (see Figure 1). The graph has $5 \cdot 2^4 = 80$ maximal bipartite subgraphs containing one vertex from four of the pairs (see Figure 2(a)), $5 \cdot 2^2 = 20$ containing one pair and one vertex from each of the opposite pairs (see Figure 2(b)) and five containing two pairs (see Figure 2(c)). Thus it has 105 maximal bipartite subgraphs and gives a lower bound of $105^{n/10} \approx 1.5926^n$ using multiple copies.

□

[1] This graph is also found in a list of counterexamples to graph conjectures [3] made by Graffiti [2], a program generating such conjectures automatically. We could not find any information about which conjecture it disproved.



(a) (b) (c)

Figure 2: The three different types of maximal bipartite subgraphs.

4 Upper bound

When we look at maximal k-colourable subgraphs it is easier to look at a specific k-colouring guaranteed to exist by the following lemma:

Lemma 1. *Let M be a maximal k-colourable subgraph of a graph $G = (V, E)$. Then the vertices of M can be split into colour classes $C_1, C_2, \dots, C_k$ of non-increasing sizes s.t. for all i, j with $0 \le i < j \le k$, $G[C_{i+1} \cup C_{i+2} \cup \dots \cup C_j]$ is a maximal $(j-i)$-colourable subgraph of $G[V \setminus (C_1 \cup C_2 \cup \dots \cup C_i)]$.*

Proof. Look at all possible k-colourings of M having the colour classes sorted in non-increasing order. Label each with a vector, having as coordinates the sizes of the colour classes in reverse order, i.e. the smallest one first. We claim that the lexicographically smallest labelled colouring $C_1, C_2, \dots, C_k$ satisfies the conditions of the lemma.

Suppose conversely that there exists i, j with $0 \le i < j \le k$ s.t. $G[C_{i+1} \cup C_{i+2} \cup \dots \cup C_j]$ is *not* a maximal $(j-i)$-colourable subgraph of $G[V \setminus (C_1 \cup C_2 \cup \dots \cup C_i)]$. Since it is $(j-i)$-colourable there exists a vertex v in the remaining graph s.t. $G[C_{i+1} \cup C_{i+2} \cup \dots \cup C_j \cup \{v\}]$ remains $(j-i)$-colourable. Now $v \in M$; otherwise, $M \cup \{v\}$ is k-colourable, and thus contradicts the maximality of M. Then $v \in C_l$ for some $l > j$. Pick a $(j-i)$-colouring of $G[C_{i+1} \cup C_{i+2} \cup \dots \cup C_j \cup \{v\}]$ together with the colouring $C_1, \dots, C_i, C_{j+1}, \dots, C_{l-1}, C_l \setminus \{v\}, C_{l+1}, \dots, C_k$ of the remaining graph. They form a k-colouring of M. Now the l'th colour class is smaller than in the original colouring, and the proceeding ones are of the same size. Thus the vector having as coordinates the sizes of the colour classes in reverse order is lexicographically smaller than the label of the original colouring. Since sorting the vector only makes it smaller, the label of the new colouring is lexicographically smaller then the label of the original colouring. This is a contradiction, and thus the lemma is true. □

We are now in a position to prove our upper bound:

Theorem 2. *Any graph contains at most $n \cdot 12^{n/4} \approx n \cdot 1.8613^n$ maximal bipartite subgraphs. Moreover, there is an algorithm that takes as input a graph and outputs all its maximal bipartite subgraphs in time $\mathcal{O}(1.8613^n)$.*

Proof. Let G be an arbitrary graph and M a maximal bipartite subgraph thereof. Setting $i = 0, j = 1$ and $i = 1, j = 2$ in Lemma 1 we can assume that the vertices of M consists of a maximal independent set M_1 of G and a maximal independent set M_2 of the remaining graph having $|M_2| \leq |M_1|$. Thus to find all maximal bipartite subgraphs, our algorithm generates all maximal independent sets of G and for each generates all no larger maximal independent sets of the remaining graph. If their union is a maximal bipartite subgraph the algorithm outputs it.[2] Let $MIS_{\leq k}(G)$ ($MIS_{=k}(G)$) denote the set of all maximal independent sets of size at most (exactly) k of G. Then the number of maximal bipartite subgraphs of G is at most

$$\sum_{k=1}^{n} \sum_{M \in MIS_{=k}(G)} \left| MIS_{\leq k}\big(G[V \setminus V(M)]\big) \right|$$

