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FAULTY RANDOM GEOMETRIC NETWORKS*

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ABSTRACT
In this paper we analyze the computational power of random geometric networks in the presence of random (edge or node) faults considering several important network parameters. We first analyze how to emulate an original random geometric network G on a faulty network F. Our results state that, under the presence of some natural assumptions, random geometric networks can tolerate a constant node failure probability with a constant slowdown. In the case of constant edge failure probability the slowdown is an arbitrarily small constant times the logarithm of the graph order. Then we consider several network measures, stated as linear layout problems (Bisection, Minimum Linear Arrangement and Minimum Cut Width). Our results show that random geometric networks can tolerate a constant edge (or node) failure probability while maintaining the order of magnitude of the measures considered here. Finally we show that, with high probability, random geometric networks with (edge or node) faults do have a Hamiltonian cycle, provided the failure probability is constant. Such capability enables performing distributed computations based on end-to-end communication protocols.

1. Introduction

The use of distributed computing in wireless networks is a computation model that is gaining increasing importance in computer science and telecommunication. In this setting, the processors, scattered geographically, communicate through transmitters, effectively forming a *broadcast network*. The following setting arises in applications of broadcast networks: A set of stations are located in some geographical area. These stations can compute, send and receive messages in synchronous steps. All the transmitters have the same power, but their effective broadcast range is limited to some specified distance r, that is, two stations can only communicate if their distance is at most r. Unit disk graphs provide a convenient way to model this setting: A graph is a *unit disk graph* if each node can be mapped to a point in the plane in such a way that two nodes are adjacent if and only if their distance is at most some specified bound r. Several researchers have shown that some important problems on broadcast networks are, in fact, classic problems restricted to unit disk

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graphs [7]. However, as pointed in [8], unit disk graphs assume that no interference from weather, mountains or other obstacles affects the communication between two stations. Also, it does not take into account the possibility that individual stations go down because of problems with power supply, mechanical damages, sabotage, etc. On the other hand, failures could model a surcharge of the processors or links.

The advent of mobile computing and of cellular phones introduces an uncertainty with respect to the positions of the stations. Assuming that the stations are homogeneously distributed in the plane is a simplified way to cope with that changing environment. Random geometric graphs come at hand to model such a situation: A *random geometric graph* is a unit disk graph whose nodes are points uniformly distributed in the unit square. Although random geometric graphs are popular objects in empirical studies, the available theoretical knowledge on random geometric graphs is more limited (see [10] for a survey and [11] for recent results on layout measures).

In this paper we analyze the computational power and several network parameters of a random geometric network in the presence of random faults. We analyze both edge failures and node failures keeping in mind that, as said, edge failures can be interpreted as the inability to communicate between two stations because of the presence of some unexpected obstacle and node failures can be interpreted as stations that become inoperative.

We start by analyzing the emulation problem [14], that is, how to emulate an original random geometric network G on a faulty network F produced by random removal of edges (or nodes) of G. In the case of random graphs in the classical $G_{n,p}$ model, the emulation problem presents a similar behavior, in the sense that a random network can be emulated with constant slowdown [16,12]. Our results state that, under the presence of some natural assumptions, random geometric networks with n nodes can tolerate a constant node failure probability with a constant slowdown. In the case of constant edge failure probability the slowdown is $\delta \log n$, where δ is an arbitrarily small positive constant.

We continue by considering several relevant network measures that can be stated as *linear layout problems* [11]. Given a graph with n vertices, a *layout* on the graph is a bijection between the vertex set and the set of naturals from 1 to n. Graph layout problems seek for a layout that minimizes a measure associated with each problem. The particular measures that we consider include *Bisection*, *Minimum Linear Arrangement* and *Minimum Cutwidth*, which play an important role in the evaluation of computer networks [1,14,13]. The definition of these problems is given in Section 2. Our results show that random geometric networks can tolerate a constant edge (or node) failure probability maintaining the order of magnitude of the measures considered here.

Finally, we consider Hamiltonian cycles in faulty random geometric networks. Specifically, we prove that, with high probability, random geometric networks with faulty nodes or faulty edges do have a Hamiltonian cycle. This is important, because with such capability it is possible to build a path to perform distributed computations based on *end-to-end communication protocols*, which allow distributed algorithms to treat an unreliable network as a reliable channel [17].

2. Preliminaries

Let G be a graph and let F be a subgraph of G; G represents a fault-free network and F represents the resulting network after the occurrence of (edge or node) faults in G. In this paper we differentiate between edge faults and node faults: graphs with faulty edges are the result of removing edges from an original graph; graphs with faulty nodes are the result of removing nodes from an original graph together with the edges incident to the removed nodes. In graph theoretical terms, a graph with faulty nodes refers to a node induced subgraph of an original graph and a graph with faulty edges refers to a subgraph of an original graph.

