The complexity of approximating the oriented diameter of chordal graphs*

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Abstract

The oriented diameter of a (undirected) graph G is the smallest diameter among all the diameters of strongly connected orientations of G. We study algorithmic aspects of determining the oriented diameter of a chordal graph. We

- give a linear time algorithm such that, for a given chordal graph G, either concludes that there is no strongly connected orientation of G, or finds a strongly connected orientation of G with diameter at most twice the diameter of G plus one;
- prove that the corresponding decision problem remains NP-complete even when restricted to a small subclass of chordal graphs called split graphs;
- show that unless $P = NP$, there is neither a polynomial-time absolute approximation algorithm nor an α-approximation (for every $\alpha < \frac{3}{2}$) algorithm computing oriented diameter of a chordal graph.

Keywords: Diameter, orientation, chordal graph, approximation algorithm.

*The work of IR and MM is partially supported by FONDAP on Applied Mathematics, Fondecyt 1990616 and Fondecyt 1010442. Part of this job was done while FVF was a postdoc at CMM, supported by FONDAP. FVF acknowledges support by EC contract IST-1999-14186, Project ALCOM-FT (Algorithms and Complexity - Future Technologies).

1 Introduction

When the linkage structure of communication networks is modeled by the graph, the diameter of the graph corresponds to the maximum number of links over which a message between two nodes must travel. In cases where the number of links in a path is roughly proportional to the time delay or signal degradation encountered by messages sent along the path, the diameter is then involved in the complexity analysis for the performance of the networks.

A variety of interrelated diameter problems are discussed in the literature (see the survey of Chung [3] for further references), including the problem of finding orientations for undirected or mixed graphs to minimize diameters. This problem has a long history. In 1939 Robbins [16] proved that an undirected graph G admits a strongly connected orientation if and only if G is connected and bridgeless. More recently, Chung, Garey and Tarjan [4] provided a linear-time algorithm for testing whether a graph has a strong orientation and finding one if it does.

Chvátal and Thomassen [5], studied the following question: How are the diameter of a graph G and the diameter of a strongly connected orientation of G related? This leads to the following problem:

ORIENTED DIAMETER PROBLEM (ODP): Given a graph G, find a strongly connected orientation H with the smallest diameter.

This question is very basic and natural. We refer to Chapter 2 of Roberts book [17] and to Chapter 2 of Bang-Jensen & Gutin book [1] for some nice applications and discussions of oriented diameter problem.

1.1 Definitions

Let G be either a simple graph or a digraph with vertex set $V(G)$ and edge set $E(G)$. By $\{u, v\}$ we denote the undirected edge with ends in u and v and by (u, v) we denote the directed arc, directed from u toward v. The distance $d_G(u, v)$ between two vertices u and v of G is the length of the shortest path (the shortest directed path if G is directed) between u and v in the graph G (from u to v if G is directed). If there is no path from u to v then we put $d_G(u, v) = +\infty$. The diameter of a graph G, denoted by $\mathrm{diam}(G)$, is defined to be the maximum distance between two vertices of G. Thus $\mathrm{diam}(G) = \max\{d_G(u, v) : u, v \in V(G)\}$. We denote by $\vec{d}_H(u, v) = \max\{d_H(u, v), d_H(v, u)\}$.

An *orientation* of an undirected graph G is a directed graph whose arcs correspond to assignments of directions to the edges of G. An orientation H of G is *strongly connected* if every two vertices in H are mutually reachable in H ($\mathrm{diam}(H) < +\infty$). An edge e in a connected graph G is called a *bridge* if $G - e$ is not connected. A connected graph G is *bridgeless* if $G - e$ is connected for every edge e, i.e. there is no bridge in G. For a graph G let us define its oriented diameter.

$$OD(G) = \min\{\mathrm{diam}(H) \colon H \text{ is an orientation of } G\}.$$

As it was proved by Robbins in 1939, if G is not connected or has a bridge, then there is no strongly connected orientation of G. In that case $OD(G) = +\infty$. Further we consider only connected bridgeless graphs and strongly connected orientations.

We say that an algorithm $\mathcal{A}$ is an (α, k)*-approximation algorithm* for ODP if for every graph G it runs in polynomial time and outputs an orientation H of G such that $\text{diam}(H) \leq \alpha OD(G) + k$.

