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On Radiocoloring Hierarchically Specified Planar Graphs: $\mathcal{PSPACE}$-completeness and Approximations

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Abstract. Hierarchical specifications of graphs have been widely used in many important applications, such as VLSI design, parallel programming and software engineering. A well known hierarchical specification model, considered in this work, is that of Lengauer introduced in [23, 21, 25], referred to as *L-specifications*. In this paper we discuss a natural restriction on the L-specifications resulting to graphs called Distance-3-Separated ($DS3$). This class is characterized by a polynomial time (to the size of the specification of the graph) testable combinatorial property.

In this work we study the Radiocoloring Problem (RCP) on $DS3$ L-specified hierarchical planar graphs. The RCP consists in assigning colors to the vertices of a graph, such that any two vertices of distance at most two get different colors. In its optimization version considered here, we wish to minimize the number of colors used. This problem is equivalent to the problem of vertex coloring the square of a graph G, G^2, where G^2 has the same vertex set as G and there is an edge between any two vertices of G^2 if their distance in G is at most 2.

We first show that RCP is $\mathcal{PSPACE}$-complete for $DS3$ L-specified hierarchical planar graphs. Second, we present a polynomial time 3-approximation algorithm for RCP as well as a more efficient 4-approximation algorithm for graphs of this class.

We note that, the best currently known approximation ratio for the RCP on ordinary (non-hierarchical) planar graphs is 2 ([1, 12]). Note also that the only known results on any kind of coloring problems have been shown for a special kind of hierarchical graphs (unit disk graphs).
Keywords: radiocoloring, hierarchical graphs, networks, $\mathcal{PSPACE}$-completeness, approximation algorithms, planarity.

1 Introduction, Our Results and Related Work

1.1 Motivation

Many practical applications of graph theory and combinatorial optimization in CAD systems, mechanical engineering, VLSI design, transportation networks, software engineering ([9]), finite analysis [20], and datalog queries [37] involve the processing large (but regular) objects constructed in a systematic manner from smaller and more manageable components. As a result, the graphs that abstract such circuits (designs) also have a regular structure and are defined in a systematic manner using smaller graphs. The methods for specifying such large but regular objects by small specification are referred to as *succinct specifications*. One way to succinctly represent objects is to specify the graph hierarchically. Hierarchical specifications are more concise in describing objects than ordinary graph representations. Henceforth, they allow the overall design of an object to be partitioned into a collection of modules,

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which are much more manageable task than producing the complete design in one step. A well known hierarchical specification model, considered in this work, is that of Lengauer introduced in [23, 21, 25], referred to as *L-specifications.*

In modern networks, Frequency Assignment Problems (FAP) have important applications in efficient bandwidth utilization, by trying to minimize the number (or the range) of frequencies used in a way that however keeps the interference of nearby transmitters at an acceptable level. Problems of assigning frequencies in networks are usually abstracted by variations of coloring graphs. An important version of Frequency Assignment Problems is the Radiocoloring Problem (RCP). RCP consists in assigning colors (frequencies) to the vertices (transmitters) of a graph (network), so that any two vertices of distance at most two get different colors. A possible objective of such an assignment, considered here, is to minimize the number of distinct frequencies used.

In this work we study the RCP problem on the well known model of L-specified hierarchical graphs of Legnauer ([23, 21, 25]). Note that RCP is equivalent to the problem of vertex coloring the square of a graph G, G^2, where G^2 has the same vertex set as G and there is an edge between any two vertices of G^2 if their distance in G is at most 2. We study here planar hierarchical graphs.

Also, our focus on coloring the square of a hierarchical planar graph is inspired by real networks, especially wireless and large ones, that may be structured in a hierarchical way and are planar.

1.2 Summary of Our Results

We investigate the computational complexity and provide efficient approximation algorithms for the RCP on a class of L-specified hierarchical planar graphs, called Distance-3-Separated graphs. In particular:

1. We prove that the decision version of the RCP for L-specified hierarchical planar graphs is $\mathcal{PSPACE}$-complete.
2. We present two approximation algorithms for RCP The algorithms offer alternative trade-offs between the quality and the efficiency of the solution achieved. The first one is a simple and very efficient (it needs small, i.e. of low order, polynomial time) 4-approximation algorithm, while the second one achieves a better solution; it is a 3-approximation algorithm, but is less efficient, although polynomial.

A critical observation about the $\mathcal{PSPACE}$-completeness constructions exploited in the $\mathcal{PSPACE}$-completeness proofs of this work, is that such constructions have to exhibit some *locality characteristics* in order to be of polynomial time to the size of the L-specification of the hierarchical graph. Another important issue for the $\mathcal{PSPACE}$-completeness reductions is whether an already known $\mathcal{NP}$-completeness proof for the same problem, that fulfills the desired locality characteristics, can be modified so that to apply for a hierarchical graph G. Unfortunately, in our case, there was no such a 'local' $\mathcal{NP}$-completeness reduction available. Therefore, we provide a new $\mathcal{NP}$-completeness reduction for the RCP of ordinary planar graphs which exhibits locality properties and then, based on this construction, we design the $\mathcal{PSPACE}$-completeness reduction of the radiocoloring problem of L-specified hierarchical planar graphs.

As it concerns the algorithmic part of the paper, remark that approximation algorithm for hierarchically specified graphs has to compute a feasible solution to the problem for the hierarchical graph, of size exponential to its specification, using only time and space polynomial to its specification. Our approximation algorithms get as input the L-specification of a hierarchical graph and, in time and space polynomial to the size of this specification, compute a feasible solution for the RCP on the hierarchical graph. Moreover, the solution output is produced using the hierarchical specification, henceforth, requiring only polynomial space to be described.

We note that the class of $DS3$ L-specified hierarchical graphs considered here can lead to graphs that are exponentially large in the size of their specification. Moreover, note that this class is a subclass of the class of L-specified hierarchical graphs considered in [27], called k-level-restricted graphs.

1.3 Related Work and Comparison

In the past, several researchers studied the complexity and the approximability of hierarchical graphs specified using the L-specification. In a fundamental work, Lengauer and Wagner [25] proved that the following problems are $\mathcal{PSPACE}$-complete for L-specified hierarchical graphs: 3-COLORING,

HAMILTONIAN CIRCUIT AND PATH, MONOTONE CIRCUIT VALUE, NETWORK FLOW, ALTERNATING GRAPH ACCESSIBILITY and INDEPENDENT SET. However, their $\mathcal{PSPACE}$-completeness results and in particular 3-COLORING, which is of interest in our case, are for general graphs. For L-specified graphs, Lengauer et al ([21–23]) have given efficient algorithms to solve several graph theoretic problems including 2-coloring, min spanning forest and planarity testing.

We remark that the $\mathcal{PSPACE}$-completeness proof of planar 3-COLORING of $DS3$ L-specified hierarchical graphs provided in this work is not implied by known $\mathcal{PSPACE}$-completeness results of the same problem for similar (but different) classes of planar graphs. This is so because our $\mathcal{PSPACE}$-completeness proof for planar 3-COLORING concerns a subclass of the L-specified hierarchical planar graphs studied in [28] for the same problem. Moreover, the $\mathcal{PSPACE}$-completeness proof of planar 3-COLORING of [30] for L-specified hierarchical graphs which are simultaneously planar and unit disks concerns a different class of hierarchical planar graphs than the class of $DS3$ L-specified hierarchical planar graphs considered here.

Marathe et al in [28] investigate the existence and non-existence of polynomial time approximations for several $\mathcal{PSPACE}$-hard problems for L-specified hierarchical graphs including maximum independent set, minimum vertex cover, minimum edge dominating set, MAX 3SAT and MAX CUT. For these problems (which try to find appropriate subsets of a set), [27] presents approximation schemes for hierarchically specified graphs when restricted on planar instances.

Note, however, that most of the work done so far on approximations of $\mathcal{PSPACE}$-complete problems, has basically addressed such 'finding a subset' problems and not coloring problems. The methodologies applied for such problems, such as maximum independent set (MIS) in [27], do not directly apply to coloring problems, since they exclude from a solution some vertices of each graph of the L-specification of the hierarchical graph, something not allowed in graph coloring problems. To our knowledge, the only work studying approximations to coloring problems on hierarchical graphs is the work of [30].

The best currently known approximation ratio for the RCP on ordinary (non-hierarchical) planar graphs (which are much simpler to color than the hierarchical ones) is 2 ([1, 12]). Also, the only known results on any kind of coloring problems have been shown for the vertex coloring for a special kind of hierarchical graphs (unit disk graphs) achieving a 6-approximation solution ([30]).

2 Preliminaries

In this work we study the Radiocoloring problem (see definition 14, in the Appendix). Its optimization version, where the objective is to minimize *the number* of colors used, is defined as follows:

Definition 1. Min order RCP: *Given a graph $G(V,E)$, find a radiocoloring assignment of G, i.e. a coloring function $\Lambda : V \rightarrow N^*$ assigning integers (colors) to the vertices of G such that $\Lambda(u) \neq \Lambda(v)$ if $d(u,v) \leq 2$, where $d(u,v)$ is the distance between u and v in G, that uses a minimum number of colors. The number of different integers in such an assignment, is called the* order *of RCP on G and is denoted here by $\lambda(G)$, i.e. $\lambda(G) = |\Lambda(V)|$.*

This work studies the min order RCP. For simplicity reasons, in the sequel we refer to it as the RCP. Remark the following:

Proposition 1. *The min order RCP of a given graph G is equivalent to the problem of coloring the square of the graph G, G^2, where G^2 has the same vertex set as G and there is an edge between any two vertices of G^2 if their distance in G is at most 2.*

We study hierarchical graphs as specified by Lengauer [23].

Definition 2. *(***L-specifications, [23]***) An L-specification $\Gamma = (G_1, \cdots, G_i, \cdots, G_n)$, where n is the number of levels in the specification, of a graph G is a sequence of labeled undirected simple graphs G_i called* cells. *The graph G_i has m_i edges and n_i vertices. The p_i of the vertices are called pins. The other $(n_i - p_i)$ vertices are called inner vertices. The r_i of the inner vertices are called nonterminals. The $(n_i - r_i)$ vertices are called terminals. The remaining $n_i - p_i - r_i$ vertices of G_i that are neither pins nor nonterminals are called* explicit *vertices.*

Each pin of G_i has a unique label, its name. The pins are assumed to be numbered from 1 to p_i. Each nonterminal in G_i has two labels (v, t), a name and a type. The type t of a nonterminal in G_i is a symbol from $G_1, \cdots, G_{i-1}$. The neighbours of a nonterminal vertex must be terminals. If a *nonterminal vertex v is of type G_j in G_i, $1 \leq j \leq i-1$, then v has degree p_j and each terminal vertex that is a neighbor of v has a distinct label (v, l) such that $1 \leq l \leq p_j$. We say that the neighbor of v labeled (v, l) matches the l-th pin of G_j.*

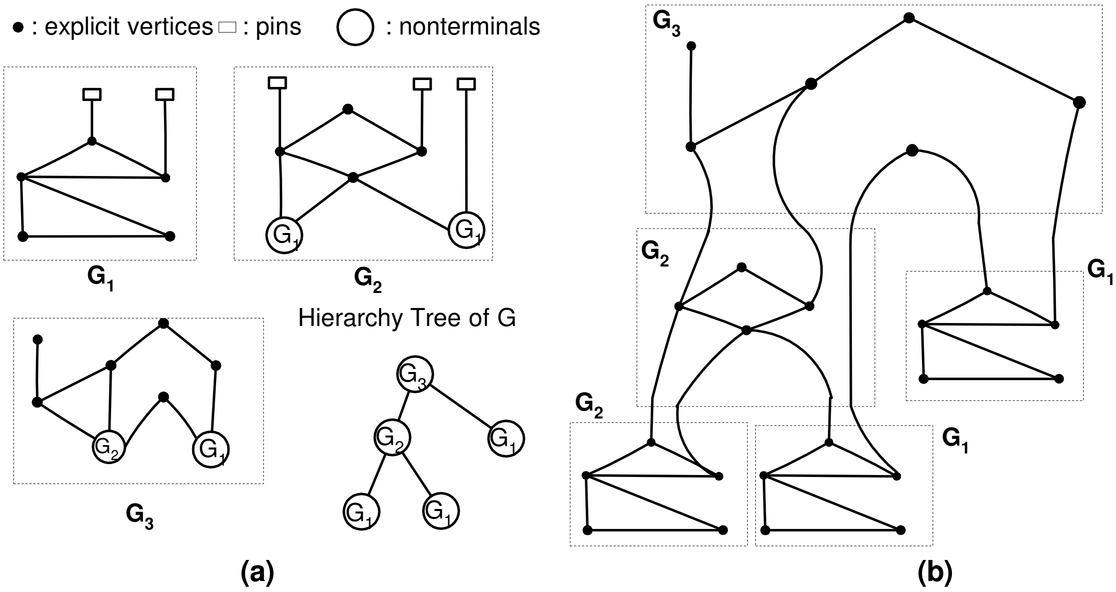

Fig. 1. (a) An L-specification $\Gamma = (G_1, G_2, G_3)$ of a graph G and its hierarchy tree $HT(G)$. **(b)** The expansion of the graph G.