Let $G = (V, E)$ be a graph and let $f \in (0, 1)$ denote its node failure probability. Let V' be a subset of V where for all $v \in V$, $\mathbf{Pr}[v \in V'] = 1 - f$, independently for all nodes in V. Then, we say that the subgraph of G induced by V' is a graph with random faulty nodes. Given G and f, we denote this probability space as $\mathcal{FN}(G, f)$.

Analogously, let $G = (V, E)$ be a graph and let $f \in (0, 1)$ denote its edge failure probability. Let E' be a subset of E where for all $e \in E$, $\mathbf{Pr}[e \in E'] = 1 - f$, independently for all edges in E. Then, we say that the subgraph of G induced by E' is a graph with random faulty edges. Given G and f, we denote this probability space as $\mathcal{FE}(G, f)$.

The *emulation* of a fault-free network G on a faulty network F consists in performing the computations performed by the nodes of G in the nodes of F and ensuring that the direct communication across edges of G is simulated across paths in F. The *slowdown* of the emulation of G in F is defined as the length of time it takes to the faulty network F to emulate one step (communication and computation) of the original network G. The formal parameters to measure the quality of an emulation are: the *dilation* (the length of the largest associated path), the *congestion* (the maximal number of paths that share an edge of F) and the *load* (the maximum number of nodes of G that are mapped to a node of F). In what follows, we denote by $d(\phi)$, $c(\phi)$ and $l(\phi)$ the dilation, congestion and load of an embedding ϕ. It is well known that an emulation with dilation d, congestion c and load l has slowdown $O(d + c + l)$.

Given as input an undirected graph $G = (V, E)$ with $n = |V|$, a *layout* φ on G is a one-to-one function $\varphi : V \to \{1, \dots, n\}$. Given a layout φ on G, let us define the sets $L(i, \varphi, G) = \{u \in V(G) : \varphi(u) \le i\}$ and $R(i, \varphi, G) = \{u \in V(G) : \varphi(u) > i\}$. Any layout induces a length assignment to the edges of the graph and a series of edge-cuts placed between consecutive positions. Their definitions are: $\lambda(uv, \varphi, G) = |\varphi(u) - \varphi(v)|$ where $uv \in E(G)$, and $\theta(i, \varphi, G) = |\{uv \in E(G) : u \in L(i, \varphi, G) \wedge v \in R(i, \varphi, G)\}|$, where $i \in \mathbb{N}$. Given a graph $G = (V, E)$ with $n = |V|$, the problems we consider are the following:

- Minimum Linear Arrangement (MinLA): find $\mathrm{MINLA}(G) = \min_\varphi \mathrm{LA}(\varphi, G)$ where $\mathrm{LA}(\varphi, G) = \sum_{uv \in E} \lambda(uv, \varphi, G) = \sum_{i=1}^{n} \theta(i, \varphi, G)$.
- Minimum Cutwidth (Cutwidth): find $\mathrm{MINCW}(G) = \min_\varphi \mathrm{CW}(\varphi, G)$ where $\mathrm{CW}(\varphi, G) = \max_{i=1}^{n} \theta(i, \varphi, G)$.
- Edge Bisection (Bisection): find $\mathrm{MINEB}(G) = \min_\varphi \mathrm{EB}(\varphi, G)$ where $\mathrm{EB}(\varphi, G) = \theta(\lfloor n/2 \rfloor, \varphi, G)$.

The three problems are known to be **NP**-complete in general (see [11] for more details on the complexity of these problems).

A graph is a *unit disk graph* if each node can can be mapped to a point in the plane in such a way that two nodes are adjacent if and only if their distance is at most some specified bound r.[†] Equivalently, a unit disk graph can be defined as the intersection graph of a set of disks with radius r [7].

Observe that we have to fix which kind of norm is used to measure distances in the plane. Under the l_p norm ($p \geq 1$), the distance $\|x - y\|_p$ between two points $x = (x_1, y_1)$ and $y = (x_2, y_2)$ is $(|x_1 - x_2|^p + |y_1 - y_2|^p)^{1/p}$. Under the l_∞ norm, their distance is $\max\{|x_1 - x_2|, |y_1 - y_2|\}$. For sake of simplicity, we use the l_∞ norm all through this paper. However, all our results are valid for any other l_p norm.

In [9] it is proven that the Cutwidth problem remains **NP**-complete even when restricted to unit disk graphs.