An $(\alpha, 0)$-approximation algorithm for ODP is called an α*-approximation algorithm* for ODP and an $(1, k)-$approximation algorithm for ODP is called an *absolute approximation algorithm* for ODP.

A *chord* of a cycle C in G is an edge not in C that has both end in C. A *chordless cycle* in G is a cycle of length more than three that has no chord. A graph G is *chordal* if it contains no chordless cycle.

1.2 Known results

Chvátal and Thomassen [5] showed that ODP is NP-hard for general graphs. Therefore, there is a natural interest to investigate the complexity issues of ODP for different graph classes.

Some graph classes for which ODP can be computed are known. However, the direction of previous studies were mainly directed on finding graph classes for which oriented diameter is equal to the diameter of a graph. Koh and Tay [11, 12, 13] study ODP for the Cartesian products of several simple graph classes. Šoltés [20] obtained some results on complete bipartite graphs and Gutin [9] investigated n-partite complete graphs. Konig, Krumme & Lazard [14] study the orientation problem on the torus. The case of planar grids was studied by Roberts & Xu [18].

Chung [3] discusses some implications and results on ODP in context of network routing. The surveys of Hedetniemi, Hedetniemi & Liestman [10] and Fraigniaud & Lazard [7] stress the importance of ODP in study communication networks problems like broadcasting and gossiping. Discussion on one-way street arrangements problems and it relation to ODPcan be found in the book of Roberts [17].

Chordal graphs form very well investigated class of graphs. They have well understood nice properties and many NP hard problems like COLORING, CLIQUE, INDEPENDENT SET can be solved fast when the input is restricted to chordal graphs. We refer to Golumbic book [8] for the introduction and to Brandstädt, Le, & Spinrad book [2] for more recent results on chordal graphs.

Not much was known about the algorithmic aspects of ODPD for chordal graphs. Chvátal and Thomassen [5] proved that every bridgeless connected graph G admits a strongly connected orientation H with the property

(P) If an edge $\{u, v\}$ belongs to a cycle in G of length k, then (u, v) or (v, u) belongs to a directed cycle in H of length at most $(k-2)2^{[(k-1)/2]} + 2$.

For a graph G where every edge belongs to a triangle, Property (P) tell us that the oriented diameter of G is at most 3 times the diameter of G. Connected bridgeless chordal graphs are graphs for which every edge belongs to a triangle. Therefore, Property (P) suggests the existence of a 3-approximation algorithm for ODP when restricted to chordal graphs. The searching of both better approximation algorithms and hardness results for ODP when restricted to chordal graphs, motivates this paper.

1.3 Our Contribution

In Section 2 we show that for every chordal graph G there exists computable in linear time orientation H such that for every pair of vertices u and v, $\vec{d}_H(u,v) \leq 2d_G(u,v)+1$. Notice that this result implies that for chordal graphs ODP is $(2,1)-$approximable. To show that the bound is sharp we construct an infinite sequence of chordal graphs such that for every graph G from this sequence any orientation of G has diameter at least $2\,\mathrm{diam}(G_n)+1$.

In Section 3 we prove that ODP remains NP-hard in the subclass of chordal graphs called split graphs. Moreover, we prove two non approximability results: first, for every $\alpha < \frac{3}{2}$ ODP is not $\alpha-$approximable in the class of split graphs; second, there is no absolute approximation algorithm for ODP when restricted to chordal graphs.

2 Positive results

Our algorithmical contribution is stated in the following theorem.

Theorem 2.1. *There is a linear time $(2,1)$-approximation algorithm for* ODP *in the class of chordal graphs.*

Theorem 2.1 follows from the next much stronger result.

Theorem 2.2. *For every bridgeless connected chordal graph G there exists computable in linear time orientation H such that, for every pair of vertices u and v, $\vec{d}_H(u,v) \leq 2d_G(u,v)+1$.*

Notice that Theorem 2.2 is an improvement of the bound of Chvátal and Thomassen applied to chordal graphs. Moreover, as stated in Theorem 2.3, this bound is the best possible.