See Figure 1 for an example of an L-specification. Note that a terminal vertex may be a neighbor of several nonterminal vertices. Given an L-specification Γ, $N = \sum_{1 \leq i \leq n} n_i$ denotes the *vertex number* and $M = \sum_{1 \leq i \leq n} m_i$ denotes the *edge number* of Γ. The size of Γ, denoted by $size(\Gamma)$, is $N + M$.

Definition 3. *(***Expansion of an L-specificied hierarchical graph, [23]***) Let any L-specified hierarchical graph, given by $\Gamma = (G_1, \cdots, G_n)$. The* expanded graph $E(\Gamma)$ *(i.e. the graph associated with Γ) is iteratively obtained as follows:*

$k = 1$: $E(\Gamma) = G_1$.

$k > 1$: Repeat the following step for each nonterminal v of G_k say of the type G_j: delete v and the edges incident on v. Insert a copy of $E(\Gamma_j)$ by identifying the l-th pin of $E(\Gamma_j)$ with the node in G_k that is labeled (v, l). The inserted copy of $E(\Gamma_j)$ is called a subcell of G_k.

For example, the expansion of the hierarchical graph G of Fig. 1 is shown in Fig. 1. To each L-specification $\Gamma = (G_1, \cdots, G_n)$, we associate a labeled rooted unoriented tree $HT(\Gamma)$ depicting the insertions of the copies of the graphs $E(\Gamma_j)$ $(1 \leq j \leq n-1)$, made during the construction of $E(\Gamma)$ as follows:

Definition 4. *(***Hierarchy Tree of an L-specification, [23]***) Let $\Gamma = (G_1, \cdots, G_n)$, be an L-specification of the graph $E(\Gamma)$. The hierarchy tree of Γ, denoted by $HT(\Gamma)$, is a labeled rooted unordered tree defined as follows:*

1. *Let r the root of $HT(\Gamma)$. The label of r is G_n. The children of r in $HT(\Gamma)$ are in one-to-one correspondence with the nonterminal vertices of G_n as follows: The label of the child s of r in $HT(\Gamma)$ corresponding to the nonterminal vertex (v, G_j) of G_n is (v, G_j).*
2. *For all other vertices s of $HT(\Gamma)$ and letting the label of $s = (v, G_j)$, the children of s in $HT(\Gamma)$ are in one-to-one correspondence with the nonterminal vertices of G_j as follows: The label of the child t of s in $HT(\Gamma)$ corresponding to the nonterminal vertex (w, G_l) of G_j is (w, G_l).*

We consider hierarchical planar graphs as studied in [21]:

Definition 5. (**Strongly planar hierarchical graph, [21]**) *An L-specified hierarchical graph G given by $\Gamma = (G_1, \ldots, G_n)$ is strongly planar if $E(\Gamma)$ has a planar embedding such that for each $E(\Gamma_i)$ all pins of it occur around a common face and the rest of $E(\Gamma_i)$ is completely inside this face.*

Actually, we study a class of strongly planar hierarchical graphs, where additionally to the above condition, *all graphs G_i, $1 \leq i \leq n$, are planar*. We call this class as **subcell strongly planar hierarchical graphs**. In the sequel, and when there is no ambiguity, we refer to such graphs simply as *hierarchical planar graphs*.

Moreover, we concentrate on a class of L-specified hierarchical graphs, called Distance-3-Separated ($DS3$) graphs, defined in the sequel using the following:

Consider an L-specified hierarchical graph G, given by $\Gamma = (G_1, \ldots, G_n)$. For each graph G_i ($1 \leq i \leq n$), we define the following subgraphs:

Definition 6. *The* inner subgraph of graph *G_i, denoted by $\mathbf{G_{in\ i}}$ is the subgraph induced by the explicit vertices of G_i not connected to any pin or nonterminal of G_i. Call the vertices of this subgraph as* inner vertices of G_i.

Definition 7. *The* outer-up subgraph of graph *G_i, denoted by $\mathbf{G_{outUp\ i}}$ is the subgraph induced by the explicit vertices of G_i connected to at least one pin of G_i. Call the vertices of this subgraph as* outer up vertices of G_i.

Definition 8. *The* outer-down subgraph of graph *G_i, denoted by $\mathbf{G_{outDown\ i}}$ is the subgraph obtained by the explicit vertices of G_i connected to at least one nonterminal of G_i. Call the vertices of this subgraph as* outer down vertices of G_i.

For each G_i, $1 \leq i \leq n$, let also $\mathbf{G_{first\ inner\ i}}$ be the subgraph of G_i obtained by the neighbours of vertices of $G_{outUp\ i}$ in G_i and let $\mathbf{G_{last\ inner\ i}}$ be the subgraph of G_i obtained by the neighbours of vertices of $G_{outDown\ i}$ in G_i. For an example of the subgraphs defined above for a hierarchical graph, see Figure 7 in the Appendix, section C.

Remark 1. Generally, an explicit vertex of G_i might belong to both outer-up, outer-down subgraphs of G_i. However, note that in the class of graphs considered in this work (see the following definition), an explicit vertex, which is not inner, is either outer up or outer down.

Definition 9. (**Distance-3-Separated, DS3**) *We call Distance-3-Separated graphs the class of L-specified hierarchical graphs of which any explicit vertex of G_i, $1 \leq i \leq n$, belongs either to $G_{outDown\ i}$ or $G_{outUp\ i}$ or none of them, but not to both of them and any vertex of $G_{outDown\ i}$ is located at distance at least 3 from any vertex of $G_{outUp\ i}$.*

Observe that the $DS3$ class of hierarchical graphs is testable in time polynomial to the size of the L-specification of a hierarchical graph.

Note that another class of L-specified hierarchical graphs, studied by Marathe et al. [16], is defined as follows:

Definition 10. *([16]) An L-specified graph G, with $\Gamma = (G_1, \ldots, G_n)$, is* **k-level-restricted** *if for all edges $(u;v)$ of E_i (E_i is the edge set of G_i), either (1) nu and nv are the same vertex of $HT(G_i)$, or (2) one of nu or nv is an ancestor of the other in $HT(G_i)$ and the length of the path between nu and nv in $HT(G_i)$ is no more than k.*

Proposition 2. *The class of $DS3$ hierarchical graphs is a subclass the k-level-restricted graphs.*

Proof. A $DS3$ graph is an 1-level-restricted graph, which additionally has the property that for each G_i $1 \leq i \leq n$ any outer up vertex of it is at distance at least 3 from any outer down vertex of it. □

Definition 11. *The* **maximum degree of a hierarchical graph** *G, $\Delta(G)$, is the maximum degree of a vertex in the expansion of the graph, $E(\Gamma)$.*

The following definitions and results are needed by the approximation algorithms proposed in section 4:

Definition 12. (**k-outerplanar graph** G **[2]**) *A k-outerplanar graph G is defined recursively by taking an embedding of the planar graph G, finding the vertices in the exterior face of the graph and removing those vertices and the edges incident to them. Then, the remaining graph should be a $(k-1)$-outerplanar graph. A 1-outerplanar graph is an outerplanar graph.*

Definition 13. *([4]) The class of* **k-trees** *is recursively defined as follows:*

(a) A clique of size $k+1$ is a k-tree.

(b) If $G(V,E)$ is a k-tree and k vertices $v_1, v_2, \cdots, v_k$ induce a complete subgraph of G, then $G' = (V \bigcup \{w\}, E \bigcup \{(v_i, w) : 1 \leq i \leq k\})$ is a k-tree, where w is a new vertex not contained in G.

(c) All k-trees can be formed with rules (a) and (b).

Any subgraph of a k-tree is called *partial k-tree*. The minimum value of k for which a graph is a subgraph of a k-tree is called the *treewidth* of the graph.

Theorem 1. *[5] Any k-outerplanar graph is a $3k-1$ bounded treewidth graph.*

Theorem 2. *[39] Let $G(V,E)$ be a k-tree of n vertices given by its tree-decomposition, let C be a set of colors, and let $\alpha = |C|$. Then, it can be determined in polynomial time $T(n,k)$, where $T(n,k) = O(n(2\alpha+1)^{2^{2(k+1)(l+2)+1}} + n^3)$, $l = 2$ and $n = |V|$, whether G has a radiocoloring that uses the colors of set C, and if such a radiocoloring exists, it can found in the same time.*

3 The Complexity of the Radiocoloring Problem

In this section, we study the complexity of RCP on L-specified hierarchical planar graphs. A critical observation about the $\mathcal{PSPACE}$-completeness constructions utilized in the $\mathcal{PSPACE}$-completeness proofs, is that such constructions have to exhibit some *locality characteristics*. That is, the new graph G' obtained by the initial graph G in the reduction should be computed based on local transformations applied on the vertices or edges of each graph G_i of the L-specification of the graph G without involving the transformation of the whole graph G. Moreover, the transformation should be such that the graph G_i', resulting by applying the transformation on G_i, to be the same for all appearances of G_i in the expansion of G. These properties enable to compute graphs G_i', $1 \leq i \leq n$ only once, and from those graphs to be able to construct the new hierarchical graph G' with the desired properties, in time polynomial to the size of the L-specification of the initial graph G.

Another important issue for the $\mathcal{PSPACE}$-completeness reductions is whether an already known $\mathcal{NP}$-completeness proof for the same problem, that fulfills such locality characteristics, can be modified so that to apply for a hierarchical graph G. This technique have been used in previous papers to get $\mathcal{PSPACE}$-completeness results for a number of problems considered (e.g. [30]).

Unfortunately, in our case, there is no such 'local' $\mathcal{NP}$-completeness reduction. The corresponding $\mathcal{NP}$-completeness reductions need to be adapted for the hierarchical case are the reductions of [35, 32]. However, although the reduction of [35] is local that of [32] is not. Henceforth, they can not be used to get the $\mathcal{PSPACE}$-completeness of L-specified hierarchical planar graphs.

For these reasons, we have developed a new $\mathcal{NP}$-completeness proof for the problem of min order radiocoloring of ordinary planar graphs which reduces it from the problem of 3-coloring planar graphs. The construction satisfies the desired locality characteristics and thus, we can utilize it to get the $\mathcal{PSPACE}$-completeness proof of the RCP for L-specified hierarchical planar graphs.

3.1 The $\mathcal{NP}$-completeness of the radiocoloring problem for planar graphs

In this section we provide a new $\mathcal{NP}$-completeness proof for the problem of radiocoloring for ordinary (non-hierarchical) planar graphs, satisfying certain desired locality characteristics. We remark that this reduction is the only one that works for the cases where $\Delta(G) < 7$, in contrast to the only known $\mathcal{NP}$-completeness proof of [35]. In fact, we prove that even for graphs of maximum degree $\Delta(G) = 3$, it is $\mathcal{NP}$-complete to decide whether G is 4-radiocolorable or not. Note interestingly, that the only graphs not included in this $\mathcal{NP}$-completeness result, are graphs that contain only lines and cycles.

This construction will be utilized in the next section to prove the $\mathcal{PSPACE}$-completeness of RCP for hierarchical planar graphs.

Theorem 3. *The following decision problem is $\mathcal{NP}$-complete:*

Input: A planar graph $G(V,E)$.