Let r be a positive number and let V be any set of points in the unit square ($[0,1]^2$). A *geometric graph* $G = (V; r)$ with node set V and radius r is the graph $G = (V, E)$ where $E = \{\{u, v\} \mid u, v \in V \wedge 0 < \|u - v\| \leq r\}$. By definition, a geometric graph is a unit disk graph.

Let $(r_i)_{i \geq 1}$ be a sequence of positive numbers. For any natural n, the class of *random geometric graphs with n nodes and radius r_n*, denoted by $\mathcal{G}(n, r_n)$, is a probability space over the set of all geometric graphs $G = (\mathcal{V}_n; r_n)$ where $\mathcal{V}_n$ is a set of n independently and uniformly distributed random points in $[0,1]^2$. Notice that a geometric graph with faulty nodes is also a geometric graph, while a geometric graph with faulty edges is not.

In the remainder of this paper we restrict our attention to the particular case where the radius is of the form

$$r_n = \sqrt{\frac{a_n}{n}} \quad \text{where} \quad r_n = o(1) \quad \text{and} \quad a_n = \Omega(\log n).$$

So, from now on the term *random geometric graph* shall refer to this selection of radius. In fact, we will say that $a_n = b_n \log n$ for some function b_n that satisfies $b_n \to \infty$ and $b_n = o(n/\log n)$. It is important to remark that through this choice, the construction of sparse but connected graphs is guaranteed: Defining the connectivity distance $\rho_n = \rho_n(\mathcal{X}_n)$ as the smallest radius r such that the graph $G(\mathcal{X}_n, r)$ is connected, it is known [2] that $\rho_n \sqrt{n/\log n}$ converges to $\frac{1}{2}$ almost surely; that is,

[†] Most definitions in the literature state that $r = 1$, but r can be any arbitrary unit as an appropriate scaling fixes that.

with probability 1 the sequence $(X_i)_{i\geq 1}$ yields a sequence $(\rho_n)_{n\geq 1}$ with this limit behavior.

Recall that a sequence of events $\mathcal{E}_n$ occurs *with high probability* if $\mathbf{Pr}[\mathcal{E}_n] \to 1$. Also recall that a Hamiltonian cycle in a graph is a cycle that visits all nodes once.

3. Preliminaries

In this section we deal with the emulation properties of independent random node faults on random geometric networks.

Definition 1 *Let $G = (V, E)$ be a geometric graph with n points and radius r_n. Dissect the unit square into $4\lceil 1/r_n\rceil^2$ boxes of size $1/(2\lceil 1/r_n\rceil)\times 1/(2\lceil 1/r_n\rceil)$ placed packed in $[0,1]^2$ starting at $(0,0)$. By construction, all the boxes exactly fit in the unit square, and any two points of V in neighboring boxes (including diagonals) will be connected by an edge in G because $1/(2\lceil 1/r_n\rceil) \leq r_n/2$. Given $\epsilon \in (0,1)$, let us say that G is* ϵ-homogeneous *if every box of this dissection contains at least $(1-\epsilon)\frac{1}{4}a_n$ points and at most $(1+\epsilon)\frac{1}{4}a_n$ points.*

The following lemma states that, with high probability, random geometric graphs drawn from $\mathcal{G}(n, r_n)$ are homogeneous.

Lemma 1 *For any constant $\epsilon \in (0,1)$, $\mathbf{Pr}[\mathcal{G}(n,r_n)$ is ϵ-homogenous$] \to 1$.*

Proof. For any n large enough, let $G = G_n$ be a random geometric graph drawn from $\mathcal{G}(n, r_n)$. Choose a box in the dissection and let Y be the random variable counting the number of points of G in this box. As the points in G are i.u.d.,

$$\mathbf{E}[Y] = \frac{n}{4\lceil 1/r_n\rceil^2} \sim \frac{nr_n^2}{4} = \tfrac{1}{4}a_n.$$

Using Chernoff's bounds [15], we obtain

$$\begin{aligned}\mathbf{Pr}\left[Y \geq (1+\epsilon)\tfrac{1}{4}a_n\right] &\leq \mathbf{Pr}\left[Y \geq (1+\tfrac{1}{2}\epsilon)\mathbf{E}[Y]\right] \leq \exp\left(-\left(\tfrac{1}{2}\epsilon\right)^2 \mathbf{E}[Y]/3\right)\\ &\leq \exp\left(-\tfrac{1}{13}\epsilon^2\tfrac{1}{4}a_n\right) = n^{-\epsilon^2 b_n/52}\end{aligned}$$

and

$$\begin{aligned}\mathbf{Pr}\left[Y \leq (1-\epsilon)\tfrac{1}{4}a_n\right] &\leq \mathbf{Pr}\left[Y \leq (1-\tfrac{1}{2}\epsilon)\mathbf{E}[Y]\right] \leq \exp\left(-\left(\tfrac{1}{2}\epsilon\right)^2 \mathbf{E}[Y]/2\right)\\ &\leq \exp\left(-\tfrac{1}{9}\epsilon^2\tfrac{1}{4}a_n\right) = n^{-\epsilon^2 b_n/36} \leq n^{-\epsilon^2 b_n/52}.\end{aligned}$$