Theorem 2.3. *For every $n \geq 1$ there exists a chordal graph G_n such that* $\mathrm{diam}(G_n) = 2n+1$ *and* $\mathrm{diam}(H) \geq 2\,\mathrm{diam}(G_n)+1$ *for every orientation H of G_n.*

The proof of this theorem has been moved to the appendix.

The rest of this section is devoted to the proof of Theorem 2.2. The proof of this theorem is indirect. First we prove that every 2-connected chordal graph has a special orientation that can be obtained from his perfect elimination ordering (Lemma 2.7). Then we use this orientation to prove the theorem for 2-connected graphs and only then extend the result on bridgeless graphs.

Let us begin with some definitions. For a given chordal graph G and an orientation H of its edges we say that an arc in H is *good* if it belongs to a directed triangle and it is *bad* otherwise. A good orientation is an orientation leaving every arc good. Let K_n be the complete graph with n vertices.

In order to orient chordal graphs we need first to construct good orientations of complete graphs K_n for $n \geq 5$.

Lemma 2.4. *For every $n \geq 5$ there exists a good orientation of K_n. Moreover, every good orientation of K_n can be extended to a good orientation of K_{n+1} and this extension can be found in linear time.*

Proof. Let us think that K_{n+1} is obtained from K_n by adding new vertex v and let H_n be a good orientation of K_n.

If n is even then a good orientation of K_{n+1} can be obtained from orientation of K_n by forming $\frac{n}{2}$ directed triangles using all edges adjacent to v. The orientation of every triangle is induced by the arc from H_n. Clearly, this orientation can be done in $O(n)$ steps.

Suppose that n is odd. Observe first that for any $n \geq 4$ and every orientation H of K_n there are three vertices in K_n inducing a triangle that is not strongly connected. Let a, b and c be such vertices for the orientation H_n. W.l.o.g. we may think that the arcs in H are of the form (a, b), (a, c) and (b, c). The remaining $n-3$ edges adjacent to v are in $(n-3)/2$ triangles, each of the triangles having one arc in K_n. We orient these edges as in the previous case. So to obtain the orientation of K_{n+1} one should choose four arbitrary vertices in K_n and find three vertices that do not undue strongly connected triangle. This can be done in constant number of steps. And the orientation of the remaining $n-3$ edges can be done in $O(n)$ steps. $\square$

Notice that for $n = 4$ the statement of Lemma 2.4 is not true. In terms of diameter we have the following corollary which will be used in Section 3.

Corollary 2.5. *For every $n \geq 4$ there exists an orientation of K_n with diameter 2 if $n \geq 5$, and with diameter 3 if $n = 4$.*

We first consider 2-connected chordal graphs. A connected graph G is said to be 2-*connected* if for every vertex v, the graph $G - \{v\}$ is connected. At the end of the section we show how to orient a chordal graph G by using the orientations of its 2-connected components.

A vertex v in a graph G is called *simplicial* if the graph induced by its neighborhood $N_G(v)$ is a clique. By the classical result of Dirac [6], chordal graphs have been characterized as those having a *perfect elimination ordering.* This is a vertex ordering $\{v_1, \ldots, v_n\}$ such that for every $i \in \{1, \ldots, n\}$, the vertex v_i is simplicial in $G_i := G[v_1, \ldots, v_i]$ (where $G[S]$ denotes the graph induced by the vertex set S). A perfect elimination ordering of a chordal graph can be found in linear time by using LexBFS (see the pioneering paper of Rose, Tarjan & Lueke [19]).

The idea of our construction is to orient all the edges incident to v_i in $G[v_1, \ldots, v_i]$ sequentially (following the perfect elimination ordering). If a chordal graph doesn't contain a maximal clique of size 4 then using Lemma 2.4 one can construct an orientation of G with diameter at most $2\,\mathrm{diam}(G)$ easily. The main problem we have to deal with is the existence of the non good orientable cliques K_4 in chordal graphs.

Let $\delta = (v_1, \ldots, v_n)$ be a perfect elimination ordering of chordal graph G. We say that vertex v_i is *super-simplicial* (subject to δ) if $N(v_i) \cap \{v_i, v_{i+1}, \ldots, v_n\} = \emptyset$. Notice that every super-simplicial vertex is simplicial but not vice versa.

