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On-Line Edge-Coloring with a Fixed Number of Colors*

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Resumé We investigate a variant of on-line edge-coloring in which there is a fixed number of colors available and the aim is to color as many edges as possible. We prove upper and lower bounds on the performance of different classes of algorithms for the problem. Furthermore, we determine the performance of two specific algorithms, *First-Fit* and *Next-Fit*.

1 Introduction

The Problem. In this paper we investigate the on-line problem EDGE-COLORING defined in the following way. A number k of colors is given. The algorithm is given the edges of a graph one by one, each one specified by its endpoints. For each edge, the algorithm must either color the edge with one of the k colors, or reject it, before seeing the next edge. Once an edge has been colored the color cannot be altered and a rejected edge cannot be colored later. The aim is to color as many edges as possible under the constraint that the resulting coloring must be a proper k-coloring.

Note that the problem investigated here is different from the classical version of the edge coloring problem, which is to color *all* edges with as *few* colors as possible. In [2] it is shown that, for the on-line version of the classical edge coloring problem, the greedy algorithm (the one that we call *First-Fit*) is optimal.

The Measures. To measure the quality of the algorithms, we use the competitive ratio which was introduced in [5] and has become a standard measure for on-line algorithms. For the problem EDGE-COLORING addressed in this paper, the competitive ratio of an algorithm $\mathbb{A}$ is the worst case ratio, over all possible input sequences, of the number of edges colored by $\mathbb{A}$ compared to the number of edges colored by an optimal off-line algorithm.

In some cases it may be realistic to assume that the input graphs are all k-colorable. Therefore, we also investigate the competitive ratio in the special case where it is known that the input graphs are k-colorable. This idea is similar to what was done in [1] and [3]. In these papers the competitive ratio is investigated on input sequences that can be fully accommodated by an optimal off-line algorithm with the resources available (in this paper the resource is, of course, the colors). Such sequences are called accommodating sequences. This is generalized in [4], where the competitive ratio as a function of the amount of resources available is investigated.

This paper illustrates an advantage of analysing accommodating sequences, apart from tailoring the measure to the type of input. A common technique when constructing a difficult proof is to start out investigating easier special cases. In our analysis of the general lower bound on the competitive ratio, the case of k-colorable input graphs was used as such a special case.

The Algorithms. We will mainly consider fair algorithms. A *fair* algorithm is an algorithm that never rejects an edge, unless it is not able to color it without violating the constraint that the resulting coloring must be a proper k-coloring. Two natural fair algorithms are *Next-Fit* and *First-Fit* described in Sections 4 and 5 respectively.

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The Results. In Section 3 we show that any fair algorithm has a competitive ratio of at least $2\sqrt{3}-3 \approx 0.4641$. Furthermore we show that no deterministic fair algorithm is more than $\frac{1}{2}$-competitive, and that no algorithm can be more than $\frac{4}{7}$-competitive, even if we allow randomization. In Section 6 we show that, in the case of k-colorable graphs, any fair algorithm is $\frac{1}{2}$-competitive and that no deterministic algorithm is more than $\frac{2}{3}$-competitive.

The performance of the algorithm *Next-Fit* matches the lower bound on the competitive ratio for fair algorithms in both the general case and in the special case where the input graphs are all k-colorable. The algorithm *First-Fit* is only slightly better. It has a competitive ratio of at most $\frac{2}{9}(\sqrt{10}-1) \approx 0.4805$ in general and exactly $\frac{k}{2k-1}$ on k-colorable graphs.

The Graphs. The lower bounds on the competitive ratio proven in this paper are valid even if we allow multigraphs. The adversary graphs used for proving the upper bounds are all simple graphs. Thus, the upper bounds are valid even if we restrict ourselves to simple graphs. Furthermore, the adversary graphs are all bipartite except one which could easily be changed to a bipartite graph. Thus, the results are all valid for bipartite graphs too.

2 Notation and Terminology

$K_{m,n}$ denotes the complete bipartite graph in which the two independent sets contain m and n vertices respectively.

A *proper coloring* of a graph is a coloring in which no two adjacent edges achieve the same color. A *k-coloring* is a coloring using at most k colors. We label the colors $1, 2, \ldots, k$ and let $C_k = \{1, 2, \ldots, k\}$.

An *r-regular* graph is a graph in which every vertex has degree r. A biregular graph is a graph in which each vertex has one of two possible vertex degrees.

The terms *fair*^{D}, *fair*^{R}, *on-line*^{D}, and *on-line*^{R} denote arbitrary on-line algorithms from the classes "fair deterministic", "fair, possibly randomized ", "deterministic, possibly not fair", and "possibly randomized and/or not fair", respectively, for the EDGE-COLORING problem. The term *off-line* denotes an optimal off-line algorithm for the problem.

3 The Competitive Ratio

We begin this section with a formal definition of the competitive ratio for the problem EDGE-COLORING.

Definition 3.1. *For any algorithm $\mathbb{A}$ and any sequence E of edges, let $\mathbb{A}(E)$ be the number of edges colored by $\mathbb{A}$ and let $OPT(E)$ be the number of edges colored by an optimal off-line algorithm. Furthermore, let $0 \leq c \leq 1$.*

An on-line algorithm $\mathbb{A}$ is c-competitive *if there exists a constant b such that*

$$\mathbb{A}(E) \geq c \cdot OPT(E) - b, \text{ for any sequence } E \text{ of edges.}$$

The competitive ratio *of $\mathbb{A}$ is $C_{\mathbb{A}} = \sup\{c \mid \mathbb{A} \text{ is } c\text{-competitive}\}$.*

3.1 A Tight Lower Bound for Fair Algorithms

In this section a tight lower bound on the competitive ratio for fair algorithms is given. Note that it is not possible to give a *general* lower bound greater than 0. The intuition behind the proof is that each edge colored by *fair*^{R} is worth one unit of some value. If the total value can be distributed to the edges colored by *off-line* such that each edge receives at least the fraction c of a unit, then the number of edges colored by *fair*^{R} is at least c times the number of edges colored by *off-line*. Thus, after giving each colored edge one unit of value, we remove the fraction c of a unit from each edge colored by both *fair*^{R} and *off-line*. Now, we have "payed" for the edges colored by both algorithms and with the value left we must pay for the edges colored by *off-line* only. Therefore the value left on each colored edge e is distributed equally among its endpoints and from there redistributed to the edges adjacent to e that are colored by *off-line* only. If each of these edges receives a total value of at least c, we have payed at least c for each edge colored by *off-line*.

Theorem 3.2. *For any fair on-line algorithm* $\mathbb{A}$ *for* EDGE-COLORING, $C_{\mathbb{A}} \geq 2\sqrt{3} - 3 \approx 0.4641$.