Question: Is $\lambda(G) \leq 4$?

Proof. It can be easily seen that the problem belongs in $\mathcal{NP}$. Let any planar graph G. We reduce RCP from the 3-COLORING problem of planar graphs, which is known to be $\mathcal{NP}$-complete ([14]). I.e. we will construct in polynomial time a new graph G', which is 4-radiocolorable if and only if G is 3-colorable. The reduction employs the component design technique.

The construction replaces every vertex u of degree d_u of the initial graph G with a component, called 'cycle node'. The cycle node obtained by a vertex of degree d_u is said to be 'a cycle node of degree d_u'. An instance of it is shown in Figure 2. A cycle node of degree d_u is constructed as follows:

1. Add a cycle of $3d_u$ vertices, called 'outer cycle node'. Group the first three vertices into a group named a and call them a_1, a_2, a_3, the second triad of vertices into a group named b and call them b_1, b_2, b_3, end so on until the d_u-th triad of it, which is named using the d_u-th latin letter. Call the first vertex of each group, $a_1, b_1, c_1, \ldots$, as 'first' vertex, the second $a_2, b_2, c_2, \ldots$, as 'second' and the third $a_3, b_3, c_3, \ldots$ as 'third'.
2. For each group $a, b, c, \ldots$ of vertices of the outer cycle node, add one more vertex, called the 'fourth' and label it as $a_4, b_4, c_4, \ldots$ accordingly.
3. Now connect the fourth vertex of each group with the second (e.g. a_2), and third (e.g. a_3) vertex of the group, as shown in Figure 2.
4. For each such sequence of four vertices, add one more vertex, call it 'fifth' and name it as $a_5, b_5, c_5, \ldots$ accordingly.
5. Connect the 'fifth' vertex of each group to the 'first' vertex of the group.
6. Add another cycle of vertices, call it 'inner cycle node'. The cycle consists of d_u triads of vertices as the outer cycle. The j-th group is named as the $d_u + j$-th letter of the latin alphabet. In the example of Figure 2 these are groups e, f, g, h. The vertices of each group, assume one called x, are called as follows: The middle vertex of the group is called 'first' and is labeled x_1, the right one (clockwise) is called 'second' and is labeled x_2 and the left one (clockwise) is called 'third' and is labeled x_3.
7. Similarly to the outer cycle node, add one more vertex for each triad of the inner cycle, call it 'fourth' and label it based on from which group of vertices it is obtained by; for the example of Figure 2 they are labeled as e_4, f_4, g_4, h_4. Then, connect the fourth vertex (e.g. e_4) of each group with the second (e.g. e_2) and the third (e.g. e_3) vertex of the group. (See Figure 2.)
8. For each group of four vertices of the outer cycle, assume the i-th, add an edge connecting the fourth vertex (e.g.a_4) of the group to the first vertex (e.g. e_1) of the i-th group of the inner cycle.

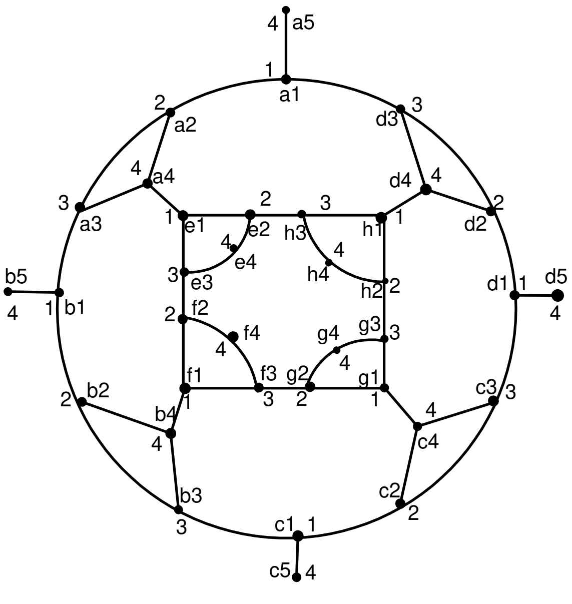

Fig. 2. The 'cycle node' component for a node of degree 4 and a radiocoloring of it using 4 colors.

In the sequel, we explain how the cycle node is used to construct in polynomial time from any planar graph G a new planar graph G', with the desired properties. We consider a planar embedding of graph G. The new graph G' is constructed as follows:

1. Replace each vertex of degree d_u of the initial graph G, with a cycle node of degree d_u. Note that the cycle node of degree d_u has d_u groups of vertices in the outer and inner cycles of it.

2. For each vertex u of graph G, number the edges incident to u, in increasing, clockwise, order.
3. For every edge of the initial graph $e = (u, v)$ connecting u and v, let u_i be the number of edge e given by vertex u and let v_j be the number of the edge e given by vertex v.
   Then, take the fifth vertex of the u_i-th group of the cycle node of vertex u and the fifth vertex of the v_j-th group of the cycle node of vertex v and collapse them to a single vertex uv. Note that, by this operation the first vertices of the cycle nodes of the two vertices are connected by uv.

An example of a graph G and the new graph G' obtained is shown in Figure 3 of the Appendix, section A (depicted in a compact way). It can easily be seen that the new graph G' is a planar graph. We next prove two Lemmas showing that G' is 4-radiocolorable if and only if the initial graph G is 3-colorable.

Lemma 1. *If* $\chi(G) \leq 3$ *then* $\lambda(G') \leq 4$.

Lemma 2. *If* $\lambda(G') \leq 4$ *then* $\chi(G) \leq 3$.

For the proofs of lemmas 1 and 2, see Appendix, section A.

This completes the $\mathcal{NP}$-completeness proof. □

3.2 The $\mathcal{PSPACE}$-completeness of Radiocoloring for Hierarchical Planar Graphs

The $\mathcal{PSPACE}$-completeness reduction, for the RCP of L-specified hierarchical planar graphs, utilizes the construction of the $\mathcal{NP}$-completeness proof of the radiocoloring problem for ordinary planar graphs, given in section 3.1, Theorem 3 exploiting its locality properties in order to achieve a construction of time polynomial to the size of the L-specification of the hierarchical graph.

The $\mathcal{PSPACE}$-completeness proofs use the following *categorization of the edges of* G_i *into 3 kinds*: (a) edges connecting an explicit vertex of G_i to another explicit vertex, called *explicit edges*, (b) edges connecting an explicit vertex with a pin of G_i, called *pin edges*, (c) edges connecting an explicit vertex of G_i with a nonterminal called by G_i, which we call *nonterminal edges*.

The $\mathcal{PSPACE}$-completeness of 3-coloring for hierarchical planar graphs

In order to be able to utilize the $\mathcal{NP}$-completeness reduction of Theorem 3 to prove the $\mathcal{PSPACE}$-completeness of RCP for hierarchical planar graphs, we need to know that the 3-COLORING problem of L-specified hierarchical planar graphs is $\mathcal{PSPACE}$-complete.

Theorem 4. *The following decision problem is* $\mathcal{PSPACE}$*-complete:*
Input: A subcell strongly planar hierarchical graph G*, given by the L-specification* $\Gamma = (G_1, \ldots, G_n)$.
Question: Is $\chi(G) \leq 3$*?*

Proof. We adapt the $\mathcal{NP}$-completeness construction of [13]. See Appendix, section B, for a full proof. □

Comment. A very similar result has been proved in [30] for a restricted class of L-specified hierarchical graphs which are simultaneously planar and unit disks. However their reduction does not clearly address the strongly planar graphs. Notice that our class is slightly different and neither a subset nor a superset of their class.

The $PSPACE$-completeness of RCP for hierarchical planar graphs

Before introducing the $\mathcal{PSPACE}$-completeness Theorem, the following observation is needed. We denote by $d(u)_H$ the degree of vertex u in graph H.

Observation 1. Let any graph G_i of the L-specification of a hierarchical graph G. Then, for any explicit vertex u in G_i, it holds that $d(u)_{G_i} = d(u)_{E(\Gamma)}$ □

Theorem 5. *The following decision problem is* $\mathcal{PSPACE}$*-complete:*
Input: A $DS3$ *subcell strongly planar hierarchical graph* G*, given by the L-specification* $\Gamma = (G_1, \ldots, G_n)$.
Question: Is $\lambda(G) \leq 4$*?*

Proof. **Membership in $\mathcal{PSPACE}$:** Similar to the $\mathcal{PSPACE}$-membership of 3-coloring a hierarchical planar graph of Theorem 4.

$\mathcal{PSPACE}$-completeness: We reduce the RCP for hierarchical planar graphs from the 3-COLORING of subcell strongly planar hierarchical graphs proved to be $\mathcal{PSPACE}$-complete in Theorem 4. Our reduction uses that construction of Theorem 3 used to prove the $\mathcal{NP}$-completeness of RCP for ordinary planar graphs.

Let any L-specified hierarchical planar graph G, given by $\Gamma = (G_1, \ldots, G_n)$. For each graph $G_i = (V_i, E_i)$ of its L-specification we construct a new graph $G'_i = (V'_i, E'_i)$ using the rules of Theorem 3:

1. Each explicit vertex u of degree d_u of G_i is replaced by a 'cycle node' of degree d_u defined in Theorem 3.
2. Take each edge $e = (u, v)$ of G_i connecting two explicit vertices u and v, and take the corresponding fifth vertices of the cycle nodes of vertices u and v. Collapse it to a single vertex uv connecting the corresponding 'first' vertices of the two cycle nodes. See Figure 6(a) in the Appendix, section B.
3. All nonterminal vertices of G_i are present in G'_i. However, the type of a nonterminal, assume one of type G_j is changed to G'_j.
4. Take each $e = (u, t)$ of G_i connecting an explicit vertex u to a nonterminal vertex t called by G_i, assume of type G_j. Assume moreover, that the edge has label (t, l), i.e. u matches the l-th pin of the graph G_j. Then, connect the corresponding fifth vertex f of the cycle node of u to the nonterminal t and associate it with the label (t, l) (i.e. vertex f matches the l-th pin of G_j). See Figure 6(b) in the Appendix, section B.
5. All pins vertices of G_i are present in G'_i.
6. Take each edge $e = (u, p)$ of G_i connecting an explicit vertex u to a pin p of G_i, numbered as the l-th. Then, remove the corresponding fifth vertex of the cycle node of vertex u and connect the corresponding first vertex of the cycle node to the pin p. See Figure 6(c) in the Appendix, section B.

Observe that the above procedure applied on all graphs $G_1, G_2, \ldots, G_n$ of G, only once for each G_i, results to a set of graphs $G'_1, G'_2, \ldots, G'_n$ that define a new L-specified hierarchical graph G', given by $\Gamma' = (G'_1, G'_2, \ldots, G'_n)$. To see why note that the same graph G'_i results applying the above procedure in any appearance of the graph G_i in the hierarchical tree $HT(\Gamma)$ of G. Note also that the graph G'_i calls the same set of terminals as G_i. Finally, observe that the graph G'_i obtained by G_i by above procedure have the same pins as G_i. The next lemmas prove that the new graph is 4-radiocolorable if and only if the initial graph is 3-colorable.

Lemma 3. *If $\chi(G) \leq 3$ then $\lambda(G') \leq 4$.* □

Lemma 4. *If $\lambda(G') \leq 4$, then $\chi(G) \leq 3$.* □

The two lemmas are proved using the same arguments as those in the proofs of lemmas 1, 2, respectively, and the observation that the expansion of the graph G' obtained, $E(\Gamma')$, is the same graph as the graph obtained by the construction of Theorem 3 when applied on the expansion of the initial hierarchical graph G.

Next, we prove that the resulting graph is a subcell strongly planar hierarchical graph: Observe that, (i) the initial graph G is a hierarchical and the transformation introduces no crossings, satisfying the second condition of subcell strongly planar hierarchical graphs, (ii) the transformation keeps the pins of the resulting graph in the outside face of the graph, satisfying the first condition of such graphs).

