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Limit results for Markovian models of TCP

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Abstract. We study in this paper the throughput of a TCP connection performing ideal congestion avoidance. With respect to existing studies published on the same topic, we determine upper and lower bounds for the throughput of a TCP connection experiencing an arbitrary constant loss. We then derive exact convergence results when the loss probability tends to 0. A remarkable property of the results obtained in this paper is that they justify a posteriori constants, which appear in the approximation of the throughput and which has been observed earlier via simulation or experiments. The analysis is performed by assuming a finite and infinite maximum congestion window size.

Keywords. TCP. Markovian models of congestion avoidance. Limit results for throughput and stationary probability.

I - Introduction

The immense commercial success of the Internet has motivated over the past few years numerous studies to understand the behavior of Internet applications. In particular, special attention has been paid to modeling TCP friendly protocols in the presence of packet loss, and especially to deriving the throughput of a TCP bulk data transfer. In the absence of timeouts, it was observed in [4, 3, 7] that the throughput of a TCP connection is given by

$$\rho = \frac{c}{R\sqrt{p}}, \tag{1}$$

where c is an empirical constant taking values 1.22 or 1.31, depending on whether the receiver uses delayed acknowledgements or not, R is the round trip time experienced by the packets the connection, and p is the the expected number of window reduction events per packet sent; p is not exactly equal to the packet loss probability but is the frequency of loss per packet sent [6]; as a first approximation, p may nevertheless be taken equal to the packet loss probability. Finally, note that in the above formula the MTU is taken equal to one. A formal analysis of the empirical formula (1) has been performed by Ott *et al.* in [6, 9].

The analysis of the throughput of a TCP bulk data transfer has been the subject of further research, in particular by including in models a maximum window size and timeouts. In Padhye *et al.* [10, 11], it was shown via the analysis of a finite state Markov model that the throughput of a TCP connection is given by

$$\frac{w}{R} \wedge \frac{1}{R\sqrt{\frac{2bp}{3}} + T_0 \min\left\{1, 3\sqrt{\frac{3bp}{8}}\right\} p(1+32p^2)}, \tag{2}$$

where $x \wedge y = \inf\{x, y\}$, w is the maximum window size, T_0 is the timeout value, and b is the number of packets acknowledged by one ACK received by the sender. The above formulae have been obtained by analyzing a finite state Markov chain and by approximating its stationary distribution.

In this paper, along the same line of investigations as [6, 9], we study the Markovian structure of the process associated with the evolution of the sender window size over time. We specifically consider the behavior of a TCP connection performing ideal congestion avoidance. In a first step, we analyze the system without assuming that the packet loss probability is small and we obtain upper and lower bounds for the throughput of the connection.

Secondly, we establish rigorous convergence results for the associated Markov process when the packet loss probability tends to zero. We give the *explicit* probability distribution of the stationary size of the congestion window in the limit. These convergence results are obtained for both finite and infinite sender window sizes. Up to now most of the studies assumed an infinite maximum window size, some truncation approximation had to be used to include this constraint.

The remarkable property of the system analyzed here is that the empirical constant $c = 1.31$ appears in this paper as a universal constant, which is a characteristics of the additive increase multiplicative decrease scheme performed in congestion avoidance. Note that 1.31 is really the constant appearing in equation (1) according to the experiments and simulations (see [4] for example), and not the constant $\sqrt{3/2}$ as suggested by equation (2).

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The organization of this paper is as follows: In Section 2, the model is described and upper and lower bounds for the throughput of a TCP connection experiencing an arbitrary constant loss probability are obtained. In Section 3, exact convergence results for the throughput are established for a finite or infinite maximum congestion window size.

II - Model description and preliminary results

We consider a persistent unicast TCP bulk data transfer. Let R denote the round trip time experienced by packets of the TCP connection under consideration. Time is then naturally divided into slots of length R. Let W_n denote the congestion window size for the nth time interval. If $W_n = x$ then the sender can send up to x packets over the nth time interval. To simplify the notation, the MTU is taken equal to one.

We assume that the probability p that one packet is lost is equal to a constant. In the following we assume that the probability that a packet is lost is a constant for all packets and that there is no correlations between successive loss events. This is clearly a simplification, which has been used in most of the analyzed models (see [7]). Nevertheless, even if routers available today on the market implement RED packet discard policies, implying burst packet loss, the above assumption is still relevant when TCP connections traverse a "low bandwidth" access network, where the rate of TCP connections is naturally limited by the link rates of the access network.