Bevis. Let $G = (V, E)$ be an arbitrary graph and assume that the edges are given in some order. Let E_{c} denote the set of edges colored by $fair^R$, let E_{u} denote the set of edges colored by *off-line* and not by $fair^R$, and let E_{d} denote the set of edges colored by both *off-line* and $fair^R$. Thus, $E_{\mathrm{u}} \cup E_{\mathrm{d}}$ are the edges colored by *off-line*, and $E_{\mathrm{d}} \subseteq E_{\mathrm{c}}$. Similarly, for any vertex x, let $d_{\mathrm{c}}(x)$ denote the number of edges incident to x that are colored by $fair^R$, let $d_{\mathrm{u}}(x)$ denote the number of edges incident to x that are not colored by $fair^R$, and let $d_{\mathrm{d}}(x)$ denote the number of edges incident to x that are colored by both $fair^R$ and *off-line*. Let c be a constant such that $0 \leq c < \frac{1}{2}$. Then $fair^R$ is c-competitive for any c such that $|E_{\mathrm{c}}| \geq c(|E_{\mathrm{d}}| + |E_{\mathrm{u}}|)$, or $|E_{\mathrm{c}}| - c|E_{\mathrm{d}}| \geq c|E_{\mathrm{u}}|$.

For each vertex $x \in V$, define $m(x) = \frac{1}{2}\big(d_{\mathrm{c}}(x) - c d_{\mathrm{d}}(x)\big)$. Note that $\sum_{x \in V} m(x) = |E_{\mathrm{c}}| - c|E_{\mathrm{d}}|$. For each $x \in V$ such that $d_{\mathrm{u}}(x) \geq 1$, define $m_{\mathrm{u}}(x) = \frac{m(x)}{d_{\mathrm{u}}(x)}$. Then,

$$\sum_{(x,y) \in E_{\mathrm{u}}} \big(m_{\mathrm{u}}(x) + m_{\mathrm{u}}(y)\big) \;\leq\; \sum_{(x,y) \in E_{\mathrm{u}}} \big(m_{\mathrm{u}}(x) + m_{\mathrm{u}}(y)\big) \;+\; \sum_{d_{\mathrm{u}}(x) = 0} m(x) \;=\; \sum_{x \in V} m(x) = |E_{\mathrm{c}}| - c|E_{\mathrm{d}}|.$$

Thus, if $m_{\mathrm{u}}(x) + m_{\mathrm{u}}(y) \geq c$ for any edge $(x, y) \in E_{\mathrm{u}}$, then

$$c|E_{\mathrm{u}}| \;\leq\; \sum_{(x,y) \in E_{\mathrm{u}}} \big(m_{\mathrm{u}}(x) + m_{\mathrm{u}}(y)\big) \;\leq\; |E_{\mathrm{c}}| - c|E_{\mathrm{d}}|$$

and $fair^R$ is c-competitive.

The inequalities below follow from two simple facts. (1) For any vertex $x \in V$, $d_{\mathrm{d}}(x) + d_{\mathrm{u}}(x) \leq k$, since *off-line* can color at most k edges incident to x. (2) For each edge $(x, y) \in E_{\mathrm{u}}$, $d_{\mathrm{c}}(x) + d_{\mathrm{c}}(y) \geq k$, since $fair^R$ is a fair algorithm.

For any edge $(x, y) \in E_{\mathrm{u}}$,

$$\begin{aligned} m_{\mathrm{u}}(x) + m_{\mathrm{u}}(y) &= \frac{1}{2}\left(\frac{d_{\mathrm{c}}(x) - c d_{\mathrm{d}}(x)}{d_{\mathrm{u}}(x)} + \frac{d_{\mathrm{c}}(y) - c d_{\mathrm{d}}(y)}{d_{\mathrm{u}}(y)}\right) \\ &\overset{(1)}{\geq} \frac{1}{2}\left(\frac{d_{\mathrm{c}}(x) - c d_{\mathrm{d}}(x)}{k - d_{\mathrm{d}}(x)} + \frac{d_{\mathrm{c}}(y) - c d_{\mathrm{d}}(y)}{k - d_{\mathrm{d}}(y)}\right) \\ &\overset{(2)}{\geq} \frac{1}{2}\left(\frac{d_{\mathrm{c}}(x) - c d_{\mathrm{d}}(x)}{k - d_{\mathrm{d}}(x)} + \frac{k - d_{\mathrm{c}}(x) - c d_{\mathrm{d}}(y)}{k - d_{\mathrm{d}}(y)}\right) \end{aligned}$$

Let $m_x = \frac{d_{\mathrm{c}}(x) - c d_{\mathrm{d}}(x)}{k - d_{\mathrm{d}}(x)}$ and $m_y = \frac{k - d_{\mathrm{c}}(x) - c d_{\mathrm{d}}(y)}{k - d_{\mathrm{d}}(y)}$, and assume w.l.o.g. that $d_{\mathrm{c}}(x) \leq \frac{k}{2}$. For $d_{\mathrm{c}}(x) \leq kc$, m_x is a monotonically decreasing function of $d_{\mathrm{d}}(x)$, and, for $d_{\mathrm{c}}(x) > kc$, m_x is a monotonically increasing function of $d_{\mathrm{d}}(x)$. Similarly, since $k - d_{\mathrm{c}}(x) \geq \frac{k}{2} > kc$, m_y is a monotonically increasing function of $d_{\mathrm{d}}(y)$.

We can now conclude that, for $kc < d_{\mathrm{c}}(x) \leq \frac{k}{2}$,

$$m_{\mathrm{u}}(x) + m_{\mathrm{u}}(y) \geq \frac{1}{2}\left(\frac{d_{\mathrm{c}}(x)}{k} + \frac{k - d_{\mathrm{c}}(x)}{k}\right) = \frac{1}{2} > c.$$

For $d_{\mathrm{c}}(x) \leq kc$,

$$m_{\mathrm{u}}(x) + m_{\mathrm{u}}(y) \geq \frac{1}{2}\left(\frac{d_{\mathrm{c}}(x) - c d_{\mathrm{c}}(x)}{k - d_{\mathrm{c}}(x)} + \frac{k - d_{\mathrm{c}}(x)}{k}\right),$$

since $d_{\mathrm{d}}(x) \leq d_{\mathrm{c}}(x)$. For $d_{\mathrm{c}}(x) \leq k$, this function attains its minimum value at $d_{\mathrm{c}}(x) = k(1 - \sqrt{1-c})$, and the minimum value is $\frac{1}{2}(c-1) + \sqrt{1-c}$, which is greater than or equal to c as long as $c \leq 2\sqrt{3} - 3$. $\square$

In section 4 it is shown that values of k exist for which the competitive ratio of *Next-Fit* is arbitrarily close to $2\sqrt{3} - 3$. Thus, the result in Theorem 3.2 is tight. The next theorem in conjunction with Theorem 3.2, shows that all fair algorithms must have very similar competitive ratios.