Finally, we prove that, the resulting graph is $DS3$ hierarchical graph: Take any outer up vertex u of any G'_i of G', connected to a pin p. In the worst case, the vertex is also an outer down vertex. Let v the nonterminal connected to v. Thus, the degree of the vertex is at least 2. By the transformation, it is replaced by a cycle node of degree at least 2. Recall that a first vertex of the cycle node, call it f, is connected to pin p of G'_i and a fifth vertex, call it t, of the cycle node is connected to nonterminal v of G'_i. Observe now, that the first vertex f is located at distance at least 4 from the fifth vertex t of the cycle node. Since, we get the same result for each graph G_i, $1 \leq i \leq n$, we conclude that the resulting graph G' is a $DS3$ subcell strongly planar hierarchical graph.

□

4 Efficient approximations to RCP for a class of L-specified hierarchical planar graphs

In this section we present two approximation algorithms for the min order RCP on $DS3$ subcell strongly planar hierarchical graphs: a simple and fast algorithm, that achieves an approximation ratio of 4, and a more sophisticated one which, being still polynomial, achieves a 3-approximation ratio. These algorithms offer alternative options that trade-off the efficiency of the algorithm and the quality of the solution achieved. Both algorithms utilize a bottom up methodology of radiocoloring an L-specified hierarchical planar graph G, given by $\Gamma = (G_1, \ldots, G_n)$. By this method the algorithms radiocolor graphs G_1, G_2, and so on up to G_n in G's specification. Actually, they compute at most $n - i$ radiocolorings for a subgraph of each graph G_i, $1 \leq i \leq n$, and use these radiocolorings for all copies of G_i in the expansion of G, $E(\Gamma)$. This enables them to run in time only polynomial to the size of the L-specification of G.

More analytically, we desire to compute only one radiocoloring assignment for each G_i, and use this in all appearances of G_i in the expansion of G, $E(\Gamma)$. However, due to the structure of L-specified hierarchical graphs, the distance two neighborhood of the outer vertices of each G_i, may differentiate for every call of G_i by other graphs G_j. Henceforth, a radiocoloring for such a vertex (the outer ones) may becomes invalid due to a change of the distance two neighborhood of the vertex. Since each graph G_i may be called by at most $n - i$ other graphs, we need to compute at most $n - i$ radiocoloring of those (outer) vertices.

Moreover, we need to guarantee that the different radiocolorings of the outer part of G_i does not introduce any implication in the radiocoloring of its inner part. This, i.e. by having only one radiocoloring for the inner part of G_i, we manage to have also no implications to the radiocoloring of the subtree of G_i, $HT(G_i)$.

Based on this design approach, both of the two algorithms partition carefully and differently (each of them), each G_i into three part: (1) *the inner part*, (2) *the outer up* and (2) *the outer down part*. Remark, that these subgraphs might be different from the inner, outer up, outer down subgraphs of G_i defined in the Definitions 6, 7 and 8.

They radiocolor the inner part of G_i only once, using, each of them, a different method. Both of them, they group and radiocolor the outer down part of it together with the outer up parts of the graphs called by it, using a known 2-approximation algorithm.

4.1 A simple 4-approximation algorithm

We first provide a simple and efficient algorithm, (HRC_1) that achieves a 4-approximation for RCP on $DS3$ subcell strongly planar hierarchical graphs.

Let A_1, B_1 two disjoint sets of colors of size $2\Delta(G) + 25$ each, where $\Delta(G)$ is the maximum degree of G.

Overview of the HRC_1 Algorithm: First, the algorithm defines for each G_i its inner, outer up and outer down parts to be equal to the inner, outer up and outer down subgraphs $G_{in\ i}$, $G_{outUP\ i}$ $G_{outDown\ i}$, respectively.

Then, it radiocolors the inner part of the graph G_i using a known 2-approximation algorithm e.g. [1, 12] using the color set A_1. Also, it radiocolors the outer down part of the graph G_i together with the outer up parts of its children using the 2-approximation algorithm with the color set B_1.

1. *The outer part of* G_i: is equal to the outer up subgraph of G_i, $G_{outUp\ i}$.
2. *The inner part of* G_i: is equal to the inner subgraph of G_i, $G_{in\ i}$.
3. *Outer down part of* G_i: is equal to the graph obtained by the outer down subgraph of G_i and the outer up vertices of its children.

Theorem 6. *Algorithm $HRC_1(G)$ produces a radiocoloring of a hierarchical planar graph G of class DS3 in time $O(n^5)$, achieving an approximation ratio of 4.*

Proof. See Appendix, section C. For an illustration of how the colors sets A_1, B_1 are used by the algorithm, see Figure 8 in the Appendix. □

4.2 A 3-approximation Algorithm HRC_2

Overview of the Algorithm: We provide a more sophisticated radiocoloring algorithm, that achieves a 3-approximation ratio rather than 4, for L-specified hierarchical planar graphs of class DS3. The basic idea of the algorithm, called HRC_2 is to partition the vertices of each graph G_i into outerplanar *levels* using a BFS (similar to [2, 18]).

The *outer up part of* G_i is constituted by the first level of the BFS tree resulted. Thus, it is equal to the outer up subgraph of G_i, $G_{outUp\ i}$. The *outer down part of* G_i is constituted by the graph induced by the vertices of the BFS tree of levels D up to the end of the tree, where D is the first level of the tree having an outer down vertex of G_i. The *inner part of* G_i *is the rest of the graph* G_i.

More analytically, the levels obtained by the partition of graph G_i exhibit some nice properties. Exploiting those characteristics, we manage to radiocolor every two successive levels of the inner part of G_i optimally interchanging color sets A, B. We prove that any two successive levels, call them a double level, is a 4-outerplanar graph. Thus, by Theorem 1, it is a bounded treewidth graph. Consequently, applying Theorem 2, we can radiocolor each double level optimally in polynomial time.

Moreover, by the way the partition of the graph into levels is constructed, the vertices of any double level are located at distance one or two only from its previous and next double levels and at distance more than two from any other level. This enables us to radiocolor successive double levels of G_i by interchanging sets of colors A, B, where $|A| = |B| = OPT(G)$ and $OPT(G)$ is the optimal number of colors needed to radiocolor G.

Finally, the outer down part of G_i is radiocolored together with the outer up parts of its children using a known (2-approximation) radiocoloring algorithm for ordinary planar graphs using color sets A and C, where $|C| = |A| = OPT(G)$. By the way the partition is constructed, the neighbours of this subgraph are only double levels radiocolored using set B. This, enables us to use, in total, for the whole radiocoloring of the hierarchical graph G only sets A, B, C, achieving a 3-approximation.

Theorem 7. *HRC_2 Algorithm produces a valid radiocoloring of a hierarchical planar graph G of class DS3 using at most $3\,OPT(G)+50$ colors, achieving a 3 approximation ratio. It runs in $O(n^2 \cdot T(n,k) + n^5)$ time, where $T(n,k)$ is a polynomial time function for the optimal radiocoloring of a k-tree of size n, specified in [39] ($k = 11$), (see Theorem 2).*

For the proof of Theorem 7 see Appendix, section D. □

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APPENDIX

Definition 14. Radiocoloring Problem (RCP): *Given a graph $G(V, E)$, find a radiocoloring assignment of G, i.e. a coloring function $\Lambda : V \rightarrow N^*$ assigning integers (colors) to the vertices of the graph G such that $\Lambda(u) \neq \Lambda(v)$ if $d(u, v) = 2$ and $|\Lambda(u) - \Lambda(v)| \geq 2$ if $d(u, v) = 1$, where $d(u, v)$ is the distance between u and v in G.*

A Claims and Lemmas used by Theorem 3 (The NP-completeness proof of RCP of ordinary planar graphs)

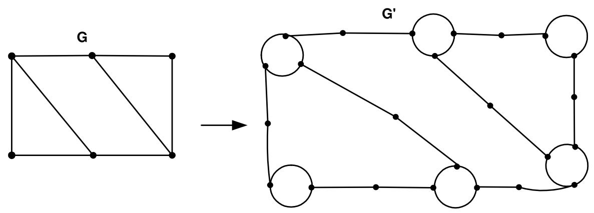

Fig. 3. An instance of the graph G and the new graph G' obtained by the construction.

Lemma 1. *If $\chi(G) \leq 3$ then $\lambda(G') \leq 4$.*

Proof. Consider a 3-coloring of the initial graph G, using colors $\{1, 2, 3\}$. Let the following radiocoloring assignment on the graph G' using 4 colors:

For any vertex u of the initial graph colored i, $i \in \{1, 2, 3\}$: (i) Color the 1-st vertices of the cycle node of vertex u in G', (vertices $a_1, b_1, c_1, \ldots$) with color i. Let $j, j' \in \{1, 2, 3\}$ with $j \neq j'$, $j, j' \neq i$. (ii) Color all 2-nd vertices of the cycle node of u with color j. (iii) Color all 3-rd vertices of the cycle node of u with color j'. (iv) Color all 4-th vertices of the cycle node of u with color 4. (v) Color all 5-th vertices with the same color as the 4-th vertices, color 4.

We now prove that the suggested radiocoloring assignment is valid. An example for such a radiocoloring for a cycle node of degree 4 is shown in Figure 2. It can be easily seen that there is no conflict between any two vertices of the cycle node.

Consider now any neighbour cycle nodes, i.e. nodes that are obtained by neighbour vertices in G. We argue that there is no conflict between the colors of any two vertices of the two cycle nodes. To see why, recall that the 'fifth' vertex, call it f, connecting the two cycle nodes is colored 4. Also, the corresponding 'first' vertices of the two cycle nodes, connected to the 'fifth' vertex f, are colored using colors of set $\{1, 2, 3\}$. Each of them is forbidden to take, besides the colors of the vertices of its cycle node, also the color of the other first vertex in the other cycle node. This color is different from its color since the two vertices took the color that their corresponding vertices take in the 3-coloring of G. Thus, there is no conflict between any two vertices of neighbour cycle nodes. Consequently, the suggested radiocoloring of G' is valid. □

The next four claims are needed to prove that if $\lambda(G') \leq 4$, then $\chi(G) \leq 3$.

Claim 1. If $\lambda(G') \leq 4$ then for any cycle node u, all of its first vertices (of its outer and inner cycle) take the same color.

Proof. Consider any radiocoloring assignment of G'. Assume that the a_1-vertex of a cycle node u is colored by j, and w.l.o.g. let $j = 1$. Then, we get:

- The vertices a_2, a_3, a_4 of the first group of vertices of the outer cycle, should take colors $2, 3, 4$ (not necessarily in this order).
- Then, the first vertex of the first group of vertices of the inner cycle (e_1 in the example of Figure 2), should get color 1.
- The first vertex of the second group of vertices of the outer cycle, b_1 should get color 1. This is because b_1 is forbidden to take colors $2, 3, 4$ by vertices a_2, a_3, a_4.
- Thus, vertices b_2, b_3, b_4 of the second group of vertices of the outer cycle, should take the colors $2, 3, 4$ (not necessarily in this order).

– The first vertex of the second group of vertices of the inner cycle (f_1 in the example of Figure 2), should get color 1.
  Continuing in the same way for the remaining groups of vertices of the cycle node, for a group of vertices of the outer cycle, assume one named x, we get that:
– The 2nd, 3th and 4th vertex of group x of vertices of the outer cycle, should take colors $2, 3, 4$ (not necessarily in this order).
– Then, the first vertex of group x of vertices of the inner cycle, should get color 1.
– Also, the first vertex of the group $x + 1$ of vertices of the outer cycle, should get color 1 (we use $x + 1$ here to denote the letter following x in the latin alphabet). This is because this vertex is forbidden to take colors $2, 3, 4$ by the 2nd, 3th and 4th vertex of the group x of vertices of the outer cycle.

We conclude that for any cycle node of the graph G', if one first vertex is colored j then all first vertices of it (of the outer and inner cycle of it) get the color j. □

Claim 2. If $\lambda(G') \leq 4$ then for any cycle node u, all of its fourth vertices (of its outer and inner cycle) take the same color.