Under the above assumption, we are led to consider the following stochastic process $\{W_n\}$ taking values in $\mathbb{N}$ and defined as follows: if $W_0 = x \in \mathbb{N}$,

$$W_1 = \begin{cases} x+1 & \text{with probability } \theta^x \\ \lfloor x/2 \rfloor & \text{otherwise,} \end{cases} \tag{3}$$

where $\theta = 1-p$. In this section, we assume that there is no upper limit of the congestion window size.

The mean drift starting from x is

$$\mathbb{E}(W_1) - x = \theta^x - x\left(1-\theta^x\right)/2;$$

this quantity converges to $-\infty$ as x gets large, Foster's criterion (see Meyn and Tweedie [5] for example) entails that the process $\{W_n\}$ is ergodic; $\{\pi_n\}$ denotes the stationary probability distribution of the stochastic process $\{W_n\}$.

Proposition 1 *The stationary probability distribution $\{\pi_n\}$ of the congestion window size can be written as*

$$\pi_n = \theta^{n(n-1)/2} c_n(\theta), \quad n \geq 0,$$

where

$$c_1(\theta) = \frac{c_0(\theta)}{1-\theta}, \quad \frac{1}{1-\theta} \leq \frac{c_n(\theta)}{c_0(\theta)} \leq \alpha_n(\theta),\ n \geq 1. \tag{4}$$

with $\alpha_n(\theta) = 1/\prod_{k=1}^{n}(1-\theta^{(3k-1)k/2})$ for $n \geq 1$. The sequence $\{c_n(\theta)\}$ converges to a finite limit $c(\theta)$ when $n \to \infty$.

Proof. The equilibrium equations for the probability $\{\pi_n\}$ give the recurrence relation, for $n \geq 1$,

$$\pi_n = \theta^{n-1}\pi_{n-1} + (1-\theta^{2n})\pi_{2n} + (1-\theta^{2n+1})\pi_{2n+1} \tag{5}$$

and $\pi_1 = \pi_0/(1-\theta)$ In particular, the above equation shows that $\pi_n > \pi_{n-1}\theta^{n-1}$ for all $n \geq 1$, so that if we write $\pi_n = \theta^{n(n-1)/2}c_n(\theta)$, we find that $c_n(\theta) > c_{n-1}(\theta)$ for all $n \geq 1$.

Moreover, by summing all the equalities in (5) up to n, we find that

$$\sum_{k=0}^{n} \pi_k = \sum_{k=0}^{n-1} \pi_k \theta^k + \sum_{k=1}^{2n+1} \pi_k (1-\theta^k),$$

or equivalently:

$$\pi_n \theta^n = \sum_{k=n+1}^{2n+1} \pi_k (1-\theta^k). \tag{6}$$

In particular for $n \geq 1$,

$$\pi_{2n}(1-\theta^{2n}) + \pi_{2n+1}(1-\theta^{2n+1}) \leq \pi_n \theta^n.$$

Inserting this inequality in (5) yields

$$\pi_n \leq \pi_{n-1}\theta^{n-1} + \pi_n\theta^n, \quad n \geq 1.$$

and then,

$$c_{n-1}(\theta) < c_n(\theta) \leq \frac{c_{n-1}(\theta)}{1-\theta^n}.$$

The equality shows that the sequence $\{c_n(\theta)\}$ is increasing and bounded and, then is converging as $n \to \infty$.

Moreover, from relation (6), we have

$$\begin{aligned} c_n(\theta)\theta^{(n+1)n/2} &= \sum_{m=n+1}^{2n+1} c_m(\theta)\theta^{m(m-1)/2}(1-\theta^m) \\ &\geq c_{n+1}(\theta) \sum_{m=n+1}^{2n+1} \theta^{m(m-1)/2}(1-\theta^m) \\ &= c_{n+1}(\theta)\left(\theta^{n(n+1)/2} - \theta^{(n+1)(2n+1)}\right), \end{aligned}$$

and then, $c_{n+1}(\theta) \leq c_n(\theta)/1 - \theta^{(3n+2)(n+1)/2}$ for $n \geq 0$. This completes the proof. $\square$

An easy consequence of the above result is the following corollary for the mean window size