3.2 An Upper Bound for Fair Deterministic Algorithms

Theorem 3.3. *For any deterministic fair algorithm* $\mathbb{A}$ *for* EDGE-COLORING, $C_{\mathbb{A}} \leq \frac{1}{2}$.

Bevis. We construct a simple graph $G = (V_1 \cup V_2, E)$ in two phases. In Phase 1, only vertices in V_1 are connected, and in Phase 2, vertices in V_2 are connected to vertices in V_1. For every vertex $x \in V_1$, a set $C_{\mathrm{u}}(x)$ of unused colors is maintained. Let $|V_1| = n$ for some large integer n.

In Phase 1 the adversary gives an edge between two unconnected vertices x and y in V_1 for which $C_{\mathrm{u}}(x) \cap C_{\mathrm{u}}(y) \neq \emptyset$. Since a common unused color exists, *fair*^{D} will color the edge. The color used by *fair*^{D} is deleted in both $C_{\mathrm{u}}(x)$ and $C_{\mathrm{u}}(y)$. This process is repeated until no two unconnected vertices with a common unused color can be found. At that point Phase 1 ends. Since there is no unconnected pair $x, y \in V_1$ such that $C(x) \cap C(y) \neq \emptyset$, the following holds true. For each color c and each vertex x such that $c \in C_{\mathrm{u}}(x)$, x is already connected to all other vertices y with $c \in C_{\mathrm{u}}(y)$. Since x can be connected to at most k other vertices, there are at most k vertices $y \neq x$ such that $c \in C_{\mathrm{u}}(y)$. Thus, $\sum_{x \in V_1} C_{\mathrm{u}}(x) \leq k(k+1)$.

In Phase 2 additional kn vertices are added to G. For each vertex $x \in V_1$, k new vertices are added to V_2 and each of the new vertices are connected to x by an edge. From Phase 2, *fair*^{D} gets at most $k(k+1)$ edges, but *off-line* rejects all edges from Phase 1 and accepts all edges from Phase 2, giving a performance ratio of at most

$$\frac{\frac{1}{2}(nk - k(k+1)) + k(k+1)}{nk} = \frac{nk + k(k+1)}{2nk} = \frac{1}{2} + \frac{k+1}{2n}$$

If we allow n to be arbitrarily large, this can be arbitrarily close to $\frac{1}{2}$. □

3.3 A General Upper Bound

Now follows an upper bound on the competitive ratio for any type of algorithm for EDGE-COLORING, fair or not fair, deterministic or randomized.

Theorem 3.4. *Any algorithm for* EDGE-COLORING *is at most* $\frac{4}{7}$*-competitive.*

Bevis. In Fig. 1, the structure of the adversary graph is depicted. Each box contains k vertices. When two boxes are connected, there are k^2 edges in a complete bipartite graph between the $2k$ vertices inside the boxes. Note that such a graph can be k-colored. The edges of the graph are divided into n levels, level $1, \ldots, n$. The adversary gives the edges, one level at a time, according to the numbering of the levels. The edges of level i are given in three consecutive phases:

1. H_i: Internal (horizontal) edges at level i. In total k^2 edges.
2. V_i: Internal (vertical) edges between level i and level $i+1$. In total $2k^2$ edges.
3. E_i: External edges at level i. In total $2k^2$ edges.

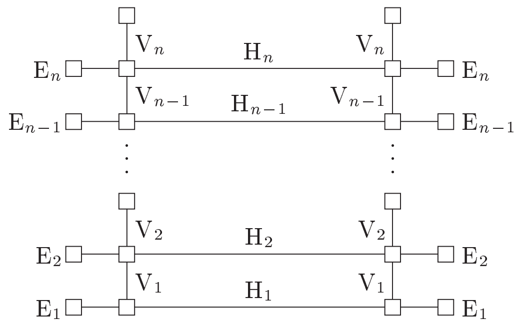
Vertices that are endpoints of internal edges are called internal vertices.

Let X_{H_i} be a random variable counting how many edges *on-line*^{R} will accept from the set H_i, and let X_{V_i} and X_{E_i} count the accepted edges from V_i and E_i respectively.

For $i = 0, \ldots, n$, let EXT_i and INT_i be random variables counting the sum of all external and internal edges, respectively, accepted by *on-line*^{R} after level i is given, i.e., $\mathrm{EXT}_i = \sum_{j=1}^{i} X_{\mathrm{E}_j}$ and $\mathrm{INT}_i = \sum_{j=1}^{i} (X_{\mathrm{V}_j} + X_{\mathrm{H}_j})$. Note that $\mathrm{EXT}_0 = \mathrm{INT}_0 = 0$.

If the adversary stops giving edges after Phase 1 of level i, *off-line* will color $k^2(2i-1)$ edges in total. These are the edges in the sets $\mathrm{E}_1, \mathrm{E}_2, \ldots, \mathrm{E}_{i-1}$, and H_i. If the adversary stops giving edges after Phase 2 (or 3) of level i, *off-line* will color $2k^2 i$ edges. These are the edges in the sets $\mathrm{E}_1, \mathrm{E}_2, \ldots, \mathrm{E}_{i-1}$, and V_i.

Let G_i denote the graph consisting of the first i levels. Consider the subgraph G'_i of G_i colored by *on-line*^{R}. Summing over all internal vertices, the total vertex degree in G'_i is at most $2k^2 i$. An internal edge (excluding V_i) contributes to this number with two, whereas an external edge (plus edges in V_i) will only contribute with one. The expected number of edges in G_i colored by



Figur 1. Structure of the adversary graph for the general upper bound on the competitive ratio.

on-line^R is $E[\mathrm{INT}_i]+E[\mathrm{EXT}_i] = (E[\mathrm{INT}_i]-E[X_{\mathrm{V}_i}])+(E[\mathrm{EXT}_i]+E[X_{\mathrm{V}_i}]) \le \frac{1}{2}(2k^2i - E[\mathrm{EXT}_i] - E[X_{\mathrm{V}_i}]) + (E[\mathrm{EXT}_i] + E[X_{\mathrm{V}_i}]) = k^2i + \frac{1}{2}(E[\mathrm{EXT}_i] + E[X_{\mathrm{V}_i}])$.

The rest of the proof is divided into two cases.

Case 1: There exists a level $i \le n$, where $E[\mathrm{EXT}_i] > \frac{2}{7}k^2i$.