Proof. Consider any radiocoloring assignment of G'. Assume that the a_4-vertex of a cycle node u is colored by j, and w.l.o.g. let $j = 4$. By claim 1, we know that all of its first vertices are colored with the same color (different than color 4), assume 1. Furthermore, we have:

– The vertices a_2, a_3 of the first group of vertices of the outer cycle, should take colors $2, 3$ (not necessarily in this order).
– Then, the 2nd and 3rd vertices of the first group of vertices of the inner cycle, (e_2, e_3 in the example of Figure 2), should take colors $2, 3$. This is because they can not take colors $1, 4$ by vertices e_1, a_4.
– The 4th vertex of the first group of vertices of the inner cycle, (e_4 in the example of Figure 2), should take color 4. This is because it is forbidden to take colors $1, 2, 3$ of vertices e_1, e_2, e_3.
– Then, the 2nd vertex of the second group of vertices of the inner cycle node (f_2 in our example), is forbidden to take colors $1, 2$ (*or* 3)$, 4$ of vertices e_1, e_3, e_4. Thus, it can take color 3 *or* 2.
– The 4th vertex of the second group of vertices of the inner cycle, (f_4 in the example of Figure 2), should take color 4. This is because it is forbidden to take colors $1, 2, 3$ of vertices f_1, f_2, e_3.
– Then, the 3rd vertex of the second group of vertices of the inner cycle node, (f_3 in our example) is forbidden to take colors $1, 3$ (*or* 2)$, 4$ by vertices f_1, f_2, f_4. Thus, it can take color 3 *or* 2.
– Thus, the 4th vertex of the second group of vertices of the outer cycle, b_4, is forbidden to take colors $1, 2, 3$ by vertices f_1, f_2, f_3. Hence, vertex b_4 can only take color 4.
  Continuing in the same way for the rest groups of vertices of the cycle node, for a group of vertices of the outer cycle, assume one named x, we get that:
– The vertices x_2, x_3 of group x of vertices of the outer cycle, should take colors $2, 3$ (not necessarily in this order). This is because vertices x_1, x_4 are colored $1, 4$.
– Take the group of vertices of the inner cycle corresponding to group x. Assume that is named x'. The 2nd and 3rd vertices of group x', vertices x'_2, x'_3, should take colors $2, 3$, not necessarily in this order. This is because they can not take colors $1, 4$ of vertices x'_1, x_4.
– The 4th vertex of group x of vertices of the inner cycle, (x'_4), should take color 4. This is because it is forbidden to take colors $1, 2, 3$ by vertices x'_1, x'_2, x'_3.
– Then, the 2nd vertex of group $x' + 1$ of vertices of the inner cycle node ($x' + 1$ denotes the letter following x' in the latin alphabet), $(x' + 1)_2$ is forbidden to take colors $1, 2$ (*or* 3)$, 4$ by vertices x'_1, x'_3, x'_4. Thus, it can take color 3 (*or* 2).
– The 4th vertex of group $x' + 1$ of vertices of the inner cycle, $(x' + 1)_4$, should take color 4. This is because it is forbidden to take colors $1, 2, 3$ of vertices $(x' + 1)_1, (x' + 1)_2, x'_3$.
– Then, the 3th vertex of group $x' + 1$ of vertices of the inner cycle node, ($(x' + 1)_3$ in our example) is forbidden to take colors $1, 3$ (*or* 2)$, 4$ by vertices $(x' + 1)_1, (x' + 1)_2, (x' + 1)_4$. Thus, it can take color 3 (*or* 2).
– Thus, the 4th vertex of group $x + 1$ of vertices of the outer cycle, $(x + 1)_4$, is forbidden to take colors $1, 2, 3$ by vertices $(x' + 1)_1, (x' + 1)_2, (x' + 1)_3$. Hence, vertex $(x + 1)_4$ can take only color 4.

We conclude that for any cycle node of the graph G', if one fourth vertex is colored j then all fourth vertices of it (of the outer and inner cycle of it) get the color j.

□

Claim 3. If $\lambda(G') \leq 4$ for any cycle node u, all of its fifth vertices take the same color and this color is the same as the color of the fourth vertices of the cycle node.

Proof. Consider any radiocoloring assignment of G'. By claim 1, we know that all of its first vertices are colored with the same color, assume w.l.o.g. color 1. Moreover, by claim 2, we know that all of its fourth vertices are colored with the same color (different than the color of the first vertices) assume 4. Furthermore, we have:

- The vertices a_2, a_3 of the first group of vertices of the outer cycle, should take colors $2, 3$, assume a_2 takes color 2 and a_3 takes color 3.
- Then, the 2nd vertex of the second group of vertices of the outer cycle, b_2, should take color 2. This is because it is forbidden to take colors $1, 3, 4$ by vertices b_1, a_3, b_4.
- The 5th vertex of the second group of vertices of the outer cycle, b_5, should take color 4. This is because it is forbidden to take colors $1, 2, 3$ by vertices b_1, b_2, a_3.
- The 3rd vertex of the second group of vertices of the outer cycle, b_3, should take color 3. This is because it is forbidden to take colors $1, 2, 4$ by vertices b_1, b_2, b_4.
  Continuing in the same way for the rest groups of vertices of the cycle node, for a group of vertices of the outer cycle, assume one named x, we get that:
- The 2nd vertex of group x of vertices of the outer cycle, x_2, should take color 2. This is because it is forbidden to take colors $1, 3, 4$ by vertices $x_1, (x-1)_3, x_4$.
- Then, the 5th vertex of group x of vertices of the outer cycle, x_5, should take color 4. This is because it is forbidden to take colors $1, 2, 3$ by vertices $x_1, x_2, (x-1)_3$.
- The 3rd vertex of the second group of vertices of the outer cycle, x_3, should take color 3. This is because it is forbidden to take colors $1, 2, 4$ by vertices x_1, x_2, x_4.

We conclude that for any cycle node of the graph G', if one fifth vertex is colored j then all fifth vertices of it (of the outer and inner cycle of it) get the color j.

□

Claim 4. If $\lambda(G') \leq 4$ then all fifth vertices of all cycle nodes take the same color.

Proof. Consider a vertex of the initial graph G and the cycle node in G' corresponding to it, let u. Let a fifth vertex of it; assume w.l.o.g. that it is colored 4. Then, by Claim 3, all fifth vertices of the cycle node get color 4.

Consider any neighbour to u cycle node, v. The cycle nodes of the two nodes are connected through a fifth vertex of their cycle nodes, let it be uv. By claim 3 all fifth vertices of the cycle node v get the same color as vertex uv, i.e. 4. Recall that vertex uv is connected to a first vertex of node u, call it u'_1 and to a first vertex of the cycle node v, call it v'_1.

Applying the same arguments to the rest of the neighbours of u, we get that all fifth vertices of their cycle nodes get color 4. Continuing, in the same way, for each neighbour, we finally get that all fifth vertices of all cycle nodes of the whole graph get the same color.

□

Lemma 2. *If $\lambda(G') \leq 4$ then $\chi(G) \leq 3$.*

Proof. Consider any radiocoloring of G' using 4 colors. Consider a vertex of the initial graph G and the cycle node in G' corresponding to it, u. Let a fifth vertex of it; assume w.l.o.g. that it is colored 4. Then, by Claim 4, all fifth vertices of all cycle nodes of G' get the same color, 4.

Since every first vertex of any cycle node of G' is at distance one from a fifth vertex (colored 4), it is forbidden to take color 4. Thus, any first vertex of any cycle node of G' can take only three of the four colors of the palette, that is the colors $\{1, 2, 3\}$. By claim 1, all first vertices of any cycle node colored i, get the same color i, $i \in \{1, 2, 3\}$.

Now return to G. Let the following coloring of G, f: Assign to each vertex u of the graph G the color that the first vertices of the corresponding to it cycle node take in the radiocoloring of G' considered. We argue that this is a valid 3-coloring of the graph G. To see why observe that, as we have shown above, all first vertices of G' take only 3 colors. Thus, the assignment f of G uses 3 colors. Moreover, recall also that, by claim 1, all first vertices of a cycle node colored i get the same color i. Thus, in G', for each cycle node u, all of it first vertices (colored with the same color) 'see' the colors of the corresponding first vertices of all neighbour cycle nodes of u. This is equivalent to the colors that the corresponding to u vertex in G 'see' by all of its neighbours. Hence, since the first vertices of G' have no conflicts with their neighbour first vertices, there is no conflict with the colors of any vertex in G and its neighbours. Thus, if G' can be radiocolored with 4 colors, then there is a 3-coloring of G. □

B $\mathcal{PSPACE}$-completeness of hierarchical planar 3-COLORING

Theorem 4. *The problem of 3-COLORING an L-specified hierarchical subcell strongly planar graph G, is $\mathcal{PSPACE}$-complete.*

Proof. **Membership in $\mathcal{PSPACE}$:** For a given hierarchical planar graph G one chooses nondeterministically a color from $\{1, 2, 3\}$ for every vertex of $E(\Gamma)$ and checks whether this is really a coloring of $E(\Gamma)$. If this is done in a depth-first manner in the expansion tree $E(\Gamma)$, then such an algorithm can be performed in polynomial space.

$\mathcal{PSPACE}$-completeness: We reduce the problem of 3-COLORING an L-specified hierarchical planar graph from the 3-COLORING of L-specified hierarchical general graphs adapting the $\mathcal{NP}$-completeness construction of [13]. The 3-COLORING of L-specified hierarchical general graphs is known to be $\mathcal{PSPACE}$-complete ([25]).

Let any L-specified hierarchical general graph G. From the L-specification of G, $\Gamma = (G_1, G_2, \ldots, G_n)$, we construct in polynomial time to the size of the L-specification of G, a new L-specified hierarchical planar graph G', defined by $\Gamma' = (G'_1, G'_2, \ldots, G'_n)$.

First recall that in the transformation of [13] each edge having a crossing is replaced by a component of Figure 4, called '*diamond*', where H is the subgraph presented in the same Figure. The component has as many 'diamonds' as the number of crossings of the edge.

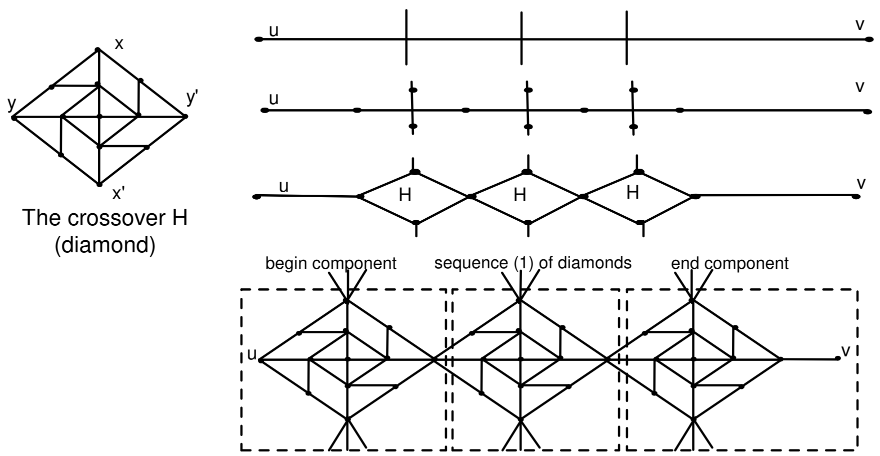

Fig. 4. The Crossover H (diamond) and the transformation of an edge to a sequence of diamonds.

Observe also that the component that replaces an edge is not a symmetric graph. We call the left part of the component shown in Figure 4 as *begin component*, the internal part that contains the sequence of diamonds, also shown in Figure 4, as *sequence of diamonds* and the right part of the component, shown in Figure 4, as *end component.*

Let $G_i = (V_i, E_i)$ be any graph of the L-specification of G. For each G_i we consider a topology such that all of its pins are located in the boundary face of the graph.

For each graph G_i, we construct a new graph $G'_i = (V'_i, E'_i)$ using the following rules:

1. The graph G'_i has the same explicit and nonterminal vertices as the graph G_i. However, the type of each nonterminal called by G_i, assume G_j, is changed to G'_j.
2. The transformation of each edge of G_i is almost the same as that of [13]. Each kind of edges of G_i (explicit, nonterminal, pin edges) is transformed as follows:
   (a) For each explicit edge uv having i crossings, replace it by a component of Figure 4 having i diamonds. The begin of the component can be either vertex u or v.
   (b) For each nonterminal edge connecting an explicit vertex u with a nonterminal node, assume one of type G_j:
   – If the edge has no crossing, then replace it by the begin component shown in Figure 5 (a).
   – If the edge has k crossings ($k \geq 1$), then replace it by a begin component and a sequence of $k - 1$ diamonds as shown in Figure 5 (b) for the case where $k = 2$.