The number of boxes is certainly smaller than n, so by Boole's inequality, the probability that for some box the number of points in the box is less than $(1-\epsilon)\frac{1}{4}a_n$ or bigger than $(1+\epsilon)\frac{1}{4}a_n$, is bounded by $2n^{1-b_n\epsilon^2/52}$. As $b_n \to \infty$, the complementary event tends to 1 as n tends to infinity. □.

Definition 2 *For any constants $\epsilon \in (0,1)$ and $f \in (0,1)$, let G be a ϵ-homogeneous geometric graph with n points and radius r_n. Let F be a node induced subgraph of G. Again, dissect the unit square into $4\lceil 1/r_n\rceil^2$ boxes of size $1/(2\lceil 1/r_n\rceil) \times 1/(2\lceil 1/r_n\rceil)$. Let us say that F is* (ϵ, f)-friendly *if every box of this dissection* *contains at least* $(1-\epsilon)^2(1-f)\frac{1}{4}a_n$ *points of* F *and at most* $(1+\epsilon)^2(1-f)\frac{1}{4}a_n$ *points of* F.

The following lemma states that for all homogeneous geometric graph G, with high probability, node induced subgraphs of G drawn from $\mathcal{FN}(G,f)$ are friendly.

Lemma 2 *For any constants* $\epsilon \in (0,1)$ *and* $f \in (0,1)$ *and for all* n, *let* G_n *be a* ϵ*-homogeneous geometric graph with* n *nodes and radius* r_n. *Then,*

$$\lim_{n\to\infty} \mathbf{Pr}\left[\mathcal{FN}(G_n,f) \text{ is } (\epsilon,f)\text{-friendly}\right] = 1.$$

Proof. For any n, let F_n be drawn from $\mathcal{FN}(G_n,f)$. Choose a box in the dissection; let y_n be the number of points of G_n in this box and let Z_n be the random variable counting the number of points of F_n in this box. As G_n is ϵ-homogeneous, we have that $(1-\epsilon)\frac{1}{4}a_n \leq y_n \leq (1+\epsilon)\frac{1}{4}a_n$. On the other hand, as Z_n is a sum of y_n Bernoulli variables with parameter $1-f$,

$$(1-\epsilon)\tfrac{1}{4}a_n(1-f) \leq \mathbf{E}\left[Z_n\right] = y_n(1-f) \leq (1+\epsilon)\tfrac{1}{4}a_n(1-f).$$

Using Chernoff's bounds as in the proof of Lemma 1 we can show concentration around this expectation. Also, using Boole's inequality as in the proof of Lemma 1, we can bound the probability over all the boxes and prove it tends to 1 as n tends to infinity. $\square$.

We introduce now the so-called locality-preserving emulation that we will use to emulate a geometric graph G on a node induced subgraph of G.

Definition 3 *Let* G *be a geometric graph with* n *points and radius* r_n *and let* F *be a node induced subgraph of* G. *We define a* locality-preserving emulation ψ *of* G *on* F *by the following algorithm:*

(i) *Dissect the unit square into* $\eta_n = 4\lceil 1/r_n\rceil^2$ *boxes of size* $1/(2\lceil 1/r_n\rceil) \times 1/(2\lceil 1/r_n\rceil)$.

(ii) *For all* $i = 1,\ldots,\eta_n$, *let* α_i *and* β_i *denote the number of points of* G *and* F *respectively in box* i.

(iii) *If* $\beta_i = 0$ *for some* i, *abort.*

(iv) *For all* $u \in V(G)$ *in box* i, *let* $\psi(u) = v$ *where* v *is a node chosen between all nodes of* $V(F)$ *contained in box* i *in such way that* $|\{w \in V(G) : \psi(w) = v\}| \leq \lceil \alpha_i/\beta_i \rceil$.

(v) *For all* $\{u,v\} \in E(G)$, *let* $\psi(\{u,v\}) = \langle\{\psi(u),\psi(v)\}\rangle$.