We need the following technical lemma about super-simplicial vertices.

Lemma 2.6. *Let $\delta = (v_1, \ldots, v_n)$ be a perfect elimination ordering of a 2-connected chordal graph G. Then if v_i is not super-simplicial and $N(v_i) \cap \{v_1, v_2, , \ldots, v_{i-1}\} \neq \emptyset$ then there are $k > i > l$ such that v_k, v_i, v_l is a clique in G.*

Proof. Let v_p and v_q, $p > i > q$, be vertices adjacent to v_i. If $\{v_p, v_q\} \in E(G)$ then $\{v_p, v_q, v_i\}$ induce a clique and the lemma is proved. If $\{v_p, v_q\} \notin E(G)$ then vertices $\{v_p, v_q, v_i\}$ belong to a cycle C in G (G is 2-connected). We choose C to have the shortest length among all cycles containing $\{v_p, v_q, v_i\}$. Notice that the length of C is at least 4. The cycle contains at least on vertex which is before (in δ) v_i and at least one vertex that is after v_i. Therefore, there are two adjacent vertices $v_{p'}$, $v_{q'}$ with $p' > i > q'$. Because C is the shortest cycle, the only chords in this cycle are the edges adjacent to v_i. Then chordality of G implies that v_i is adjacent to $v_{p'}$ and $v_{q'}$ which concludes the proof of the lemma. $\square$

Lemma 2.7. *There exists a linear time algorithm that given a 2-connected chordal graph G and a perfect elimination ordering $\{v_1, \ldots, v_n\}$ of $V(G)$ computes an orientation H with the following properties.*

(a) Every maximal clique in G has at most one bad arc in H.

(b) If (u, v) is a bad arc in H then u is a simplicial vertex (in the perfect elimination ordering) of $V(G)$.

(c) For every $v \in V(G)$, $\vec{d}_H(v, v_1) \leq 2d_G(v, v_1)$.

(d) Every clique in H has diameter at most 3.

Proof. Iteratively, for $k = 3, \ldots, n$ we construct an orientation H_k of $G_k = G[v_1, \ldots, v_k]$ with the following properties.

(P1) Every bad arc belongs to a maximal clique (in H_k) of size four or two.

(P2) At most one arc is bad in each maximal clique.

(P3) If (u, v) is a bad arc in H then u is either a super-simplicial vertex in G or the vertices u, v are used in some step $j > k$ to form a new clique, i.e. $u, v \in N_{G_j}[v_j]$ for some $j > k$.

Clearly, for $k = 3$ such an orientation exists. We extend the orientation H_k to an orientation H_{k+1} of G_{k+1} satisfying properties (P1), (P2) and (P3).

Suppose that $N_{G_{k+1}}(v_{k+1}) = \{u_1, ..., u_r\}$. Notice that $r \geq 2$ since G is 2-connected.

1. If $r > 4$ then by (P1) we have that every arc in $H_k[u_1, ..., u_r]$ is good. We use Lemma 2.4 to get a good orientation of $G[u_1, ..., u_r, v_{k+1}]$.

2. For $r = 4$ we use Lemma 2.4 to get a good orientation of $G[u_1, ..., u_r, v_{k+1}]$. For $r = 2$ we orient new edges in a directed triangle following the orientation given to $\{u_1, u_2\}$. In both cases the bad arc in H_k (if any) belongs to one of the directed triangles in $H[u_1, ..., u_r, v_{k+1}]$ and is good in H_{k+1}.

3. For $r = 3$ we consider three cases.

   (i) If $H_k[u_1, u_2, u_3]$ contains a bad arc, say (u_1, u_2), then we direct the new edges obtaining the following arcs: (u_2, v_{k+1}) and (v_{k+1}, u_1). Moreover, if $(u_1, u_3) \in H_k$ then we add (u_3, v_{k+1}) to H_{k+1}. Otherwise we add (v_{k+1}, u_3) to H_{k+1}. Then the arcs (u_2, v_{k+1}), (v_{k+1}, u_1) and (u_1, u_2) are in a directed triangle and the arc between v_{k+1} and u_3 is also in a directed triangle.