Note that in both cases, the edge connecting u to the nonterminal G_j, is replaced by a component that has three edges connected to the nonterminal G_j. As we will show in the next case (c), the pins of the nonterminal will also be triplicated so that each such edge will match the appropriate pin.

(c) For each pin edge connecting an explicit vertex v with a pin of G_i, p:
– If the edge has no crossing, then replace it by the end component of Figure 5 (c).
– If the edge has k crossings ($k \geq 1$), then replace it by an end component and a sequence of $k-1$ diamonds as shown in Figure 5 (d) for the case where $k = 2$.
Note that, again, in both cases, the edge connecting v to the pin p, is replaced by a component that has three pins. In this way, the pins of G'_i are triplicated compared to the pins of G_i.

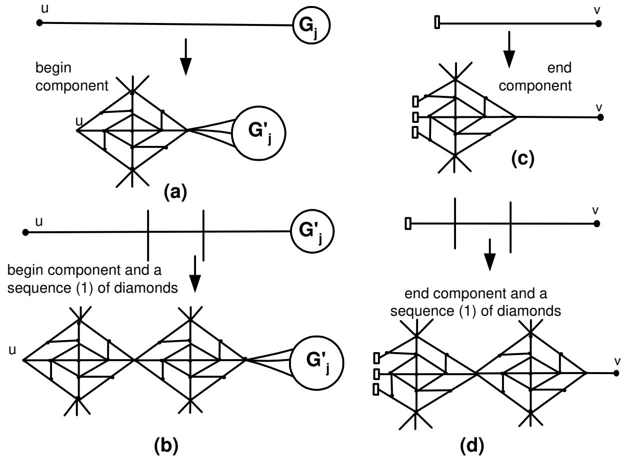

Fig. 5. The edge transformation of a nonterminal edge and pin edge.

We now prove that graphs $G'_1, G'_2, \ldots, G'_n$ obtained by the graphs $G_1, G_2, \ldots, G_n$ of the L-specification of G, define a new L-specified hierarchical graph G'.

Consider any graph G'_i. From it, applying the constructive definition 3, we can get the graph $E(\Gamma'_i)$. To see why take any nonterminal G'_j called by G'_i. Recall that by the transformation, each pin p of G_j is replaced in G'_j by three pins p_1, p_2, p_2. Moreover, note that each edge e of G_i connected to G_j is replaced in G'_i by three edges e_1, e_2, e_3 connected to G'_j. Hence, in $E(\Gamma'_i)$ every edge connected to a nonterminal called by it, matches the appropriate pin of the nonterminal. Finally, from G'_n applying definition 3, we get $E(\Gamma'_n)$. Thus, in this way, graphs $G'_1, \ldots, G'_n$ define a new hierarchical graph G' specified by $\Gamma' = (G'_1, \ldots, G'_n)$ using the L-specification. Moreover, observe that since the transformation eliminates all crossings the new hierarchical graph obtained is a planar graph. Finally, note that since the transformation is applied only once to each G_i, the construction of G' is performed in polynomial time to the size of the L-specification of G.

Next, we prove that the new planar graph G' obtained by G is 3-colorable if and only if G is 3-colorable. In particular, we will show that the expansion of the hierarchical planar graph G', $E(\Gamma')$ obtained by G exhibits the same properties as the graph obtained by the expansion of G, $E(\Gamma)$, when the transformation of [13] is applied on it.

To see why, consider the expansion $E(\Gamma'_i)$ of the graph G'_i obtained by graph G_i. This graph is the same as the corresponding graph obtained using the construction of [13], except that: (1) In [13] only the edges that have a crossing are replaced by the component of Figure 4, while in our construction, all edges corresponding to pin or nonterminal edges are replaced by the component of Figure 4. (2) An edge in the construction of [13] having k ($k \geq 1$) crossings is replaced by a sequence of k diamonds. In our construction it is replaced by a sequence of at most $k+1$ diamonds.

We argue that these two differences do not have any influence on the desired properties of the new graph $E(\Gamma'_i)$ obtained. As it concerns the first difference, note that after the transformation of each such edge uv, the two neighbour vertices u and v 'see' the color of the other vertex as in [13]. As it concerns the second difference, note that although the diamonds of the component might be more than those of the corresponding component in [13], the property of the transformation still holds; that is the two neighbour vertices u and v 'see' the color of the other neighbour vertex as in [13].

Since the two constructions are equivalent, we can use the same arguments used in [13] to prove that the new hierarchical planar graph G' is 3-colorable if and only the initial hierarchical graph G is 3-colorable.

Finally, we prove that the resulting L-specified graph is a subcell strongly planar hierarchical graph. This is true because (i) For each G_i we consider a topology such that all of its pins are located in the boundary face of the graph. Thus, in the resulting graph the corresponding pins are also located in the boundary face of the graph. (ii) for each G_i the transformation eliminates all crossings; henceforth, we get that each G'_i is a planar graph. The above two conditions prove that the resulting L-specified hierarchical graph G' is a subcell strongly planar graph.

□

B.1 Figure used in Theorem 5

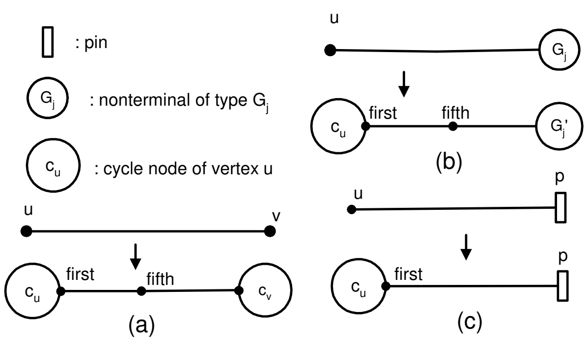

Fig. 6. Theorem 5: The transformation of graph G_i to graph G'_i.

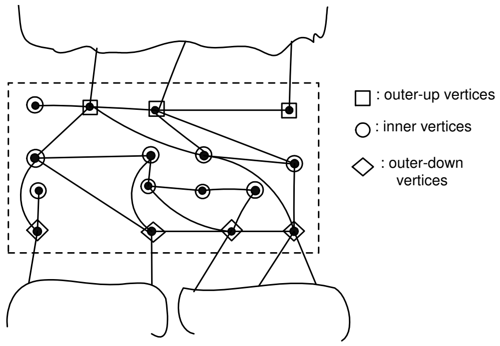

Fig. 7. An example of a graph G_i of a hierarchical graph G, considering inner, outer up and outer down subgraphs of G_i

C The HRC_1 Algorithm

C.1 Algorithm description

Let A_1, B_1 two disjoint sets of colors of size $2\Delta(G) + 25$ each, where $\Delta(G)$ is the maximum degree of G. The HRC_1 algorithm is described (in high level pseudocode) as follows:

HRC$_1$ Algorithm

`For each` G_i`,` $1 \leq i \leq n$ `do:`

1. `Let` $Gin_i = G_{in\ i}$`. Also, let` $Gin'_i = G_{outUp\ i} \bigcup G_{in\ i} \bigcup G_{outDown\ i}$

2. Apply any known radiocoloring algorithm, e.g. [1, 12], for ordinary planar graphs on Gin'_i using colors of set A_1. Ignore the coloring of vertices of Gin'_i that are not in Gin_i (outer up, down vertices of G_i).
3. Let $Gout_i = G_{outDown\ i} \bigcup_{\forall\ j\ called\ by\ G_i} \{\cup G_{outUp\ j}\}$. Also, let $Gout'_i = G_{last\ inner\ i} \bigcup G_{outDown\ i} \bigcup_{\forall\ j\ called\ by\ G_i} \{\cup G_{outUp\ j} \cup G_{first\ inner\ j}\}$
4. Apply any known radiocoloring algorithm, e.g. [1, 12], for ordinary planar graphs on $Gout'_i$ using colors of set B_1. Ignore the coloring of vertices of $Gout'_i$ not in $Gout_i$ (last inner vertices of G_i and first inner vertices of its children).

C.2 Theorem used in the Analysis of HRC_1 algorithm

Theorem 6. *Algorithm $HRC_1(G)$ produces a radiocoloring of a hierarchical planar graph G of class DS3 in time $O(n^5)$, achieving an approximation ratio of 4.*

Proof. Correctness

Observe first that the algorithm, during the radiocoloring of the subgraphs of each graph G_i, it considers 'extensions' of those subgraphs. E.g. in order to get a radiocoloring of Gin_i, it considers Gin'_i, which contains Gin_i but also subgraphs $G_{outUp\ i}$ and $G_{outDown\ i}$. This is required in order to get a valid radiocoloring of Gin_i, because, a vertex of those graphs ($G_{outUp\ i}$ and $G_{outDown\ i}$) may be connected to more than one vertex of Gin_i making those vertices of Gin_i distance two neighbours. Similar arguments prove that graphs $Gout'_i$, $Gout'_j$ (G_j calls G_i) are required in order to produce valid radiocolorings for graphs $Gout_i$, $Gout_j$, respectively.

The radiocoloring assignment for each subgraph of G is computed using a known radiocoloring algorithm ([12, 1]) for ordinary planar graphs. Since these algorithms require for the radiocoloring of a graph G with maximum degree $\Delta(G)$, at most $2\Delta(G) + 25$ colors, the sets A_1, B_1 of the same size can be used to produce a valid radiocoloring on any subgraph of G.

We next prove that there is no conflict between any two vertices of a graph G_i or a vertex of G_i and a nearby vertex of a graph that calls G_i or a graph that is called by G_i. Recall that:

Each G_i is partitioned into 3 parts:

- $G_{up\ i}$: is radiocolored together with each graph G_j that calls G_i, and the other outer up vertices of the graphs that are also called by G_j, graph $G_{out\ j}$, using color set A_1.
- $G_{is\ i}$: is radiocolored using color set B_1.
- $G_{down\ i}$: is radiocolored together with the outer up vertices of the graphs that are called by G_i, graph $G_{out\ i}$, using color set A_1.

Since $G_{up\ i}$ and $G_{in\ i}$ are radiocolored using different colors, there is no conflict between them. $G_{up\ i}$ and $G_{down\ i}$ are radiocolored using the same set of colors, A_1. However, there is no conflict because there are located at distance at least 3 apart by the definition of class DS3 of hierarchical graphs considered here. Also, there is no conflict between $G_{up\ i}$ (colored in $G_{out\ j}$) and $G_{in\ j}$ because they are radiocolored using different color sets. Finally, there is no conflict between $G_{down;i}$ (colored in $G_{out\ i}$) and $G_{in\ j}$ (G_j is a child of G_i) because they are radiocolored using a different set of colors.

The above arguments prove that the HRC_1 algorithm produces a valid radiocoloring of the hierarchical planar graph G.

□

Performance and Approximation ratio

The algorithm for each G_i computes a radiocoloring assignment using a known radiocoloring algorithm ([12, 1]). Such algorithms require $O(n^2)$ time for the radiocoloring of a graph of size n. Remark that when considering the outer down vertices of G_i and their possible children, we get a subgraph of size (number of vertices) n^2. Thus, the radiocoloring algorithm used needs time $O(n^4)$. Since this procedure is called for each G_i, $1 \leq i \leq n$, the algorithm needs $O(n^5)$ time in total.

To compute the approximation ratio of HRC_1 algorithm, recall that it uses two sets of colors of size $2\Delta(G) + 25$ each. Observe also that the maximum degree of the graph G, $\Delta(G)$ is a lower bound on the number of colors needed for the radiocoloring of the graph G. Since HRC_1 algorithm uses color sets A_1, B_1 for the radiocoloring computed, it uses $4\Delta(G) + 50$ colors in total, and henceforth, it achieves a 4-approximation ratio.