Let i denote the first level such that $E[\mathrm{EXT}_i] > \frac{2}{7}k^2i$. Assume that the number of edges colored by *on-line*^{R} is at least $\frac{4}{7}$ of the number of edges colored by *off-line*. If the adversary stops the sequence after Phase 1 of level i, the following inequality must hold:

$$E[\mathrm{INT}_{i-1}] + E[\mathrm{EXT}_{i-1}] + E[X_{\mathrm{H}_i}] \ge \frac{4}{7}k^2(2i-1). \tag{1}$$

If the adversary stops the sequence after Phase 2 of level i, the following inequality must hold:

$$E[\mathrm{INT}_i] + E[\mathrm{EXT}_{i-1}] \ge \frac{4}{7}k^2 2i. \tag{2}$$

If *on-line*^{R} is $\frac{4}{7}$-competitive, both inequalities must hold. The discussion prior to Case 1, combined with the inequality $E[X_{\mathrm{V}_{i-1}}] + 2E[X_{\mathrm{H}_i}] + E[X_{\mathrm{V}_i}] + E[X_{\mathrm{E}_i}] \le 2k^2$, gives upper bounds on the left sides in the two inequalities above:

$$\begin{aligned}
E[\mathrm{INT}_{i-1}] + E[\mathrm{EXT}_{i-1}] + E[X_{\mathrm{H}_i}] &\le k^2(i-1) + \frac{1}{2}(E[\mathrm{EXT}_{i-1}] + E[X_{\mathrm{V}_{i-1}}]) + E[X_{\mathrm{H}_i}] \\
&\le k^2(i-1) + \frac{1}{2}E[\mathrm{EXT}_{i-1}] + k^2 - \frac{1}{2}E[X_{\mathrm{V}_i}] - \frac{1}{2}E[X_{\mathrm{E}_i}] \text{ and} \\
E[\mathrm{INT}_i] + E[\mathrm{EXT}_{i-1}] &\le k^2(i-1) + \frac{1}{2}(E[\mathrm{EXT}_{i-1}] + E[X_{\mathrm{V}_{i-1}}]) + E[X_{\mathrm{H}_i}] + E[X_{\mathrm{V}_i}] \\
&\le k^2(i-1) + \frac{1}{2}E[\mathrm{EXT}_{i-1}] + k^2 + \frac{1}{2}E[X_{\mathrm{V}_i}] - \frac{1}{2}E[X_{\mathrm{E}_i}]
\end{aligned}$$

By inserting these bounds into (1) and (2) and noting that $E[\mathrm{EXT}_{i-1}] \le \frac{2}{7}k^2(i-1)$ and $E[X_{\mathrm{E}_i}] > \frac{2}{7}k^2$, solving for $E[X_{\mathrm{V}_i}]$ gives a contradiction: The first inequality gives that $E[X_{\mathrm{V}_i}] < \frac{4}{7}k^2$, whereas the second inequality gives that $E[X_{\mathrm{V}_i}] > \frac{4}{7}k^2$. Thus, in this case *on-line*^{R} is not $\frac{4}{7}$-competitive.

Case 2: For all $i \le n, E[X_{\mathrm{E}_i}] \le \frac{2}{7}k^2i$.

The expected number of edges colored by *on-line*^{R} is

$$\begin{aligned}
E[\mathrm{INT}_n] + E[\mathrm{EXT}_n] &\le k^2n + \frac{1}{2}(E[\mathrm{EXT}_n] + E[X_{\mathrm{V}_n}]) \\
&= k^2n + \frac{1}{2}E[\mathrm{EXT}_{n-1}] + \frac{1}{2}(E[X_{\mathrm{E}_n}] + E[X_{\mathrm{V}_n}]) \\
&\le k^2n + \frac{1}{7}k^2(n-1) + \frac{1}{2}2k^2 \\
&= \frac{8}{7}k^2n + \frac{6}{7}k^2.
\end{aligned}$$

Thus, we get an upper bound on the performance ratio of $\dfrac{\frac{8}{7}k^2n + \frac{6}{7}k^2}{2nk^2} = \dfrac{4}{7} + \dfrac{3}{7n}$, which can be arbitrarily close to $\frac{4}{7}$, if we allow n to be arbitrarily large. $\square$

Thus, even if we allow probabilistic algorithms that are not necessarily fair, no algorithm is more than 0.11 apart from the worst fair algorithm when comparing competitive ratios.

4 The Algorithm *Next-Fit*

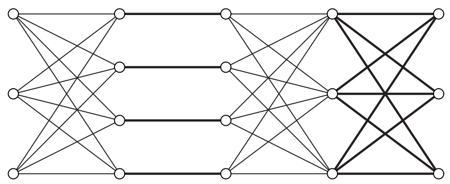
The algorithm *Next-Fit* (*NF*) is a fair algorithm that uses the colors in a cyclic order. *Next-Fit* colors the first edge with the color 1 and keeps track of the last used color c_{last}. When coloring an edge (u, v) it uses the first color in the sequence $\langle c_{\text{last}} + 1, c_{\text{last}} + 2, \ldots, k, 1, 2, \ldots, c_{\text{last}}\rangle$ that is not yet used on any edge incident to u or v, if any.

Intuitively, this is a poor strategy and it turns out that its competitive ratio matches the lower bound of section 3.1. Thus, this algorithm is mainly described here to show that the lower bound cannot be improved.

When proving upper bounds for *Next-Fit*, it is useful to note that any coloring in which each color is used on exactly n or $n + 1$ edges, for some $n \in \mathbb{N}$, can be produced by *Next-Fit*, for some ordering of the request sequence. The colors just need to be permuted so that the colors used on $n + 1$ edges are the lowest numbered colors.

Theorem 4.1. $\inf\limits_{k\in\mathbb{N}} C_{NF}(k) = 2\sqrt{3} - 3 \approx 0.4641.$

Bevis. The adversary constructs a graph G_{NF} in the following way. It chooses an $x \in C_k$ as close to $(\sqrt{3} - 1)k$ as possible and then constructs a $(k - x)$-regular bipartite graph $G_1 = (L_1 \cup R_1, E_1)$ with $|L_1| = |R_1| = k$ and a graph $G_2 = (L_2 \cup R_2, E_2)$ isomorphic to $K_{x,x}$. Now, each vertex in R_1 is connected to each vertex in L_2 and each vertex in R_2 is connected to each vertex in L_1. Call these extra edges E_{12}. The graph G_{NF} for $k = 4$ is depicted in Fig. 2.



Figur 2. The graph G_{NF} when $k = 4$, showing that $C_{NF}(4) \le \frac{13}{28} \approx 0.4643$.