(ii) If $H_k[u_1, u_2, u_3]$ has no bad arcs and v_{k+1} is not super-simplicial (with respect to the perfect elimination ordering) then by Lemma 2.6 at least one edge, say, with ends in $\{v_{k+1}, u_1\}$, is used in a step $j > k$. Then we direct edges $\{v_{k+1}, u_2\}$ and $\{v_{k+1}, u_3\}$ to form a directed triangle with arc (u_2, u_3) (or (u_3, u_2)) and we add the bad arc (v_{k+1}, u_1).

(iii) If $G[u_1, u_2, u_3]$ has no bad arcs and v_{k+1} is super-simplicial then we direct edges $\{v_{k+1}, u_2\}$ and $\{v_{k+1}, u_3\}$ to form a directed triangle with arc (u_2, u_3) (or (u_3, u_2)) and we add the bad arc (v_{k+1}, u_1), where the vertex u_2 has among all u_i the minimum distance in G to v_1.

It is easy to see that the orientation H_{k+1} satisfies properties (P1), (P2) and (P3).

Clearly the orientation H satisfies properties (a) and (b). We prove that H satisfies Property (c) by induction in k. As before let $\{u_1, \ldots, u_r\} = N_{G_{k+1}}(v_{k+1})$. Let us assume that for all $v \in G_k$

$$\vec{d}_H(v_1, v) \le 2d_G(v_1, v) \tag{1}$$

If there is no bad arcs in H connecting v_{k+1} with $u_1, \ldots, u_r$ we have that $\vec{d}_H(u_i, v_{k+1}) \le 2$ for all $i = 1, \ldots, r$ and (1) holds for G_{k+1}.

If v_{k+1} is connected to some u_i by a bad arc in H then $r = 3$, $\{u_1, u_2, u_3\}$ induces a directed triangle in H and v_{k+1} is a simplicial vertex of G. Therefore $d_G(v_1, v_{k+1}) = d_G(v_1, u_2) + 1$ where u_2 is the vertex having minimum distance to v_1 in G, among all u_i, $i = 1, 2, 3$. By the construction of H, there exists a directed triangle that contains v_{k+1} and u_2 which implies that $\vec{d}_H(u_2, v_{k+1}) \le 2$. Therefore $\vec{d}_H(v_1, v_{k+1}) \le 2d_G(v_1, u_2) + 2 = 2d_G(v_1, v_{k+1})$. Property (d) follows from (P2).

Finally, we claim that for every k the orientation of the arcs adjacent to v_{k+1} during the extension of orientation H_k to H_{k+1} can be performed in $O(|N(v_k)|)$. We assume that the set of super-simplicial vertices is known. (Clearly this set can be computed in linear time $O(\sum_{v \in V(G)} |N(v)|) = O(|E(G)|)$.)

If we are in the cases 1 or 2 then the orientation of arcs can be performed in $O(|N(v_k)|)$ steps by Lemma 2.4. If we are in case 3 then subcase (i) is performed in a constant number of steps. For subcases (ii) we should be able to find a vertex from $\{u_1, u_2, u_3\}$ which has a common neighbor with v_k in $\{v_{k+1} \ldots, v_n\}$. Every neighbor of v_k in $\{v_{k+1}, \ldots, v_n\}$ has at most 3 neighbors in $\{v_1, v_2, , \ldots, v_{k-1}\}$ and such a vertex can be found in $O(|N(v_k)|)$ steps. The subcase (iii) takes constant number of steps.

Finally, the complexity of the algorithm is $O(\sum_{1 \le k \le n} |N(v_k)|) = O(|E(G)|)$. $\square$

Lemma 2.8. *There exists a linear time algorithm that given a 2-connected chordal graph G computes an orientation H such that, for every pair of vertices u and v, $\vec{d}_H(u, v) \le 2d_G(u, v) + 1$.*

Proof. Given G the algorithm first computes (in linear time) a perfect elimination ordering and then the orientation H (in linear time) given by Lemma 2.7. We prove that H has the desired property.

Take $u, v \in V(G)$ and let P be a shortest $(u, v)-$path in G. If $d_G(u, v) = 1$ then u, v are in some clique C, $|C| \ge 3$. From Property (d) we have $\vec{d}_H(u, v) \le 3$.