□

□

D The HRC_2 Algorithm

D.1 Partitioning and Grouping procedures

The algorithm applies a number of procedures in each G_i, partitioning the graph into smaller subgraphs and then radiocolor the produced subgraphs:

1. Partitioning Procedure of G_i: We partition the graph G_i into *outerplanar* subgraphs, called *single levels* according to the following procedure: Add a new vertex in G_i, called *virtual vertex* and connect it to all vertices of $G_{outUp\ i}$. Note that this can be done without creating any crossings because each vertex of $G_{outUp\ i}$ is connected to a pin and each pin is replaced by a vertex that calls G_i in the expansion of the hierarchical graph. Then, construct a BFS tree T rooted on the virtual vertex. That is, starting from this vertex, find all of its neighbours. Next, find the neighbours of those vertices, and so on. Number the levels of the BFS tree using successive integers starting with 0 for the root. Now, partition the vertices of G_i into levels according to their distance from the root of the tree T. That is, a vertex which is at distance k from the root belongs to level k. Let M_i be the number of the levels of the partition. Also, let l_{im} the subgraph of G_i induced on vertices of level m. Finally, remove the virtual vertex.

Observation 2. The subgraph induced by vertices of level 1 resulting by the partitioning procedure on G_i is in fact the subgraph $G_{outUp\ i}$.

Let D_i the first level of the partition that contains a vertex of the outer down subgraph of G_i, $G_{outDown\ i}$ that is, $D_i = \min\{l_u : u \in G_{outDown\ i}\}$, where l_u is the level of vertex u. For simplicity reasons, and where there is no confusion, in the sequel we use the term D to refer to the level D_i of G_i.

Observation 3. By the definition of class DS3, at least the 2nd and the 3rd level of the partition of G_i consist only of inner vertices.

2. Grouping Procedure: Next, we group together the levels of the partition of G_i as follows:

1. *First Level Subgraph*: The subgraph induced by the vertices of the first level of the partition (outer up vertices). It is denoted by $G_{first\ level\ i}$.
2. *Inner Levels Subgraph*: The subgraph induced by the vertices of levels 2 to level $D-1$. Note that the subgraph for graphs of class DS3, consists only of inner vertices. Denote it by $G_{inner\ levels\ i}$.
3. *Down Levels Subgraph*: The subgraph induced by the vertices of levels D up to the last level M_i. Note that this subgraph, for graphs of class DS3, consists only of inner and outer down vertices. Denote it by $G_{down\ levels\ i}$.

3. Partitioning into double levels Procedure: Group together every two successive levels of the partition of $G_{inner\ levels\ i}$ as follows:
For $m = 1, \ldots, \frac{D-2}{2}$, group together levels $2m$ and $2m+1$ and call the subgraph obtained as *m-th double level* of G_i, denoted by L_{im}. If the number of double levels is even, we extend the last odd level with the rest of the levels of $G_{inner\ levels\ i}$, so that always the number of double levels of $G_{inner\ levels\ i}$ to be odd.

4. Constructing extended double levels Procedure: Take the single levels preceding and following level L_{im}, $l_{i\ 2(m-1)+1}$ and $l_{i\ 2(m+1)}$.

We define the extended level of $L_{i\ m}$, denoted by $Le_{i\ m}$, to be the subgraph induced by the vertices of $L_{i\ m}$ and their neighbours: $Le_{i\ m} = l_{i\ 2(m-1)+1} \bigcup L_{i\ m} \bigcup l_{i\ 2(m+1)}$.

Note that for the first double level $m=1$, we have $Le_{i\ 1} = G_{outUp\ i} \bigcup L_{i\ 1} \bigcup l_{i\ 3}$. For the last double level, we have $Le_{i\ m} = L_{i\ 2(m-1)+1} \bigcup L_{i\ m} \bigcup l_{i\ 2(m+1)}$, where $m = \frac{D-2}{2}$.

4. Constructing the extended down-levels subgraph Procedure: Consider the subgraph $G_{Down\ Levels\ i}$ of G_i. Let H the multiset of nonterminals called by G_i. For each $G_j \in H$ take the subgraphs $G_{first\ level\ j}$ and $G_{first\ inner\ j}$ of G_j. Let $GD_i = G_{Down\ Levels\ i} \bigcup_{G_j \in H} \{\cup G_{first\ level\ j}\}$. Also, let $GDe_i = l_D \bigcup G_{Down\ Levels\ i} \bigcup_{G_j \in H} \{G_{first\ level\ j} \cup G_{first\ \ inner\ level\ j}\}$.

D.2 Algorithm description

Let A, B, C three disjoint sets of colors with $|A| = |B| = |C| = max\{\Delta(G) + 25, OPT(G)\}$, where $OPT(G)$ the maximum number of colors used by an optimal radiocoloring assignment of the graph G.

HRC₂ algorithm (high level description)

`For each` G_i, $1 \leq i \leq n$, `do:`

1. `Apply the` ***partitioning Procedure*** `on` G_i. `Next, apply the` ***Grouping Procedure*** `of` G_i.
2. `Apply the` ***partitioning into double levels Procedure*** `on` $G_{inner\ levels\ i}$.
3. `Radiocolor optimally each` ***odd*** `(even) extended double level subgraphs,` $Le_{i\ m}$, $1 \leq m \leq \frac{D-2}{2}$, m `odd (even), together using color set` B `(`A`). Then, ignore the coloring of the vertices of` Le_{im} `not in` $L_{i\ m}$.
4. `Apply a known radiocoloring algorithm for ordinary planar graphs, e.g. [1, 12] on` GDe_i `using color sets` A `and` C. `Ignore the colors of the vertices of` GDe_i `not in` GD_i.

D.3 Claims, Lemmas and the Theorem used in the Analysis of HRC_2 algorithm

Lemma 5. *The colors assigned to any double level of G_i, $L_{i\ m}$, $1 \leq m \leq \frac{D-2}{2}$, do not conflict with the colors of other levels of G_i or the colors of vertices of any graph that calls G_i or it is called by G_i.*

Proof. The lemma is proved by the following three claims.

Claim 5. Any vertex of a level k is at distance more than two from any vertex in a level $> k + 2$ or $< k - 2$ of G_i.

Proof. Assume that this is not true. That is there is a vertex u at level k which at distance at most two from a vertex v at level $k + j$, $j > 2$ $(k - j)$. However, note that vertex v should be contained in level $k + 1$ or $k + 2$ $(k - 1$ or $k - 2)$ according to the way the partition is constructed, a contradiction. □

Claim 6. For any $1 < m < \frac{D-2}{2}$, the colors assigned to any double level of G_i, $L_{i\ m}$, do not conflict with the colors of other levels of G_i.

Proof. Consider any such level $L_{i\ m}$. We study the colors of the vertices outside $L_{i\ m}$.

Consider a vertex of the first level of it, $l_{i\ 2m}$. Such a vertex might be at distance one from a vertex in level $l_{i\ 2(m-1)+1}$, at distance two from a vertex at level $l_{i\ 2(m-1)}$ and at distance two from a vertex of level $l_{i\ 2m+2}$. The vertices of levels $l_{i\ 2(m-1)}$ and $l_{i\ 2(m-1)+1}$ belong to the double level $L_{i\ (m-1)}$. The vertices of level $l_{i\ 2m+2}$ belong to the double level $L_{i\ (m+1)}$. Recall that the algorithm radiocolors levels $L_{i\ m-1}$, $L_{i\ m}$ and $L_{i\ m+1}$ interchanging the set of colors A and B. Thus, there is no conflict between the vertices of level $l_{i\ 2m}$ and the vertices of levels $L_{i\ m-1}$ or $L_{i\ m+1}$.

Moreover, any vertex of $l_{i\ 2m}$ is at distance more than two from any vertex of levels $l_{i\ 2(m-2)}$ or less, that is vertices of double levels $L_{i\ m-2}$ or less. Although the algorithm radiocolors levels $L_{i\ m}$ and $L_{i\ m-2}$ with the same set of colors, the above observation implies that there is no conflict between a vertex in $l_{i\ 2m}$ and a vertex of levels $L_{i\ m-2}$ or less.

Consider now a vertex of the second level of $L_{i\ m}$, $l_{i\ 2m+1}$. Such a vertex might be at distance one or two from the following vertices outside $L_{i\ m}$: It might be at distance one from a vertex of level $l_{i\ 2m+1}$, at distance two from a vertex of level $l_{i\ 2(m+1)+1}$ and at distance two from a vertex of level $l_{i\ 2m-1}$. The vertices of levels $l_{i\ 2(m+1)}$ and $l_{i\ 2(m+1)+1}$ belong to the double level $L_{i\ (m+1)}$. The vertices of level $l_{i\ 2m-1}$ belong to the double level $L_{i\ m-1}$. Recall that the algorithm radiocolors levels $L_{i\ m-1}$, $L_{i\ m}$ and $L_{i\ m+1}$ interchanging the set of colors A and B. Thus, there is no conflict between the vertices of level $l_{i\ 2m}$ and the vertices of levels $L_{i\ m-1}$ and $L_{i\ m+1}$.

Moreover, any vertex of $l_{i\ 2m+1}$ is at distance more than two from any vertex of levels $l_{i\ 2(m+2)}$ or greater, that is vertices of double levels $L_{i\ m+2}$ or greater. Although the algorithm radiocolors levels $L_{i\ m}$ and $L_{i\ m+2}$ with the same set of colors, above observation implies that there is no conflict between a vertex in $l_{i\ 2m}$ and a vertex of levels $L_{i\ m+2}$ or greater.

□

Claim 7. The colors assigned to the first and the last double level of $G_{inner\ levels\ i}$, i.e. levels $L_{i\ 1}$ and $L_{i\ \frac{D-2}{2}}$, do not conflict with the colors of other levels of G_i or the colors of vertices of any graph that calls G_i or it is called by G_i.

Proof. Consider the first double level $L_{i\ 1}$. We study the colors of the vertices outside $L_{i\ 1}$. Recall that the algorithm radiocolors the vertices of this double level using color set B.

Consider the levels preceding level $L_{i\ 1}$:

– Consider a vertex of the first level of $L_{i\ 1}$, $l_{i\ 2}$. As it concerns the levels preceding level $L_{i\ 1}$, a vertex of $l_{i\ 2}$ might be at distance one from a vertex in level $l_{i\ 1}$ and at distance two from a vertex of a graph G_j that calls G_i. Recall that the algorithm radiocolors the vertices of level $l_{i\ 1}$ together with the vertices of the graph G_j connected to them, using the sets of colors A and C. Thus, there is no conflict between the vertices of level $l_{i\ 2}$ and the vertices of level $l_{i\ 1}$ or the vertices of any graph G_j that calls G_i.

– Consider now a vertex of the second level of $L_{i\ 1}$, $l_{i\ 3}$. The vertices of previous double levels, near the vertices of level $l_{i\ 3}$ are only a subset of the vertices at distance one or two from the vertices of level $l_{i\ 2}$. Thus, there is no conflict between the vertices of level $l_{i\ 3}$ and the vertices of any previous level outside it.

Consider the levels following level $L_{i\ 1}$:

– Consider a vertex of the second level of $L_{i\ 1}$, $l_{i\ 3}$. As it concerns the next double levels, a vertex of $l_{i\ 3}$ might be at distance one from a vertex of level $l_{i\ 4}$ and at distance two from a vertex of level $l_{i\ 5}$. The vertices of levels $l_{i\ 4}$ and $l_{i\ 5}$ belong to the double level $L_{i\ 2}$. Recall, that the algorithm radiocolors level $L_{i\ 1}$ using the set of colors B. Also, it radiocolors level $L_{i\ 2}$ with a different set of colors, A. Thus, again there is no conflict between the vertices of the two levels.

– Consider now a vertex of the first level of $L_{i\ 1}$, $l_{i\ 2}$. The vertices of next double levels, outside level $L_{i\ 1}$, near the vertices of level $l_{i\ 2}$ are only a subset of the vertices at distance one or two from the vertices of $l_{i\ 3}$. Thus, there is no conflict between the vertices of level $l_{i\ 2}$ and the vertices of any next level outside it.

Consider the last double level $L_{i\ \frac{D-2}{2}}$. We study the colors of the vertices outside the level.