Observe that Step 5 maps each edge of G on a path of only one edge in F. This edge exists in the subgraph, because if $\{u,v\} \in E(G)$, u and v belong to neighboring or identical boxes and as $\psi(u)$ belongs to the same box than u and $\psi(v)$ belongs to the same box than v, it is the case that $\psi(u)$ and $\psi(v)$ also belong to neighboring or identical boxes, and thus are connected by an edge in $E(F)$.

The following lemma states that, with high probability, our locality-preserving emulation is able to emulate a homogeneous graph G on a friendly node induced subgraph of G with constant dilation, congestion and load.

Lemma 3 *For any constants $\epsilon \in (0,1)$ and $f \in (0,1)$ and for all n, let G_n be a ϵ-homogeneous geometric graph with n points and radius r_n; let F_n be a (ϵ, f)-friendly node induced subgraph of G_n and let ψ_n be a locality-preserving emulation of G_n on F_n. Then, there exist constants c_0 and l_0 such that,*

$$\lim_{n\to\infty} \mathbf{Pr}\left[d(\psi_n) = 1 \ \wedge\ c(\psi_n) \le c_0 \ \wedge\ l(\psi_n) \le l_0\right] = 1.$$

Proof. Let n be any natural. By the definition of the locality-preserving emulation ψ, all edges in G_n are mapped in ψ_n to a path consisting of one edge in F_n; thus the dilation of the locality-preserving emulation is $d(\psi_n) = 1$. Moreover, every node $u \in V(F)$ has load at most $1 + \lceil \alpha_i/\beta_i \rceil$, which is at most $2+(1+\epsilon)(1-\epsilon)^{-2}(1-f)^{-1}$ because G_n is ϵ-homogeneous and F_n is (ϵ, f)-friendly. Thus, $l(\psi_n) \le 2+(1+\epsilon)(1-\epsilon)^{-2}(1-f)^{-1} = l_0$, which is constant, since ϵ and f are constants too. Finally, let $\{u, v\}$ be any edge in G_n; as G_n is a geometric graph, u and v must be in neighboring or identical boxes, and so $\psi_n(u)$ and $\psi_n(v)$ also must be in neighboring or identical boxes. Therefore, the congestion of $\{u, v\}$ must be at most the product of the loads of u and v and thus $c(\psi_n) \le l_0^2 = c_0$. $\square$.

The combination of the previous lemmas implies our main result for random geometric networks with node faults:

Theorem 1 *With high probability, random geometric networks, with n nodes and radius $r_n = \sqrt{a_n/n}$, with $r_n \to 0$ and $a_n = b_n \log n$ for any function $b_n \to \infty$ and $b_n = o(n/\log n)$, can be emulated with constant slowdown in the presence of random node faults, provided the failure probability is constant.*

4. Emulations of random geometric networks with faulty nodes

In this section we deal with the emulation properties of independent random edge faults on random geometric networks.

Definition 4 *Let $G = (V, E)$ be a geometric graph with n points and radius r_n. Dissect the unit square into $9\lceil 1/r_n \rceil^2$ boxes of size $w_n \times w_n$ with $w_n = 1/(3\lceil 1/r_n \rceil)$ placed packed in $[0,1]^2$ starting at $(0,0)$. By construction, all the boxes exactly fit in the unit square, and any two points of V in boxes whose centers are within distance $2w_n$ will be connected by an edge in G because $3w_n \le r_n$. Given $\epsilon \in (0,1)$, let us say that G is* ϵ-spreading *if every box of this dissection contains at least $(1-\epsilon)\frac{1}{9}a_n$ points and at most $(1+\epsilon)\frac{1}{9}a_n$ points.*

An analogous proof to the one of Lemma 1 shows that, with high probability, random geometric graphs drawn from $\mathcal{G}(n, r_n)$ are spreading:

Lemma 4 *For any constant $\epsilon \in (0,1)$, $\mathbf{Pr}\left[\mathcal{G}(n, r_n) \text{ is } \epsilon\text{-spreading}\right] \to 1$.*

Definition 5 *For any constants $\epsilon \in (0,1)$ and $f \in (0,1)$, let G be a ϵ-spreading geometric graph with n points and radius r_n. Let F be an edge induced subgraph of G. Let us say that F is* (ϵ, f)-kindly *if every node has degree at least $(1-\epsilon)^2(1-f)\frac{25}{9}a_n$ in F.*

The following lemma states that, for all spreading geometric graph G, with high probability, edge induced subgraphs of G drawn from $\mathcal{FE}(G, f)$ are kindly.