Assume first, that $k - x \le 1$. In this case $|L_1| \le |L_2| + 1$, so G_1 and G_2 can be colored by *Next-Fit* with C_{k-x} and $C_k \setminus C_{k-x}$ respectively. After this, *Next-Fit* will not be able to color any of the edges in E_{12}. It is possible however, to color all edges in $E_1 \cup E_{12}$ with k colors, because the subgraph of G_{NF} containing these edges is bipartite and has maximum degree k. Thus, for any $x \in C_k$, the competitive ratio of *Next-Fit* can be no more than

$$\frac{|E_1| + |E_2|}{|E_1| + |E_{12}|} = \frac{k(k - x) + x^2}{k(k - x) + 2kx} = \frac{k^2 - kx + x^2}{k^2 + kx}.$$

This ratio attains its minimum value of $2\sqrt{3} - 3$ when $x = (\sqrt{3} - 1)k$. Thus, by allowing arbitrarily large values of k, it can be arbitrarily close to $2\sqrt{3} - 3$.

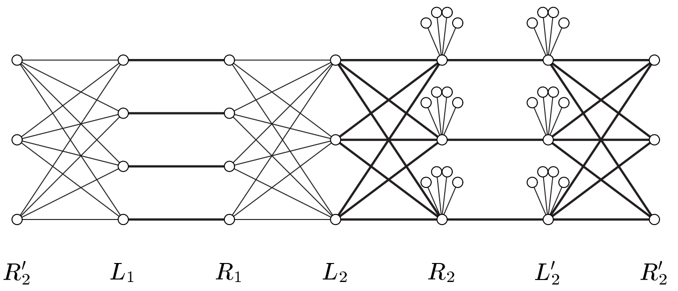
If $k - x > 1$, then $|L_1| > |L_2| + 1$ and thus it is not possible to make *Next-Fit* color all edges in G_1 using only C_{k-x}. In this case more copies of G_{NF} are needed. Let m be the smallest positive integer such that $m(k-x)$ is a multiple of k. Then mx is a multiple of k as well. In general, m copies of G_{NF}, $G_{\mathrm{C}}^1, G_{\mathrm{C}}^2, \ldots, G_{\mathrm{C}}^m$, are used. A k-coloring of the m copies of G_{NF} in which each color is used the same number of times can be obtained in the following way. In G_{C}^i, G_1 is colored with the colors $(k-x)(i-1)+1 \bmod k$, $(k-x)(i-1)+2 \bmod k$, ..., $(k-x)i \bmod k$, and G_2 is colored with the remaining colors in C_k. □

5 The Algorithm *First-Fit*

The algorithm *First-Fit* (*FF*) is a fair algorithm. For each edge e that it is able to color, it colors e with the lowest numbered color possible.

Theorem 5.1. $\inf_{k \in \mathbb{N}} C_{FF}(k) \leq \frac{2}{9}(\sqrt{10}-1) \approx 0.4805.$

Bevis. The adversary graph G_{FF} of this proof is inspired by the graph G_{NF}. It is not possible, though, to make *First-Fit* color the subgraph G_2 of G_{NF} with $C_k \setminus C_x$. Therefore, the graph is extended by the subgraph G_2' isomorphic to G_2. Each vertex in R_2 is connected to exactly $k-x$ vertices in L_2' and vice versa. Now, E_2 denotes the edges in G_2 and G_2' and the edges connecting them. Finally, $2kx$ new vertices are added, and each vertex in $R_2 \cup L_2'$ is connected to k of these vertices. Let E_3 denote the set of these extra edges. The graph G_{FF} for $k=4$ is depicted in Fig. 3.



Figur 3. The graph G_{FF} when $k=4$, showing that $C_{FF}(4) \leq \frac{25}{52} \approx 0.4808$.

If the edges in G_1 and the edges connecting G_2 and G_2' are given first (one perfect matching at a time), followed by the edges in G_2 and G_2' (one perfect matching at a time), it is obvious that *First-Fit* will color the edges in the desired way. After this, *First-Fit* will not be able to color any more edges of G_{FF}. On the other hand it is possible to k-color the set $E_1 \cup E_{12} \cup E_3$ of edges. Thus, the competitive ratio of *First-Fit* can be no more than

$$\frac{|E_1|+|E_2|}{|E_1|+|E_{12}|+|E_3|} = \frac{k(k-x)+2x^2+x(k-x)}{k(k-x)+2kx+2kx} = \frac{k^2+x^2}{k^2+3kx}.$$

This ratio attains its minimum value of $\frac{2}{9}(\sqrt{10}-1)$, when $x = \frac{1}{3}(\sqrt{10}-1)k$. Thus, for the graph G_{FF}, the optimal (from an adversary's point of view) value of x is an integer as close as possible to $\frac{1}{3}(\sqrt{10}-1)k$, and by allowing arbitrarily large values of k, the ratio can be arbitrarily close to $\frac{2}{9}(\sqrt{10}-1)$. □

6 *k*-colorable graphs

Now that we know that the competitive ratio cannot vary much between different kinds of algorithms for the EDGE-COLORING problem, it would be interesting to see what happens if we know something about the input graphs — for instance that they are all k-colorable. In this section we investigate the competitive ratio in the case where the input graphs are known to be k-colorable.

6.1 A Tight Lower Bound for Fair Algorithms

In this section a lower bound on the competitive ratio for any *fair* algorithm is given. As in the proof of Theorem 3.2 the idea is that each colored edge is worth one unit of some value. Again the value of each colored edge e is distributed equally among its endpoints and, from there, redistributed to the uncolored edges adjacent to e. If each uncolored edge receives a total value of at least one, then there are at least as many colored edges as uncolored edges.

Theorem 6.1. *Any fair algorithm for* EDGE-COLORING *is* $\frac{1}{2}$*-competitive on* k*-colorable graphs.*

Bevis. Let $G = (V, E)$ be an arbitrary k-colorable graph and assume that the edges are given in some order. Let E_{c} denote the set of edges colored by fair^R, and let E_{u} denote the set of edges *not* colored by fair^R. Similarly, for any vertex x, let $d_{\mathrm{c}}(x)$ denote the number of edges incident to x that are colored by fair^R, and let $d_{\mathrm{u}}(x)$ denote the number of edges incident to x that are not colored by fair^R. Then, fair^R is $\frac{1}{2}$-competitive if $|E_{\mathrm{c}}| \geq |E_{\mathrm{u}}|$.