Corollary 2 *The mean window size $\overline{W}(\theta)$ satisfies the inequality $m(\theta) \leq \overline{W}(\theta) \leq M(\theta)$, with*

$$m(\theta) = \frac{\frac{1}{1-\theta}\sum_{n\geq 1} n\theta^{n(n-1)/2}}{1+\sum_{n\geq 1}\alpha_n(\theta)\theta^{n(n-1)/2}},$$

$$M(\theta) = \frac{\sum_{n\geq 1} n\alpha_n(\theta)\theta^{n(n-1)/2}}{1+\frac{1}{1-\theta}\sum_{n\geq 1}\theta^{n(n-1)/2}}.$$

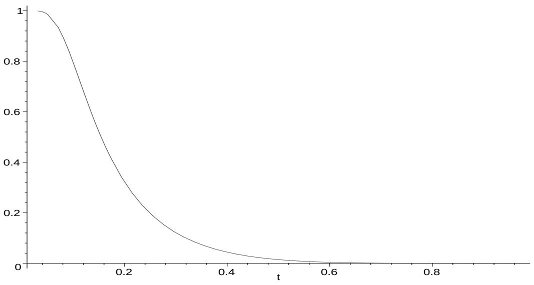

Figure 1: Relative error on the mean window size as a function of the loss rate p.

To conclude this section, notice that the above result yields a lower and upper bound for the throughput ρ of the TCP connection, since $\rho = \overline{W}(\theta)/R$. The accuracy of the bounds $m(\theta)$ and $M(\theta)$ is illustrated in Figure 1, which displays the relative error $(M(\theta) - m(\theta))/(M(\theta) + m(\theta))$ as a function of the loss rate $p = 1 - \theta$. It turns out that the bounds $m(\theta)$ and $M(\theta)$ are quite accurate for $p > 0.3$.

III - Asymptotic results when $p \to 0$

A. Infinite maximum window size

In this section we consider the case when $\theta \sim 1$ and we set $\theta = e^{-\alpha}$ with $\alpha \ll 1$. Let $\{\widehat{W}_n^\alpha\}$ be the Markov chain embedded at loss events, that is, $\widehat{W}_n^\alpha$ is the value of the congestion window over an elementary time interval (of length R) when there has been a packet loss in the preceding time interval.

If $\widehat{W}_0^\alpha = x$, then

$$\widehat{W}_1^\alpha = \left\lfloor \frac{x + G_x^\alpha}{2} \right\rfloor, \tag{7}$$

where G_x^α is a random variable whose distribution is given by, for $y \in \mathbb{N}$,

$$\begin{aligned}\mathbb{P}(G_x^\alpha = y) &= \prod_{i=x}^{x+y-1} e^{-k\alpha}\left(1 - e^{-(x+y)\alpha}\right) \\ &= e^{-\alpha x y - \alpha y(y-1)/2}\left(1 - e^{-\alpha(x+y)}\right). \end{aligned}\tag{8}$$

With the same argument as for proving the ergodicity of the process $\{W_n\}$, it is easily checked that the Markov chain $\{\widehat{W}_n^\alpha\}$ is also ergodic. Let $\{\hat{\pi}_n^\alpha\}$ denote its invariant probability on $\mathbb{N}$; in the following $\widehat{W}_\infty^\alpha$ will denote a random variable with distribution $\{\hat{\pi}_n^\alpha\}$.

Lemma 3 *For $x \in \mathbb{R}_+$, when α tends to 0 the variable $H_x^\alpha = \sqrt{\alpha}\, G^\alpha_{\lfloor x/\sqrt{\alpha}\rfloor}$ converges in distribution to a non negative random variable H_x such that*

$$\mathbb{P}\left(H_x \geq y\right) = e^{-xy - y^2/2}, \tag{9}$$

for $y \geq 0$.

Proof. For $\xi \in \mathbb{R}$, we have

$$\begin{aligned}\mathbb{E}\left[e^{\xi H_x^\alpha}\right] &= \sum_{y=0}^{\infty} \mathbb{P}\left(G^\alpha_{\lfloor x/\sqrt{\alpha}\rfloor} = y\right) e^{\xi\sqrt{\alpha} y} \\ &= \sum_{y=0}^{\infty} e^{-\alpha\lfloor x/\sqrt{\alpha}\rfloor y - \alpha y(y-1)/2 + \xi\sqrt{\alpha} y}\left(1 - e^{-\alpha\lfloor x/\sqrt{\alpha}\rfloor - \alpha y}\right).\end{aligned}$$