Lemma 5 *For any constants $\epsilon \in (0,1)$ and $f \in (0,1)$ and for all n, let G_n be an ϵ-spreading geometric graph with n nodes and radius r_n. Then,*

$$\lim_{n\to\infty} \mathbf{Pr}\left[\mathcal{FE}(G_n, f) \text{ is } (\epsilon, f)\text{-kindly}\right] = 1.$$

Proof. Let n be any natural, let G denote G_n and let u be a node in G. As G is ϵ-spreading, each box contains at least $(1-\epsilon)\frac{1}{9}a_n$ points. As u is connected to all other nodes in boxes whose centers are at distance 0, 1 or 2 from the center of the box that contains u (there are 25 such boxes), we have that $\deg_G(u) \geq (1-\epsilon)\frac{25}{9}a_n$. Let $F = F_n$ be an edge faulty random geometric graph of G drawn from $\mathcal{FE}(G, f)$. As each edge of G fails independently with probability f, we have that $\mathbf{E}\left[\deg_F(u)\right] \geq (1-\epsilon)(1-f)\frac{25}{9}a_n$. Using Chernoff's bounds we get

$$\begin{aligned}\mathbf{Pr}\left[\deg_F(u) < (1-\epsilon)\mathbf{E}\left[\deg_F(u)\right]\right] &\leq \exp(-\epsilon^2\mathbf{E}\left[\deg_F(u)\right]/2)\\ &\leq n^{-(1-f)(1-\epsilon)\epsilon^2 b_n\cdot 25/18}.\end{aligned}$$

So, applying Boole's inequality, the probability that some node in F has degree less than $(1-\epsilon)^2(1-f)\frac{25}{9}a_n$ is at most $n^{1-(1-f)(1-\epsilon)\epsilon^2 b_n\cdot 25/18}$, which tends to zero as n tends to infinity because $b_n \to \infty$. $\square$.

We introduce now the so-called randomized one-hop emulation that we will use to emulate a geometric graph G on an edge induced subgraph of G.

Definition 6 *Let G be a geometric graph with n points and radius r_n and let F be an edge induced subgraph of G. We define a* randomized one-hop emulation Ψ *of G on F by the following randomized algorithm:*

(*i*) *Dissect the unit square into $9\lceil 1/r_n\rceil^2$ boxes of size $1/(3\lceil 1/r_n\rceil)\times 1/(3\lceil 1/r_n\rceil)$.*
(*ii*) *For all $u \in V(G)$, let $\Psi(u) = u$.*
(*iii*) *For all $\{u,v\} \in E(F)$, let $\Psi(\{u,v\}) = \langle\{u,v\}\rangle$.*
(*iv*) *For all $\{u,v\} \in E(G)\setminus E(F)$, let $\Psi(\{u,v\}) = \langle\{u,w\},\{w,v\}\rangle$ where w is chosen at random among all nodes $t \in V(G)$ in the same box where u belongs such that $\{u,t\} \in E(F)$ and $\{t,v\} \in E(F)$. If not possible, abort.*

Lemma 6 *For any constants $\epsilon \in (0,1)$ and $f \in (0,1)$ and for all n, let G_n be a ϵ-spreading geometric graph with n points and radius r_n; let F_n be an (ϵ, f)-kindly edge induced subgraph of G_n and let Ψ_n be a randomized one-hop emulation of G_n on F_n. Then, for any constant $\delta > 0$,*

$$\lim_{n\to\infty} \mathbf{Pr}\left[l(\Psi_n) = 1 \ \wedge\ d(\Psi_n) \leq 2 \ \wedge\ c(\Psi_n) \leq \delta\log n\right] = 1.$$

Proof. Using probabilistic standard techniques it is straightforward to prove that, with high probability, the one-hop emulation does not abort. So, we can assume that the randomized algorithm does not abort.

Let n be any natural. By the definition of our randomized one-hop emulation, all nodes in G_n are mapped to themselves; thus the load of Ψ_n is $l(\Psi_n) = 1$. Moreover, every edge in G_n is mapped to at most two edges; thus the dilation of

Ψ_n is $d(\Psi_n) \leq 2$. For the congestion, let $\{u,v\} \in E(F_n)$, let B_u denote the box where u belongs to and let $c(\Psi_n, u, v)$ denote the congestion of the edge $\{u,v\}$ in the emulation Ψ_n (i.e., the number of edges in G_n mapped to $\{u,v\}$). Write $u \sim v$ to mean $\{u,v\} \in E(F)$. Then, by linearity of expectation,

$$\begin{aligned}\mu &= \mathbf{E}\left[c(\Psi_n, u, v)\right] \\ &\leq 1 + \sum_{w \in B_u} \mathbf{Pr}\left[w \not\sim v\right] \mathbf{Pr}\left[w \sim u\right] \mathbf{Pr}\left[\Psi_n(\{w,v\}) = \langle \{w,u\}, \{u,v\} \rangle\right] \\ &\leq 1 + (1+\epsilon)\tfrac{1}{9} a_n \cdot f \cdot (1-f) \cdot \frac{1}{(1-\epsilon)^2 (1-f) \frac{25}{9} a_n} \leq 1 + \frac{f(1+\epsilon)}{(1-\epsilon)^2} = C_{\epsilon,f}.\end{aligned}$$

So, it holds that $1 \leq \mu \leq C_{\epsilon,f}$.