Now, for each vertex $x \in V$, let $m(x) = \frac{1}{2} d_{\mathrm{c}}(x)$. Note that $\sum_{x \in V} m(x) = |E_{\mathrm{c}}|$. For each vertex $x \in V$ such that $d_{\mathrm{u}}(x) \geq 1$, define $m_{\mathrm{u}}(x) = \frac{m(x)}{d_{\mathrm{u}}(x)}$. Then,

$$\sum_{(x,y) \in E_{\mathrm{u}}} \big(m_{\mathrm{u}}(x) + m_{\mathrm{u}}(y)\big) \;\leq\; \sum_{(x,y) \in E_{\mathrm{u}}} \big(m_{\mathrm{u}}(x) + m_{\mathrm{u}}(y)\big) \;+\; \sum_{d_{\mathrm{u}}(x)=0} m(x) \;=\; \sum_{x \in V} m(x) \;=\; |E_{\mathrm{c}}|.$$

In what follows, we will prove that $m_{\mathrm{u}}(x) + m_{\mathrm{u}}(y) \geq 1$ for every edge $(x, y) \in E_{\mathrm{u}}$, giving the desired inequality:

$$|E_{\mathrm{u}}| \leq \sum_{(x,y) \in E_{\mathrm{u}}} \big(m_{\mathrm{u}}(x) + m_{\mathrm{u}}(y)\big) \leq |E_{\mathrm{c}}|.$$

Let $(x, y) \in E_{\mathrm{u}}$. Since G is k-colorable, $d_{\mathrm{c}}(x) + d_{\mathrm{u}}(x) \leq k$, yielding the first inequality below. Note that, since $d_{\mathrm{u}}(x), d_{\mathrm{u}}(y) \geq 1$, this means that $d_{\mathrm{c}}(x), d_{\mathrm{c}}(y) \leq k - 1$. Thus, the two divisions on the right hand side of the inequality are not divisions by zero. The second inequality follows from the fact that $d_{\mathrm{c}}(x) + d_{\mathrm{c}}(y) \geq k$, since fair^R is a fair algorithm. Finally, the last inequality can be shown to hold by finding the minimum with respect to $d_{\mathrm{c}}(x)$.

$$\begin{aligned} m_{\mathrm{u}}(x) + m_{\mathrm{u}}(y) &= \frac{1}{2}\left(\frac{d_{\mathrm{c}}(x)}{d_{\mathrm{u}}(x)} + \frac{d_{\mathrm{c}}(y)}{d_{\mathrm{u}}(y)}\right) \geq \frac{1}{2}\left(\frac{d_{\mathrm{c}}(x)}{k - d_{\mathrm{c}}(x)} + \frac{d_{\mathrm{c}}(y)}{k - d_{\mathrm{c}}(y)}\right) \\ &\geq \frac{1}{2}\left(\frac{d_{\mathrm{c}}(x)}{k - d_{\mathrm{c}}(x)} + \frac{k - d_{\mathrm{c}}(x)}{d_{\mathrm{c}}(x)}\right) \geq \frac{1}{2}\left(\frac{k/2}{k/2} + \frac{k/2}{k/2}\right) = 1 \end{aligned}$$

□

In Section 4 it is shown that, on k-colorable graphs, the competitive ratio of the algorithm *Next-Fit* is $\frac{1}{2}$ for all even k. Thus, the result in Theorem 6.1 is tight.

6.2 An Upper Bound for Deterministic Algorithms

Theorem 6.2. *For any deterministic algorithm* $\mathbb{A}$ *for* EDGE-COLORING, $C_{\mathbb{A}} \leq \frac{2}{3}$*, even on* k*-colorable graphs.*

Bevis. The adversary gives a $\lceil \frac{k}{2} \rceil$-regular bipartite graph $G = (L \cup R, E)$ with $|L| = |R| = N$, for some large integer N. For each vertex $x \in L \cup R$, let $C(x)$ be the set of colors with which $\textit{on-line}^D$ has colored the edges incident to x. Let $p = \sum_{i=0}^{\lceil \frac{k}{2} \rceil} \binom{k}{i}$. Then there are p possibilities $C_1, C_2, \ldots, C_p$ for $C(x)$. Let $S_i^L = \{x \in L \mid C(x) = C_i\}$ and $S_i^R = \{x \in R \mid C(x) = C_i\}$. For each i, the vertices in S_i^L are divided into $\lfloor |S_i^L|/k \rfloor$ subsets of size k and at most one subset of size $|S_i^L| - k\lfloor |S_i^L|/k \rfloor$. The same is done to S_i^R. Let $\mathcal{S}$ be the union over all i of all subsets of S_i^L or S_i^R of size k, i.e. $\mathcal{S}$ is a set of sets. Then $|\mathcal{S}| \geq 2(N - (k-1)p)/k$. Thus, if N is chosen sufficiently large, the number of vertices contained in the sets in $\mathcal{S}$ will be much larger than the number of vertices not contained in the sets in $\mathcal{S}$.

Now, for each set $S \in \mathcal{S}$, $\lfloor \frac{k}{2} \rfloor$ new vertices are created, and each of these $\lfloor \frac{k}{2} \rfloor$ vertices are connected to each vertex in S. Assume that for each vertex $x \in S$, $|C(x)| = d$. Note that $d \leq \lceil \frac{k}{2} \rceil$. Then *on-line*^{D} can color at most $k - d$ edges incident to each of the new vertices. Now, looking at the subgraph colored by *on-line*^{D} and summing the vertex degrees of the vertices in S and the $\lfloor \frac{k}{2} \rfloor$ new vertices, we get at most $kd + 2 \cdot \lfloor \frac{k}{2} \rfloor (k - d)$, which reduces to k^2 if k is even and to $k^2 - k + d \leq k^2 - \frac{1}{2}k + \frac{1}{2}$ if k is odd. Since $S \subseteq L$ or $S \subseteq R$, the whole graph is bipartite. Furthermore, it has maximum degree k. Thus, by König's Theorem [6, p. 209], it can be k-colored off-line. Looking at the whole graph, and summing the vertex degrees of the vertices in S and the $\lfloor \frac{k}{2} \rfloor$ new vertices, we get $k^2 + \lfloor \frac{k}{2} \rfloor \cdot k$ which reduces to $\frac{3}{2}k$, if k is even, and to $\frac{3}{2}k^2 - \frac{1}{2}k$, if k is odd. Now, $k^2/(\frac{3}{2}k^2) = \frac{2}{3}$, and $(k^2 - \frac{1}{2}k - \frac{1}{2})/(\frac{3}{2}k^2 - \frac{1}{2}k) = \frac{2}{3} - \frac{k-3}{9k^2-3k} \leq \frac{2}{3}$ for $k \geq 3$. Thus, by allowing arbitrarily large graphs, ratios arbitrarily close to $\frac{2}{3}$ can be obtained. □

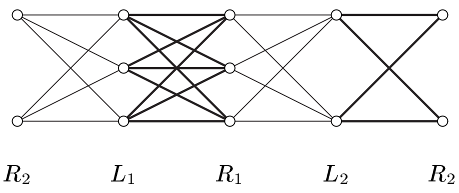
6.3 The algorithm *Next-Fit*

Theorem 6.3. *On k-colorable graphs,* $C_{NF}(k) \leq \begin{cases} \frac{1}{2}, & \text{if } k \text{ is even} \\ \frac{1}{2} + \frac{1}{2k^2}, & \text{if } k \text{ is odd} \end{cases}$