For sufficiently small α,

$$1 - e^{-\alpha\lfloor x/\sqrt{\alpha}\rfloor - \alpha y} \sim \alpha\lfloor x/\sqrt{\alpha}\rfloor + \alpha y,$$

and then, the quantity $\mathbb{E}\left[e^{\xi H_x^\alpha}\right]$ is equivalent to

$$\begin{aligned}&\sum_{y=0}^{\infty} e^{-\alpha\lfloor x/\sqrt{\alpha}\rfloor y - \alpha y(y-1)/2 + \xi\sqrt{\alpha} y}\left(\sqrt{\alpha}\lfloor x/\sqrt{\alpha}\rfloor + \sqrt{\alpha} y\right)\sqrt{\alpha} \\ &\quad \to \int_0^\infty (x+y) e^{(\xi - x)y - y^2/2}\, dy\end{aligned}$$

when $\alpha \to 0$. Lévy's theorem shows that the random variable H_x^α converges in distribution to a limit when $\alpha \to 0$, its PDF is defined by Equation (9). This completes the proof. □

Theorem 4 *When α tends to 0, $X^\alpha = \sqrt{\alpha}\,\widehat{W}_\infty^\alpha$ converges in distribution to a random variable V satisfying the following identity in distribution*

$$V \overset{dist.}{=} \frac{V + H_V}{2}, \tag{10}$$

where $\{H_x\}$ is a family of random variables independent of V whose distribution is given by (9). The identity (10) has a unique solution.

Proof. The proof is quite technical (see [2]). A sketch of the proof is as follows: if $\widehat{W}_\infty^\alpha$ is a random variable with distribution $\{\hat{\pi}_n\}$, then

$$\widehat{W}_\infty^\alpha \overset{\text{dist.}}{=} \left\lfloor \frac{\widehat{W}_\infty + G^\alpha_{\widehat{W}_\infty}}{2} \right\rfloor,$$

with the same notation as in Lemma 3. The last equation can be written as

$$X^\alpha \overset{\text{dist.}}{\sim} \frac{X^\alpha + H^\alpha_{X^\alpha}}{2},$$

when $\alpha \to 0$. If V is a limiting distribution of X^α, then it must therefore satisfy $V \overset{\text{dist.}}{=} (V + H_V)/2$, since the last equation has a unique solution in V, this completes the proof. □

The following proposition gives the density of V.

Proposition 5 *The random variable V satisfying the relation (10) has a density $h(x)$ on $\mathbb{R}_+$ given by*

$$h(x) = \frac{1}{C}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\prod_{k=1}^{n-1}(4^k - 1)} 4^n x e^{-4^n x^2/2}, \tag{11}$$

where C is the normalizing constant

$$C = \prod_{k=1}^{\infty} \left(1 - 1/4^k\right). \tag{12}$$

Proof. For $\xi > 0$, let

$$\widetilde{V}(\xi) = \mathbb{E}\left(e^{-\xi V^2/2}\right) = \int_0^{+\infty} e^{-\xi x^2/2}\, V(dx),$$

$\widetilde{V}$ is the Laplace transform of the variable $V^2/2$.

Since the variable V is the solution of the equation (10), we get that

$$\begin{aligned}
\widetilde{V}(\xi) &= \mathbb{E}\left[\exp\left(-\frac{\xi}{2}\left(\frac{V+H_V}{2}\right)^2\right)\right] \\
&= \int\int (x+y)e^{-\xi((x+y)/2)^2/2 - xy - y^2/2}\, V(dx)\, dy \\
&= \int_0^{+\infty}\int_x^{+\infty} y e^{-y^2(\xi/4+1)/2}\, dy\, e^{x^2/2}\, V(dx) \\
&= \frac{1}{\xi/4+1}\int_0^{+\infty} e^{-(\xi/4)\, x^2/2}\, V(dx) \\
&= \frac{4}{\xi+4}\widetilde{V}(\xi/4).
\end{aligned} \tag{13}$$

By iterating the above equation and by using the fact that $\widetilde{V}(0) = 1$, we obtain finally that

$$\widetilde{V}(\xi) = \prod_{k=1}^{+\infty} \frac{4^k}{\xi + 4^k}. \tag{14}$$

The right hand side of the above expression is the Laplace transform of the sum of independent exponentially distributed random variables with respective parameters 4^n for $n \geq 1$.