Suppose that $d_G(u,v) > 1$. Clearly, the inner vertices of P cannot be simplicial; therefore each arc in H associated to some inner edge of P is contained in a directed triangle in H.

If the arc associated to the edge in P incident to u is bad then u is simplicial and the arc is directed from u in the orientation H. Therefore, if the arc e associated to the edge of P incident with v is contained in a directed triangle (in H), then we get $\vec{d}_H(u,v) \leq 2d_G(u,v)$.

If the arc e is bad then v is simplicial and $e = (y,v)$. But $\vec{d}_H(y,v) \leq 3$. Hence, $\vec{d}_H(u,v) \leq \vec{d}_H(u,y) + \vec{d}_H(y,v) \leq 2d_G(u,y) + 3 = 2d_G(u,v) + 1$. □

Considering the special rôle that vertex v_1 plays in the orientation H obtained in Lemma 2.7, we say that H is rooted in v_1 and by extension we say that G is rooted in v_1.

Proof of Theorem 2.2. For a simplicial decomposition $\{v_1, \ldots, v_n\}$, we consider the 2-connected component C_0 of G that contains v_1 and we orient it as in Lemma 2.8. The set of 2-connected components has a tree-like structure T which we consider rooted in C_0. By the classical result of Tarjan [21], the tree-like structure of 2-connected component can be computed in linear time. Notice that the notion of father and sons of a 2-connected component is well defined. For every 2-connected component C we define its *father* cut vertex as the unique vertex in C which belongs to its father.

To each 2-connected component we assign an orientation as in Lemma 2.7 rooted in its father cut vertex. Let H be the orientation of G so obtained. In each 2-connected component the construction of H is done in linear time by Lemma 2.7. the orientation H is computable in linear time.

Let u and v be two vertices of G and let P be a shortest path between u and v in G. If P has no cut points then u and v lay in the same 2-connected component. From Lemma 2.8 and the construction of H we have $\vec{d}_H(u,v) \leq 2d_G(u,v) + 1$. Otherwise, let $u_1, \ldots, u_r$ be the cut points in P and C_i the 2-connected component containing u_i and u_{i+1} for $i = 1, \ldots, r-1$. Notice that for at most one i_0 neither u_{i_0} nor u_{i_0+1} are father cut vertices of C_{i_0}. From the construction of H we know that $\vec{d}_H(u_i, u_{i+1}) \leq 2d_G(u_i, u_{i+1})$ for all $i = 1, \ldots, r$ not equal to i_0 (in this case $\vec{d}_H(u_{i_0}, u_{i_0+1}) \leq 2d_G(u_{i_0}, u_{i_0+1}) + 1$). Therefore, $\vec{d}_H(u,v) \leq 2d_G(u,v) + 1$. □

3 Negative results

Our first step is to prove the NP-hardness of ODP for chordal graphs. In fact we will prove a stronger result: The NP-hardness of ODP for split graphs. A graph G is a split graph if its vertex set $V(G)$ can be partitioned into sets C and I such that C is a clique and I is an independent set. Split graphs form a subclass of chordal graphs of diameter at most 3.

Our proof, inspired by the one of Chvátal and Thomassen [5], relies on the NP-completeness of the 2-coloring problem for hypergraphs obtained by Lovász [15]. Let us recall that a hypergraph $\mathcal{H}$ is called 2-colorable if its vertices can be colored red and blue in such a way that every edge includes at least one vertex of each color.

Lemma 3.1. *For every $k \geq 0$ and for every hypergraph $\mathcal{H}$ there exists a chordal graph $G^k_{\mathcal{H}}$ (split graph for $k = 0$) such that if $\mathcal{H}$ is 2-colorable then $OD(G^k_{\mathcal{H}}) = 2(k+1)$ and if $\mathcal{H}$ is not 2-colorable then $OD(G^k_{\mathcal{H}}) = 3(k+1)$*

The proof of this lemma is moved to the appendix.

Theorem 3.2. ODP *is NP-hard for split graphs.*

Proof. Taking $k = 0$ in Lemma 3.1 the 2-coloring problem for hypergraphs can be reduced to ODP in polynomial time. $\square$

Now we prove two results concerning the hardness of approximating the oriented diameter.