Recall the general Chernoff's bounds: Let $X_1, \ldots, X_n$ be independent random variables corresponding to Bernoulli experiments, each variable with probability p_i of success and probability $q_i = 1 - p_i$ of failure. Let $X = \sum_{i=1}^{n} X_i$ be a random variable with expected value $\mu = \sum_{i=1}^{n} p_i$. Then, for any $d > 0$,

$$\mathbf{Pr}\left[X > (1+d)\mu\right] \leq \left(\frac{e^d}{(1+d)^{1+d}}\right)^{\mu}.$$

Assuming n large enough and taking $d = \frac{\delta \log n}{\mu} - 1$, in our case we have

$$\mathbf{Pr}\left[c(\Psi_n, u, v) > \delta \log n\right] \leq \left(\frac{e^{\frac{\delta \log n - \mu}{\mu}}}{\left(\frac{\delta \log n}{\mu}\right)^{\frac{\delta \log n}{\mu}}}\right)^{\mu} \leq \left(\frac{e\mu}{\delta \log n}\right)^{\delta \log n}$$

As the number of edges in F_n is obviously less than n^2, using Boole's inequality we have that the probability that some edge in F_n has congestion greater than $\log n$ is bounded above by $n^2 \left(e\mu/\delta \log n\right)^{(\delta \log n)}$, which tends to zero as n tends to infinity because $(\delta \log n)^{\delta \log n}$ grows faster than any polynomial. $\Box$.

Notice that the previous lemma can be straightened to allow any function $\delta(n)$ such that $\delta(n) \log\log n \to \infty$.

The combination of the previous lemmas implies our main result for random geometric networks with edge faults:

Theorem 2 *With high probability, random geometric networks, with n nodes and radius r_n, with $r_n \to 0$ and $a_n/\log n \to \infty$, can be emulated with slowdown $\delta \log n$, for arbitrary constant $\delta > 0$, in the presence of random edge faults, provided the failure probability is constant.*

In the case of edge faults we have to deal with a randomized emulation, therefore the slowdown is a bit wider than in the case of node faults. We suspect that a deterministic algorithm could achieve a constant congestion.

5. Layout problems on faulty random geometric graphs

In this section we deal with the BISECTION, CUTWIDTH and MINLA layout problems. We will develop the case of random geometric graphs with edge faults and finish by pointing out the results for the vertex failure case.

To obtain a lower bound, we first need an isoperimetric inequality:

Lemma 7 ([9]) *For any integer $m \geq 3$, let (A, B) be a partition of the $\{1, \ldots, m\}^2$ 2-dimensional lattice with vertical, horizontal and diagonal connections. Let $\partial_{A,B}$ be the number of elements of $A \times B$ that are neighbor pairs, including diagonal neighbors. Then, $\partial_{A,B} \geq 3 \min\left\{\sqrt{|A|}, \sqrt{|B|}\right\}$.*

The following definition captures the property that edges of a geometric graph are appropriately distributed in the unit square.

Definition 7 *For any constants $\epsilon \in (0,1)$ and $f \in (0,1)$, let $G = (V, E)$ be an ϵ-homogeneous geometric graph with n points and radius r_n. Let $F = (V, E')$ be an edge induced subgraph of G. Dissect the unit square into $4\lceil 1/r_n \rceil^2$ boxes of size $1/(2\lceil 1/r_n \rceil) \times 1/(2\lceil 1/r_n \rceil)$. Let us say that F is* (ϵ, f)-mixing *if for every two disjoint vertex sets A and B such that A is inside a box B_A (of this dissection) and B is inside a box B_B, and either $B_A = B_B$ or B_A is a neighbor of B_B, then*

$$cut(A, B) \geq (1-\epsilon)^3 (1-f) (\tfrac{1}{4} a_n)^2,$$

where $cut(A, B) = |\{(u, v) \in E' \ : \ u \in A \text{ and } v \in B\}|$.