Bevis. The adversary constructs a graph G_{NF} in the following way. First it constructs two complete bipartite graphs $G_1 = (L_1 \cup R_1, E_1)$ with $|L_1| = |R_1| = \lceil \frac{k}{2} \rceil$ and $G_2 = (L_2 \cup R_2, E_2)$ with $|L_2| = |R_2| = \lfloor \frac{k}{2} \rfloor$. G_1 can be colored with $\lceil \frac{k}{2} \rceil$ colors using each color $\lceil \frac{k}{2} \rceil$ times, and G_2 can be colored with $\lfloor \frac{k}{2} \rfloor$ colors using each color $\lfloor \frac{k}{2} \rfloor$ times. The edges in these two graphs are given in an order such that *Next-Fit* colors G_1 with $C_{\lceil \frac{k}{2} \rceil}$ and G_2 with $C_k \setminus C_{\lceil \frac{k}{2} \rceil}$. Now, each vertex in R_1 is connected to each vertex in L_2 and each vertex in R_2 is connected to each vertex in L_1. Let E_{12} denote these edges connecting G_1 and G_2. *Next-Fit* is not able to color any of the edges in E_{12}. It is, however, possible to color all edges in G_{NF} with C_k, since the graph is bipartite and has maximum degree k. Thus, in the case where the input graphs are all k-colorable, the competitive ratio of *Next-Fit* can be no more than

$$\frac{|E_1| + |E_2|}{|E_1| + |E_2| + |E_{12}|} = \frac{\lceil \frac{k}{2} \rceil^2 + \lfloor \frac{k}{2} \rfloor^2}{\lceil \frac{k}{2} \rceil^2 + \lfloor \frac{k}{2} \rfloor^2 + 2\lceil \frac{k}{2} \rceil \lfloor \frac{k}{2} \rfloor},$$

which reduces to $\frac{1}{2}$ when k is even, and to $\frac{1}{2} + \frac{1}{2k^2}$ when k is odd.



Figur 4. The graph G_{NF} when $k = 5$

□

6.4 The algorithm *First-Fit*

The following theorem is an immediate consequence of Lemma 6.5 and Lemma 6.6.

Theorem 6.4. *On k-colorable graphs,* $C_{FF}(k) = \frac{k}{2k-1}$.

Thus, for small values of k, the competitive ratio of *First-Fit* on k-colorable graphs is significantly larger than that of *Next-Fit*, but the difference tends to zero as k approaches infinity.

Lemma 6.5. *On k-colorable graphs, $C_{FF}(k) \geq \frac{k}{2k-1}$.*

Bevis. Let E be the edge set of an arbitrary k-colorable graph G. Assume that *First-Fit* is given the edges in E in some order. For $c \in C_k$, let E_c denote the set of edges that *First-Fit* colors with the color c. We will prove by induction on c that, for all $c \in C_k$, $\sum_{i=1}^{c} |E_i| \geq \frac{c}{2k-1}|E|$.

For the base case, consider $c = 1$. By the definition of *First-Fit*, each edge in $E \setminus E_1$ is adjacent to at least one edge in E_1. Furthermore, since G is k-colorable, each edge in E_1 is adjacent to at most $2(k-1)$ other edges. Thus, $|E| \leq 2(k-1)|E_1| + |E_1|$, or $|E_1| \geq \frac{1}{2k-1}|E|$.

For the induction step, let $c \in C_k$. Each edge in $E \setminus \cup_{i=1}^{c} E_i$ is adjacent to at least one edge in E_c. Moreover, since each edge in E_c is adjacent to at least $c-1$ edges in $\cup_{i=1}^{c-1} E_i$, each edge in E_c is adjacent to at most $2(k-1) - (c-1) = 2k - c - 1$ edges in $E \setminus \cup_{i=1}^{c} E_i$. Therefore, $|E_c| \geq \frac{1}{2k-c}\, |E \setminus \cup_{i=1}^{c-1} E_i|$. Thus,

$$\begin{aligned}\sum_{i=1}^{c} |E_i| \geq \sum_{i=1}^{c-1} |E_i| + \frac{|E| - \sum_{i=1}^{c-1} |E_i|}{2k-c} &= \frac{|E| + (2k-c-1)\sum_{i=1}^{c-1} |E_i|}{2k-c} \\ \geq \frac{|E| + (2k-c-1)\frac{c-1}{2k-1}|E|}{2k-c} &= \frac{c}{2k-1}|E|\end{aligned}$$

□

Lemma 6.6. *On k-colorable graphs, $C_{FF}(k) \leq \frac{k}{2k-1}$.*

Outline of the Proof. Inspired by the proof of Lemma 6.5, we construct a bipartite graph G and a *First-Fit* coloring of G such that all vertices have degree k and no edge is adjacent to more than one edge of each color. For such a graph the analysis in the proof of Lemma 6.5 is tight, meaning that *First-Fit* colors exactly $\frac{k}{2k-1}$ of the edges in G. Since G is bipartite, it can be k-colored off-line. □

The full proof of Lemma 6.6 can be found in the appendix.

7 Conclusions

We have proven that the competitive ratios of algorithms for Edge-Coloring can vary only between approximately 0.46 and 0.5 for deterministic algorithms and between 0.46 and 0.57 for probabilistic algorithms (it can, of course, be lower for algorithms that are not fair). Thus, we cannot hope for algorithms with competitive ratios much better than those of *Next-Fit* and *First-Fit*. In the case of k-colorable graphs the gap is somewhat larger: the (tight) lower bound for fair algoritms is $\frac{1}{2}$ and the upper bound for deterministic algorithms is $\frac{2}{3}$. In this case we have no upper bound on the competitive ratio for probabilistic algorithms.

We have shown that the performance of *Next-Fit* matches the lower bound on the competitive ratio in both the general case and in the special case of k-colorable graphs. Furthermore, we have found the exact competitive ratio of *First-Fit* on k-colorable graphs. For small values of k it is significantly better than that of *Next-Fit*, but for large values of k they can hardly be distinguished. In the general case, *First-Fit* is at most 0.016 better than *Next-Fit*. We believe that the competitive ratio of *First-Fit* is larger than that of *Next-Fit* but we have not proven it.