We split the sum in two

$$\sum_{k=1}^{\lfloor \frac{n}{4} \rfloor} \sum_{M \in MIS_{=k}(G)} \left| MIS_{\leq k}\big(G[V \setminus V(M)]\big) \right| + \sum_{k=\lfloor \frac{n}{4} \rfloor + 1}^{n} \sum_{M \in MIS_{=k}(G)} \left| MIS_{\leq k}\big(G[V \setminus V(M)]\big) \right|$$

and use two bounds on the number of maximal independent sets. Eppstein [1] shows that $|MIS_{\leq k}(G)| \leq 3^{4k-n}4^{n-3k}$ for any graph G. Moon and Moser [6] shows that any graph can have at most $3^{n/3}$ maximal independent sets in total. Since $|MIS_{=k}(G)| \leq |MIS_{\leq k}(G)|$ we get that the sum is at most

$$\sum_{k=1}^{\lfloor \frac{n}{4} \rfloor} 3^{4k-n}4^{n-3k}3^{4k-(n-k)}4^{(n-k)-3k} + \sum_{k=\lfloor \frac{n}{4} \rfloor + 1}^{n} 3^{4k-n}4^{n-3k}3^{(n-k)/3}$$

The expression in the first sum is increasing as a function of k, and the expression in the second is decreasing. Both equals $12^{n/4}$ if $k = n/4$, thus the sum is at most $n \cdot 12^{n/4} \approx n \cdot 1.8613^n$.

All maximal independent sets of a graph can be generated in time proportional to their number, see e.g. Johnson et al. [4]. Those of size at most k can be generated in time $\mathcal{O}\left(3^{4k-n}4^{n-3k}\right)$, as shown by Eppstein [1]. Thus in our algorithm we generate all maximal independent sets of the graph in time proportional to their number. For those of size $k \leq n/4$ we use the algorithm of Eppstein to generate the maximal independent sets of size at most k of the remaining graph, and for those of size $k > n/4$ we generate all maximal independent sets of the remaining graph in time proportional to their number. We thus get a time bound proportional to 1.8613^n □

5 Colouring

Suppose that a graph $G = (V, E)$ is k-colourable. By setting $M = V$, $i = 0$ and $j = 2$ in Lemma 1 we obtain that the graph has a k-colouring consisting of a maximal bipartite subgraph s.t. the remaining graph is $(k-2)$-colourable. Thus the algorithm of Schiermeyer [8]

[2] The union is not necessarily a maximal bipartite subgraph: In the 6-cycle, two opposite vertices form a maximal independent set, but their union with a maximal independent set of the remaining graph do not form a maximal bipartite subgraph.

checks whether a graph is k-colourable by generating all maximal bipartite subgraphs and checking whether the remaining graphs are $(k-2)$-colourable. The time complexity of checking k-colourability is proportional to the time complexity of generating all maximal bipartite subgraphs times the time complexity of checking $(k-2)$-colourability of the remaining graphs.

The time complexity of checking 4-colourability using the above algorithm is proportional to the time complexity of generating all maximal bipartite subgraphs, since 2-colourability can be checked in polynomial time. By Theorem 2 the running time is $\mathcal{O}(1.8613^n)$. This is not competitive as Nielsen [7] has a 4-colouring algorithm running in time $\mathcal{O}(1.7504^n)$. To improve the running time we need smaller upper bounds on the number of maximal bipartite subgraphs, while still being able to generate them in time proportional to their number.

6 Conclusion

We have shown that there can be at least $105^{n/10} \approx 1.5926^n$ and at most $n \cdot 12^{n/4} \approx n \cdot 1.8613^n$ maximal bipartite subgraphs of a graph, and they can be generated in time proportional to our upper bound. Maximal bipartite subgraphs can be used in 4-colouring algorithms, but to be competitive better upper bounds are needed.

We found our lower bound by testing all graphs of size $n \leq 10$ on a computer. This becomes infeasible even for slightly larger n, since the number of graphs grows very fast. To prove our upper bound we used bounds on the number of maximal independent sets of a graph. These bounds are tight, at least for the values where our expression attains its maximum. Thus new ideas are needed to prove better lower and upper bounds.

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