Theorem 3.3. *Let $\alpha < \frac{3}{2}$. Unless $P = NP$,* ODP *has no α-approximation algorithm for split graphs.*

Proof. Let $A(G)$ be the orientation assigned to a graph G by an α-approximation algorithm for ODP. If $\mathcal{H}$ is 2-colorable then $G_{\mathcal{H}}$ has diameter 2. Thus $\operatorname{diam}(A(G_{\mathcal{H}})) \leq 2\alpha < 3$. On the other hand, if $\mathcal{H}$ is not 2-colorable then every orientation of the graph $G_{\mathcal{H}}$ has diameter at least 3. Whence $\operatorname{diam}(A(G_{\mathcal{H}})) \geq 3$. Therefore, since $G_{\mathcal{H}}$ can be constructed in polynomial time, by knowing $\operatorname{diam}(A(G_{\mathcal{H}}))$ we can decide the 2-colorability of $\mathcal{H}$. $\square$

Theorem 3.4. *Unless $P = NP$, there is no absolute approximation algorithm for* ODP *when restricted to chordal graphs.*

Proof. Let us assume that there exist K and an absolute approximation algorithm for ODP such that $\operatorname{diam}(A(G)) \leq OD(G) + K$. By using this algorithm we could decide the 2-coloring problem for hypergraphs. Let $\mathcal{H}$ be a hypergraph and $k > K$. From Lemma 3.1 there exists a chordal graph $G^k_{\mathcal{H}}$ computable in polynomial time such that, if $\mathcal{H}$ is 2-colorable, then G^k_H has diameter $2k + 2$. Thus $\operatorname{diam}(A(G^k_{\mathcal{H}})) \leq 2k + 2 + K < 3k + 2$. And, if H is not 2-colorable then $G^k_{\mathcal{H}}$ has diameter $3k + 2$. Thus $\operatorname{diam}(A(G^k_{\mathcal{H}})) \geq 3k + 2$. $\square$

4 Concluding remarks

In this paper we have provided linear time $(2, 1)$-approximation algorithm for oriented diameter of chordal graphs. From another hand, we proved that for ever $\alpha < 3/2$ finding an orientation with diameter at most α times the oriented diameter is NP hard. The challenging question is to decrease the gap between these lower and upper bounds. But even existing of 2-approximation algorithm is an intersting open problem.

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A Appendix

Proof of Theorem 2.3. In Figure 1 we show a chordal graph G_2 of diameter 5 for which there is no orientation with diameter smaller than $2 \cdot 5 + 1$. This construction can be easily generalized to larger graphs. $\square$

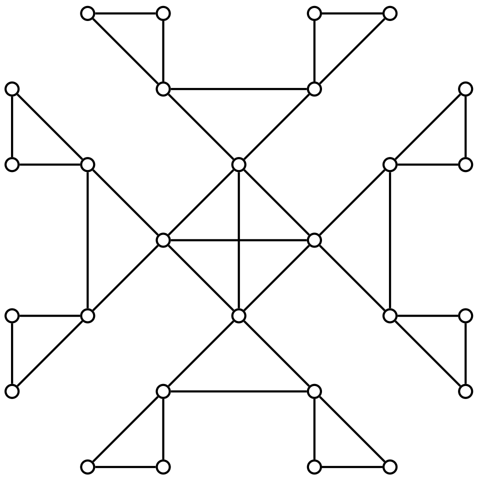

Figure 1: Connected bridgeless chordal graph of diameter 5.

Proof of Lemma 3.1. We first consider the case $k = 0$. For a given hypergraph $\mathcal{H}$, we will construct a split graph $G^0_{\mathcal{H}} = G_{\mathcal{H}}$ such that $\mathcal{H}$ is 2-colorable if and only if there is an orientation of $G_{\mathcal{H}}$ of diameter 2. Let $\mathcal{H}$ be a hypergraph with vertex set V of size n and edge set E of size m.