Consider the levels following level $L_{i\ \frac{D-2}{2}}$:

– Consider a vertex of the second level of $L_{i\ \frac{D-2}{2}}$, $l_{i\ D-1}$. As it concerns the next double levels, such a vertex might be at distance one from a vertex of level $l_{i\ D}$ and at distance two from a vertex of level $l_{i\ D+1}$. The vertices of levels $l_{i\ D}$ and $l_{i\ D+1}$ belong to the subgraph $G_{down\ levels\ i}$. Recall that the algorithm radiocolors level $L_{i\ \frac{D-2}{2}}$ using the set B. This is because the algorithm, if the number of double levels is odd, then obviously uses color set B at the last double level and otherwise (if this number is even) we are done by extending the last odd double level with the rest of the levels of G_i. It also radiocolors the vertices of $G_{down\ levels\ i}$ together with the first level subgraph of all graphs called by G_i using the sets of colors A and C. Thus, there is no conflict between the vertices of level $l_{i\ D-1}$ and the vertices of levels $l_{i\ D}$, $l_{i\ D+1}$ or greater.

– Consider now a vertex of the first level of $L_{i\ \frac{D-2}{2}}$, $l_{i\ D-2}$. The vertices of next levels, near the vertices of level $l_{i\ D-2}$ are only a subset of the vertices at distance one or two from the vertices of $l_{i\ D-1}$. Thus, there is no conflict between the vertices of level $l_{i\ D-2}$ and the vertices of any next level outside it.

Consider the levels preceding level $L_{i\ \frac{D-2}{2}}$:

– Consider a vertex of the first level of $L_{i\ \frac{D-2}{2}}$, $l_{i\ D-2}$. As it concerns the previous double levels, such a vertex might be at distance one from a vertex of level $l_{i\ D-3}$ and at distance two from a vertex of level $l_{i\ D-4}$. The vertices of levels $l_{i\ D-3}$ and $l_{i\ D-4}$ belong to the double level $L_{i\ (\frac{D-2}{2}-1)}$. Recall that the algorithm radiocolors level $L_{i\ \frac{D-2}{2}}$ using the set B. It also radiocolors the vertices of $L_{i\ (\frac{D-2}{2}-1)}$ using the set A. Thus, there is no conflict between the vertices of level $l_{i\ D-2}$ and the vertices of double levels $L_{i\ (\frac{D-2}{2}-1)}$ or less.

– Consider now a vertex of the second level of $L_{i\ \frac{D-2}{2}}$, $l_{i\ D-1}$. The vertices of previous levels, near the vertices of level $l_{i\ D-1}$ are only a subset of the vertices at distance one or two from the vertices of $l_{i\ D-2}$. Thus, there is no conflict between the vertices of level $l_{i\ D-1}$ and the vertices of any previous level outside it.

□

The previous 3 claims prove lemma 5. □

The following lemma is needed in order to manage to radiocolor optimally each double level in polynomial time.

Lemma 6 *For any double level L_{im} of $G_{inner\ levels\ i}$, $1 \le m \le \frac{D-2}{2}$ the subgraph Le_{im} is a 4-outerplanar graph.*

Proof. The lemma is proved by the following 3 claims.

Claim 8. The subgraph l_{im}, $1 \le m \le D-1$ is an outerplanar graph.

Proof. By the way the partition is constructed, using a BFS tree. □

Claim 9. The subgraph $L_{im} = l_{i\ 2m} \bigcup l_{i\ 2m+1}$, for any m, $1 \le m \le \frac{D-2}{2}$ is a 2-outerplanar graph.

Proof. The claim is proved by the way the partition is constructed, using a BFS tree and the previous claim.

Consider the last extended level $m = \frac{D-2}{2}$. For this case we have $L_{im} = l_{i\ D-2} \bigcup l_{i\ D-1}$. Note that $L_{i\ m}$ does not to belong to $G_{inner\ levels\ i}$ but to the subgraph $G_{down\ levels\ i}$. However, $L_{i\ m}$ was constructed by the partition into levels procedure applied on G_i.Thus, by the way the levels are constructed, we get the claim. □

Claim 10. The subgraph $Le_{im} = l_{i\ 2(m-1)+1} \bigcup L_{i\ m} \bigcup l_{i\ 2(m+1)}$, for any m, $1 \le m \le \frac{D-2}{2}$, is a 4-outerplanar graph.

Proof. Consider any extended level m with $1 < m < \frac{D-2}{2}$. For this case the claim is proved by combining the two previous claims.

Consider the first extended level $m = 1$. For this case we have $Le_{im} = l_{i\ 1} \bigcup L_{i\ 1} \bigcup l_{i\ 3} = G_{outUp\ i} \bigcup L_{i\ 1} \bigcup l_{i\ 3}$. Note that $G_{outUp\ i}$ does not belong to $G_{inner\ levels\ i}$ but to the subgraph $G_{first\ level\ i}$. However, $G_{outUp\ i}$ was constructed by the partitioning into levels procedure applied on G_i. Thus, by the way the levels are constructed, we get the claim.

Consider the last extended level $m = \frac{D-2}{2}$. For this case we have $Le_{im} = l_{i\ 2(m-1)+1} \bigcup L_{i\ m} \bigcup l_{i\ 2(m+1)} = l_{i\ D-3} \bigcup L_{i\ \frac{D-2}{2}} \bigcup l_{i\ D-1}$. Note that $L_{i\ m}$ does not to belong to $G_{inner\ levels\ i}$ but to the subgraph $G_{down\ levels\ i}$. However, $L_{i\ m}$ was constructed by the partition into levels procedure applied on G_i. Thus, by the way the levels are constructed, we get the claim. □

The previous 3 claims prove lemma 6.

□

Lemma 7. *The HRC_2 algorithm radiocolors any extended double level Le_{im} optimally in polynomial to the size of G_i time. This procedure produces a valid radiocoloring of double level L_{im}.*

Proof. Recall that HRC_2 gets a radiocoloring of the vertices of $L_{i\ m}$ by radiocoloring the extended level $Le_{i\ m}$. In order to prove the Lemma we have to prove that (i) the algorithm radiocolors optimally level $Le_{i\ m}$ in polynomial time (ii) By radiocoloring Le_{im}, we get an optimal radiocoloring of L_{im}.

(i) By claim 10 (used in the proof of Lemma 6, see Appendix, section E), $Le_{i\ m}$ is a 4-outerplanar graph. Thus, by Theorem 1, it is a bounded treewidth graph (11-bounded treewidth graph). Hence, by Theorem 2, it can be radiocolored optimally in polynomial time.

(ii) In order to prove that the radiocoloring of $L_{i\ m}$ resulting by the radiocoloring of $Le_{i\ m}$ is valid, we have to prove that the radiocoloring of the vertices of $L_{i\ m}$ is valid in G_i as it concerns those vertices. The subgraph $Le_{i\ m}$ consists of the vertices of the double level $L_{i\ m}$ and their neighbours. Observe that, the subgraph of G_i^2 induced by vertices of $L_{i\ m}$ is a subgraph of $Le_{i\ m}^2$. Thus, the radiocoloring of the vertices of $Le_{i\ m}$ constitutes a valid radiocoloring of the vertices of L_{im} on G_i.

□

Theorem 7. *HRC_2 Algorithm produces a valid radiocoloring of a hierarchical planar graph G of class DS3 using at most $3\ OPT(G) + 50$ colors, where $OPT(G)$ is the optimal number of colors needed for the radiocoloring of G.*

Proof. (Approximation ratio):

Consider any graph G_i in the hierarchy tree of G. The algorithm radiocolors the down levels subgraph of G_i together with the first level subgraphs of the graphs called by G_i (graph GD_i) using color sets A and C applying any known polynomial time radiocoloring algorithm. Employing the algorithm of [1, 12] we can compute such a radiocoloring in polynomial time using at most $2\Delta + 25$ colors. Hence, the sets of colors A and C of colors, with $|A| = |C| = max\{\Delta(G) + 25, OPT(G)\}$, used by the algorithm, are enough to produce a valid radiocoloring algorithm for GD_i.

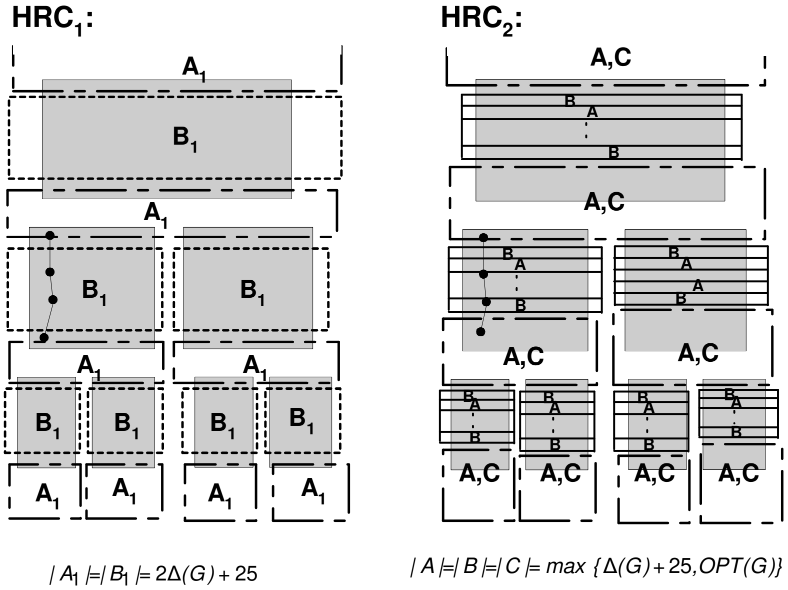

Fig. 8. The radiocoloring of a subtree of the hierarchical tree of G as specified by algorithms HRC_1, and HRC_2.

The inner levels subgraph of G_i is radiocolored using, at the first double level ($L_{i\,1}$), colors of set B, in the next double level colors of set A, etc, until the last double level ($\frac{D-2}{2}$) which is radiocolored again using colors of set B. The first and the last double level are colored using color set B because the algorithm, if the number of double levels is odd, then obviously uses color set B at the last double level and otherwise (if this number is even) we are done by extending the last odd double level with the rest of the levels of G_i. As we showed in lemmas 7 and 5, the radiocoloring computed of any double level is optimal and with no conflicts with the other levels of G_i or the vertices of the graphs that are called by it or call G_i.

But, what about the case where the inner levels subgraph consists of less than two levels? In this case a vertex of first level subgraph of G_i may be at distance 2 from a vertex of down levels subgraph of G_i. In this case since the two subgraphs are radiocolored using the same sets of colors (sets A, C) there might be conflicts between them. However, such a case does not appear in the class DS3 of hierarchical graphs considered here (definition 9), since any vertex of first level subgraph is at distance more than two from any vertex of down levels subgraph.

We conclude that for any graph G_i of the hierarchical graph G there is no conflict either between any two vertices in G_i or between a vertex of G_i and a vertex that is called by G_i or calls G_i.

Finally, since for the radiocoloring of all graphs G_i of G we use colors of sets A, B, C, the HRC_2 algorithm obtains a 3-approximation ratio for the radiocoloring of the hierarchical planar graph G.

See Figure 8 for a graphical illustration of the radiocoloring produced by the algorithm $HRC_2(G)$.

(Time complexity):

For each graph G_i we have the following computations:

The inner levels subgraph of G_i is radiocolored only once. The algorithm radiocolors the levels of the subgraph independently using the theorem of [39]. Since each level is of size $O(n)$, we require time $O(n \cdot T(n,k))$ for all levels to be optimally radiocolored. Since this procedure is called for each G_i, $1 \leq i \leq n$, the optimal radiocoloring of the levels of all graphs needs $O(n^2 \cdot T(n,k))$ time in total.

The down levels subgraph of G_i is also radiocolored only once together with the first level subgraphs of the graphs that are called by G_i using the algorithm of [12] or [1]. Such algorithms require $O(n^2)$ time for the radiocoloring of a graph of size n. Remark that when considering the down levels subgraph of G_i and the outer up vertices of the graphs called by G_i, we get a subgraph of size (number of vertices) n^2. Thus, the radiocoloring algorithm used needs time $O(n^4)$. Since this procedure is called for each G_i, $1 \leq i \leq n$, the algorithm needs $O(n^5)$ time in total.

Summing up, we get that in total the algorithm takes $O(n^2 \cdot T(n,k) + n^5)$ time in total, where $T(n,k) = O(n(2\alpha+1)^{2^{2(k+1)(l+2)+1}} + n^3)$, $l = 2$ and $\Delta(G) \leq \alpha \leq 2\Delta(G) + 25$. □