Litteratur

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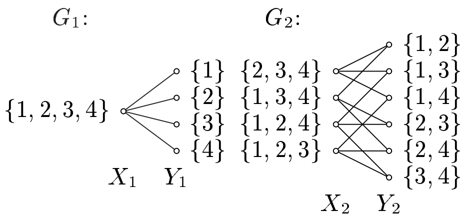
APPENDIX

Proof of Theorem 6.6. The edges of the adversary graph are given in two phases. In Phase 1, the edges of k bipartite biregular are given in an order such that *First-Fit* will color all of them. In Phase 2, the bipartite graphs are connected through edges that *First-Fit* cannot color. The resulting graph is called G. In G, every vertex has degree k, and no edge is adjacent to more than one edge of each color. For such a graph the analysis in the proof of Lemma 6.5 is tight, meaning that *First-Fit* colors exactly $\frac{k}{2k-1}$ of the edges. Furthermore, G is bipartite. Thus, *off-line* colors all of the edges.

Phase 1. The building blocks are $\lceil \frac{k}{2} \rceil$ bipartite biregular graphs, $G_1, G_2, \ldots, G_{\lceil \frac{k}{2} \rceil}$. For each i, G_i has vertex partition (X_i, Y_i). X_i has one vertex corresponding to each subset of C_k of size $k+1-i$, and Y_i has one vertex corresponding to each subset of C_k of size i (see Fig. 5). For each vertex v in G_i, let $C(v)$ denote the set of colors corresponding to v and let $\overline{C}(v) = C_k \setminus C(v)$. Each vertex $x \in X_i$ is connected to every vertex in $\{y \in Y_i \mid C(x) \cup C(y) = C_k\}$. Note that, for each edge (x, y), $|C(x)| + |C(y)| = (k+1-i) + i = k+1$. Thus, $|C(x) \cap C(y)| = 1$. We will now investigate the coloring in which each edge (x, y) receives the color in $C(x) \cap C(y)$.



Figur 5. The graphs G_1 and G_2 when $k = 4$. Next to each vertex v the color set $C(v)$ is shown.

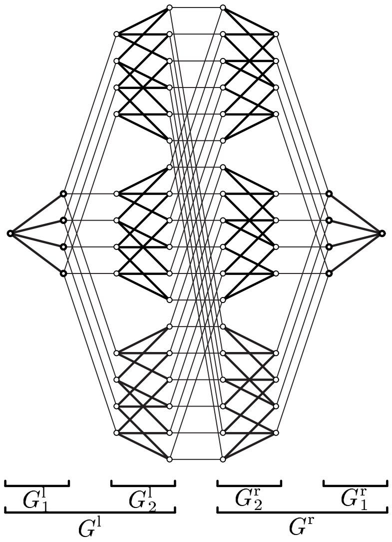
Let $x \in X_i$, for some i. Then, for each $c \in C(x)$, there is exactly one vertex $y \in Y_i$ such that $C(x) \cup C(y) = C_k$ and $C(x) \cap C(y) = \{c\}$. Similarly, if $y \in Y_i$, then, for each $c \in C(y)$, there is exactly one vertex $x \in X_i$ such that $C(x) \cup C(y) = C_k$ and $C(x) \cap C(y) = \{c\}$. This shows that the coloring is a proper coloring. It also shows that each vertex $x \in X_i$ has degree $|C(x)|$ and each vertex $y \in Y_i$ has degree $|C(y)|$.

Every edge (x, y) is adjacent to an edge of each color $c \in C_k \setminus (C(x) \cap C(y))$. Thus, the coloring will result if *First-Fit* is given the edges in order of nondecreasing color.

Finally, no edge (x, y) is adjacent to more than one edge of each color, since $|C(x)| + |C(y) \setminus (C(x) \cap C(y))| = k$ and $C(x) \cup C(y) = C_k$.

Now, k bipartite biregular graphs $G^{\mathrm{l}}_1, G^{\mathrm{l}}_2, \ldots, G^{\mathrm{l}}_{\lceil \frac{k}{2} \rceil}$, and $G^{\mathrm{r}}_1, G^{\mathrm{r}}_2, \ldots, G^{\mathrm{r}}_{\lfloor \frac{k}{2} \rfloor}$ are constructed. For $i \in \{1, 2 \ldots, \lceil \frac{k}{2} \rceil\}$, G^{l}_i consists of $\prod_{j=1}^{i-1} \frac{k-j}{j}$ copies of G_i and, for $i \in \{1, 2 \ldots, \lfloor \frac{k}{2} \rfloor\}$, G^{r}_i is isomorphic to G^{l}_i (see Fig. 6).

Phase 2. Let $i \in \{1, 2, \ldots, \lceil \frac{k}{2} \rceil - 1\}$. Then, for each pair of vertices $y \in Y_i$ and $x \in X_{i+1}$, $|C(y)| + |C(x)| = i + k + 1 - (i+1) = k$. Thus, for each vertex $y \in Y_i$, there is exactly one vertex $x \in X_{i+1}$ such that $C(y) \cup C(x) = C_k$. Since G^{l}_{i+1} contains at least $k - i$ copies of G_{i+1}, each vertex $y \in Y^{\mathrm{l}}_i$ can be connected to $k - i$ vertices in $\{x \in X^{\mathrm{l}}_{i+1} \mid C(x) \cup C(y) = C_k\}$. Since the number of copies of G_i in G^{l}_i is exactly $\frac{i}{k-i}$ times the number of copies of G_{i+1} in G^{l}_{i+1}, this can be done such that every vertex in X^{l}_{i+1} is connected to exactly i vertices in Y^{l}_i. All these edges are now added, yielding a connected graph G^{l} in which every vertex, except the vertices in $Y^{\mathrm{l}}_{\lceil \frac{k}{2} \rceil}$, has degree k. A graph G^{r} is constructed from $G^{\mathrm{r}}_1, G^{\mathrm{r}}_2, \ldots, G^{\mathrm{r}}_{\lfloor \frac{k}{2} \rfloor}$ in the same way. Note that in G^{r}, $Y_{\lfloor \frac{k}{2} \rfloor}$ plays the role of $Y_{\lceil \frac{k}{2} \rceil}$ in G^{l}. Finally, G^{l} and G^{r} are connected through edges connecting pairs



Figur 6. The graph G when $k = 4$

of vertices $y^{\mathrm{l}} \in Y^{\mathrm{l}}_{\lceil \frac{k}{2} \rceil}$ and $y^{\mathrm{r}} \in Y^{\mathrm{r}}_{\lfloor \frac{k}{2} \rfloor}$ such that $C(y^{\mathrm{l}}) \cup C(y^{\mathrm{r}}) = C_k$ and in a way so that each vertex in $Y^{\mathrm{l}}_{\lceil \frac{k}{2} \rceil} \cup Y^{\mathrm{r}}_{\lfloor \frac{k}{2} \rfloor}$ ends up having degree k. The resulting graph is denoted by G.

For each edge (x, y) given in Phase 2, $C(x) \cup C(y) = C_k$. Thus, *First-Fit* cannot color the edge. Furthermore, $|C(x)| + |C(y)| = k$. Thus, (x, y) is not connected to more than one edge of each color. This completes the proof. □