The clique C of $G_{\mathcal{H}}$ contains $n + 2m + 2$ vertices. More precisely, $C = V \cup Y$ with $V = \{v_1, v_2, \ldots, v_n\}$ being the vertex set of $\mathcal{H}$, and $Y = \{\alpha, \beta\} \cup E_1 \cup E_2$ where E_1 and E_2 are copies of the edge set E of $\mathcal{H}$. The independent set I of $G_{\mathcal{H}}$ contains $m + 1$ vertices. More precisely, $I = \{x\} \cup E$.

Now let us explain how to connect the vertices of I with those of C. The vertex x is connected to all the vertices of V. A vertex $e \in E$ is connected to a vertex $v \in V$ if and only if $v \in e$ (in the hypergraph $\mathcal{H}$). Finally, every vertex $y \in Y$ is connected to every vertex e of E. Clearly, $OD(G_{\mathcal{H}}) \leq 3$.

Let D be an orientation of $G_{\mathcal{H}}$ of diameter two. The way we color every vertex v of the hypergraph $\mathcal{H}$ is the following: If according to D, the edge connecting vertex x with v is oriented towards v, then we color it red (otherwise we color it blue). Since for every vertex $e \in E$ the distance $d_D(x, e) = d_D(e, x) = 2$, it follows that every edge e in $\mathcal{H}$ contains a red and a blue vertex.

Now let us suppose that $\mathcal{H}$ is 2-colorable. Let us denote by R and B the set of red and blue vertices in V. We partition the sets R and B as follows: $R = \{r\} \cup R'$, $B = \{b\} \cup B'$ (we just need the sets R', B', E_1, E_2 to have more than 5 elements; we can assume without loss of generality that the 2-coloring problem is restricted to instances satisfying this). In Corollary 2.5, we showed that we can orient the internal edges of R', B', E_1, E_2 in order to achieve, for each subgraph, an internal diameter 2.

The rest of the orientation is described in the following $0-1$ matrix. A value 1 in the position (P, Q) means that all the edges connecting the vertices of P with those of Q are oriented from P towards Q.

	x	r	R'	b	B'	α	β	E	E_1	E_2
x		1	1							
r				1		1	1			
R'					1				1	1
b	1					1	1			
B'	1								1	1
α			1		1			1	1	
β			1		1	1				1
E							1			1
E_1		1		1			1	1		
E_2		1		1		1			1	

The only edges that have not been oriented yet are those connecting the subsets R and B with E. By orienting them we will solve the problem of reaching E from x and reaching x from E. The solution is easy: We orient all the edges between R and E from R towards E and all the edges between B and E from E towards B.

The last problem is to reach any $e \in E$ from any other $e' \in E$. For achieving this we slightly modify the orientation between the sets E, E_1 and E_2. We identify m disjoint directed triangles of the form $e \to e_1 \to e_2 \to e$, with $e \in E$, $e_1 \in E_1$ and $e_2 \in E_2$, and we reverse the order getting $e_2 \to e_1 \to e \to e_2$. The reader should be able to verify that the diameter of D is at most 2.

For $k > 0$ the chordal graph $G^k_{\mathcal{H}}$ is constructed from several copies of the split graph $G^0_{\mathcal{H}} = (C, \{e_1, \ldots, e_m, x\})$ arranged in a tree-like structure as shown in Figure 2. In this way we are able to "amplify the gap" of the diameter of $G^0_{\mathcal{H}}$ according to the colorability of $\mathcal{H}$. $\square$

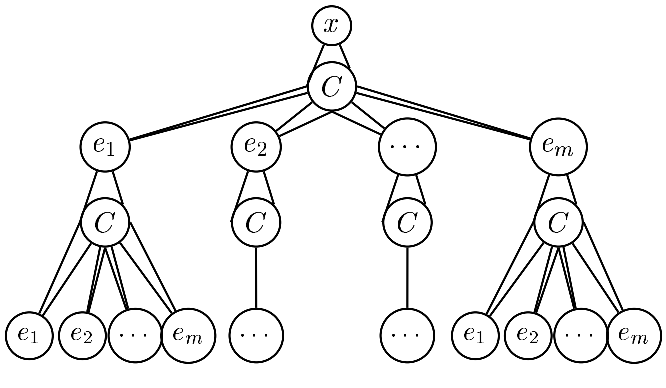

Figure 2: Tree-like structure of $G^2_{\mathcal{H}}$.