The poles of the Laplace transform (14) are equal to -4^n for $n \geq 1$. The residue at point -4^n is

$$(-1)^{n-1}\prod_{k=1}^{n-1}\frac{1}{4^k - 1}\prod_{k=1}^{\infty}\frac{1}{(1-1/4^k)}.$$

It follows that

$$\mathbb{P}\left(\frac{V^2}{2} > x\right) = \frac{1}{C}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\prod_{k=1}^{n-1}(4^k-1)}e^{-4^n x},$$

where C is defined by equation (12), and hence,

$$\mathbb{P}(V > x) = \frac{1}{C}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\prod_{k=1}^{n-1}(4^k-1)}e^{-4^n x^2/2},$$

Relation (11) follows by taking derivatives. This completes the proof. □

Similar expressions have been obtained by Ott *et al.* [8] for a continuous model to approximate the TCP congestion avoidance mechanism; the approximation is not justified by a limit theorem. Adjih *et al.* [1] obtained asymptotics for the stationary distribution of $\{W_n^\alpha\}$ via Mellin transform.

A direct consequence of Proposition 5 is the following result, which gives an estimate of the throughput of the TCP connection when there is no upper limit on the congestion window size.

Corollary 6 *The throughput ρ^α of the TCP connection is such that*

$$\lim_{\alpha\to 0}\sqrt{\alpha}\rho^\alpha = \frac{1}{R}\frac{1}{\mathbb{E}[V]}, \tag{15}$$

where $1/\mathbb{E}(V) = 1/\int x h(x)\, dx \sim 1.309833351$; the function h is defined by (11).

Proof. By definition, the throughput of the TCP connection is given by

$$\rho^\alpha = \frac{\mathbb{E}\left[\sum_{k=0}^{G^\alpha_{\widetilde{W}^\alpha_\infty}}\left(\widetilde{W}^\alpha_\infty + k\right)\right]}{R\mathbb{E}\left[G^\alpha_{\widetilde{W}^\alpha_\infty}\right]}.$$

By using the convergence results given by Theorem 4, we have $R\sqrt{\alpha}\rho^\alpha \to 1/\mathbb{E}[V]$ as $\alpha \to 0$, where we have used the fact that $\mathbb{E}[H_V] = \mathbb{E}[V]$, $\mathbb{E}[(H_V + V)^2] = 4\mathbb{E}[V^2]$, and $\mathbb{E}[V^2/2] = 1/3$. This completes the proof. □

To conclude this section, note that the above corollary justifies a posteriori the empirical constant 1.31, which has been observed in [4].

B. Finite window size

In this section, we assume that the congestion window of the TCP connection cannot exceed a finite value $\lfloor w/\sqrt{\alpha}\rfloor$. In this case, the evolution of the congestion window size $\mathcal{W}_n^\alpha$ at loss instants is governed by the equation

$$\mathcal{W}_{n+1}^\alpha = \left\lfloor \frac{1}{2}(\mathcal{W}_n^\alpha + G^\alpha_{\mathcal{W}_n^\alpha}) \wedge \frac{w}{\sqrt{\alpha}}\right\rfloor. \tag{16}$$

This new Markov chain is obviously ergodic. Let $\pi^\alpha(dx)$ denote its stationary distribution and $\mathcal{W}_\infty^\alpha$ denote a random variable with distribution $\pi^\alpha(dx)$. The random variable $\mathcal{W}_\infty^\alpha$ satisfies the following equation: ‘

$$\mathcal{W}_\infty^\alpha \stackrel{\text{dist.}}{=} \left\lfloor\left(\frac{\mathcal{W}_\infty^\alpha + G^\alpha_{\mathcal{W}_\infty^\alpha}}{2}\right)\wedge\frac{w}{2\sqrt{\alpha}}\right\rfloor. \tag{17}$$

With similar arguments used to prove Theorem 4, we see that as $\alpha \to 0$, $\sqrt{\alpha}\mathcal{W}_\infty^\alpha$ tends to a limiting random variable $\mathcal{W}$ such that

$$\mathcal{W} \stackrel{\text{dist.}}{=} \left(\frac{\mathcal{W} + H_{\mathcal{W}}}{2}\right)\wedge\frac{w}{2}. \tag{18}$$

Theorem 7 *The PDF of the random variable $\mathcal{W}$ has a mass at $w/2$ with magnitude η equal to*

$$\frac{1}{\eta} = 2e^{w^2/2} - 1 - 2e^{w^2/2}\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\frac{a_k^n}{1-4^{k+1}}\times \left(e^{2\left(1-4^{k+1}\right)w^2/4^{n+1}} - e^{2\left(1-4^{k+1}\right)w^2/4^{n+2}}\right) \quad (19)$$

and for $x \in [w/2^{n+1}, w/2^n)$,

$$\mathbb{P}(\mathcal{W} \leq x) = -2\eta e^{w^2/2}\sum_{k=0}^{n-1} a_k^n e^{-4^{k+1}x^2/2} + 1 + \eta, \quad (20)$$

where the coefficients a_n^k are defined by $a_0^1 = 1$, $a_k^n = a_0^{n-k}/\prod_{\ell=1}^{k}(1-4^\ell)$ for $n \geq 2$ and $k = 1, \ldots, n-1$, and for $n \geq 1$

$$a_0^{n+1} = a_0^n + \sum_{k=0}^{n-2}\frac{a_k^{n-1}}{1-4^{k+1}}e^{2v\left(1-4^{k+1}\right)/4^n} - \sum_{k=0}^{n-1}\frac{a_k^n}{1-4^{k+1}}e^{2v\left(1-4^{k+1}\right)/4^n}.$$

Proof. The equation (13) implies that the identity

$$\left(\frac{Z+H_Z}{2}\right)^2 \stackrel{\text{dist.}}{=} \frac{1}{2}\mathcal{E}_1 + \frac{1}{4}Z^2,$$

where $\mathcal{E}_1$ is an exponential random variable with mean unity holds for any non negative random variable Z, in particular for $Z = \mathcal{W}$. Let $\mathcal{V} = \mathcal{W}^2$ and $v = w^2/4$. Then,

$$\mathcal{V} \stackrel{\text{dist.}}{=} (\mathcal{E}_1/2 + \mathcal{V}/4) \wedge v.$$

The PDF of the random variable $\mathcal{V}$ has a mass at point v, whose magnitude is denoted by η.

Let $f(y) = \mathbb{P}(\mathcal{V} \leq y)$. We have $f(y) = 1$ for $y \geq v$, and for $y < v$,

$$f(y) = \eta\mathbb{P}\left(\mathcal{E}_1 \leq 2\left(y - v/4\right)\right) + 2\int_0^y e^{-2u} f(4(y-u))\,du.$$

From the above equation, it turns out that $f(y)$ can be written for $y \in \left[v/4^n, v/4^{n-1}\right)$, $n \geq 1$ as

$$f(y) = \sum_{k=0}^{n-1}\alpha_k^n e^{-4^k(2y)} + 1 + \eta,$$

with

$$\alpha_0^1 = 2\int_0^{v/4} e^{2u} f(4u)\,du - e^{v/2}(1+\eta). \quad (21)$$

The normalizing condition $f(v^-) + \eta = 1$ implies that $\alpha_0^1 = -2\eta e^{2v}$. Moreover, the coefficients α_n^k are related to each other as $\alpha_k^{n+1} = \alpha_{k-1}^n/(1-4^k)$ for $k \geq 1$, and

$$\alpha_0^{n+1} = 2\int_0^{v/4^{n+1}} e^{2z} f(4u)\,du - (1+\eta)e^{-2v/4^{n+1}} - \sum_{k=0}^{n-1}\frac{\alpha_k^n}{1-4^{k+1}}e^{2v\left(1-4^{k+1}\right)/4^{n+1}}. \quad (22)$$

Simple manipulations show that the coefficients α_k^n linearly depend on α_0^1 and then that $\alpha_k^n = \alpha_0^1.a_k^n$, where the coefficients a_k^n are defined in Theorem 7. The mass η is determined by using equation (21) and the fact that $\alpha_0^1 = -2\eta e^{2v}$. This completes the proof. □

An estimate for the throughput of the TCP connection is obtained by using the fact that

$$\rho^\alpha = \frac{\mathbb{E}\left(\sum_{k=0}^{\widetilde{G}^\alpha_{\mathcal{W}^\alpha_\infty}}(\mathcal{W}^\alpha_\infty + k) \wedge \frac{w}{\sqrt{\alpha}}\right)}{R\mathbb{E}\left(\widetilde{G}^\alpha_{\mathcal{W}^\alpha_\infty}\right)},$$

where $\widetilde{G}^\alpha_x$ is the first time when a loss occurs. Trite calculations give that when $\alpha \to 0$, then $R\sqrt{\alpha}\rho^\alpha$ converges to

$$\frac{1}{\mathbb{E}\left(H_\mathcal{W} 1_{\{\mathcal{W}+H_\mathcal{W} \leq w\}}\right) + \eta(w + 1/w)},$$

this quantity is upper bounded by $1/\mathbb{E}(H_\mathcal{W})$. This bound is accurate if $\exp(-w^2/2)$ is small enough.

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