In a similar way as in Lemma 2 we get the following result:

Lemma 8 *For any constants $\epsilon \in (0,1)$ and $f \in (0,1)$ and for all n, let G_n be an ϵ-homogeneous geometric graph with n nodes and radius r_n. Then*

$$\lim_{n \to \infty} \mathbf{Pr}\left[\mathcal{FE}(G_n, f) \text{ is } (\epsilon, f)\text{-mixing}\right] = 1.$$

The following lemma is the basis of our lower bounds for (ϵ, f)-mixing geometric graphs with edge faults.

Lemma 9 *For any constants $\epsilon \in (0, \frac{1}{5})$ and $f \in (0,1)$, let G be a ϵ-homogeneous geometric graph with n points and radius r_n; let F be a (ϵ, f)-mixing subgraph of G. Let φ be any layout of F. Then, for any integer i such that $\alpha = i/n \in (2\epsilon, 1-2\epsilon)$, it is the case that*

$$\theta(i, \varphi, F) \geq (1-\epsilon)^3 (1-f) \frac{3(1-5\epsilon)}{8} \min\left\{\sqrt{\alpha - 2\epsilon}, \sqrt{1-\alpha-2\epsilon}\right\} n^2 r_n^3.$$

Proof. We assign colors to points (nodes) and boxes: color the first i points in the ordering "red" and the other points "green"; color the boxes containing at most $\frac{1}{5}\epsilon a_n$ green points "red", the boxes containing at most $\frac{1}{5}\epsilon a_n$ red points "green", and the other boxes "yellow". Let Y_n be the number of yellow boxes.

Observe that $\theta(i, \varphi, F)$ is the total number of edges between opposite-color points. Let us refer to such edges as "within-box" if the points in question lie in the same box, or "between-box" if not. We consider two cases:

Case 1: $Y_n \ge 25\epsilon^{-2}n^{1/2}a_n^{-1/2}$. For each yellow box, the cost of within-box edges is at least $(1-\epsilon)^3(1-f)\epsilon^2 a_n^2/25$. Hence,

$$\begin{aligned}\theta(i,\varphi,F) &\ge (1-\epsilon)^3(1-f)(\tfrac{1}{5}\epsilon a_n)^2 \cdot Y_n \\ &\ge (1-\epsilon)^3(1-f)n^{1/2}a_n^{3/2} = (1-\epsilon)^3(1-f)n^2 r_n^3.\end{aligned}$$

Case 2: $Y_n < 25\epsilon^{-2}n^{1/2}a_n^{-1/2}$. In this case, we consider only between-box edges (between opposite colored points) which are between neighboring boxes (including diagonal neighbors). Consider a particular box containing a total of t points, r of them red. Suppose that the total number of red points in neighboring boxes is r' and the total number of green points in neighboring boxes is g'. Then, the total number of between-box edges of the type we are considering, involving points in that particular box, is $(1-\epsilon)^3(1-f)(rg'+(t-r)r') = (1-\epsilon)^3(1-f)(r(g'-r')+tr')$. Given t, r', and g', this is a linear function of r and so attains its minimum over the range $[0,t]$ either at $r=0$ or at $r=t$ (or both). Hence, it is possible to change the points in that box to either all red or all green without increasing the total number of between-box edges of the type we are considering.

Let us modify the coloring of points by going through the yellow boxes in turn, successively changing the color of points in each box either to all red or to all green, whichever does not increase the total number of between-box edges of the type we are considering. When done, there will no longer be any yellow boxes! Let R_n be the number of red boxes and G_n the number of green boxes based on this modified coloring.

By homogeneity, the number of points whose color has been changed is at most

$$Y_n \cdot (1+\epsilon)\tfrac{1}{4}a_n \le 25\epsilon^{-2}n^{1/2}a_n^{1/2} = 25\epsilon^{-2}nr_n.$$

Thus, for n so large that $25\epsilon^{-2}r_n \le \epsilon$, the number of red points in the modified coloring is at least $(\alpha-\epsilon)n$.

By definition of "green box", the number of red points in green boxes is at most $\frac{1}{5}\epsilon a_n 4\lceil 1/r_n\rceil^2 \le \epsilon n$. Thus the total number of red points in red boxes (in the modified coloring) is at least $(\alpha-2\epsilon)n$ for n large enough. By a similar argument, the number of green points in green boxes in the modified coloring is at least $(1-\alpha-2\epsilon)n$ for n large enough.

As, by homogeneity, no box can contain more than $\frac{1}{4}(1+\epsilon)a_n$ points and there are at least $(\alpha-2\epsilon)n$ red points in red boxes and $(1-\alpha-2\epsilon)n$ green points in green boxes, we have

$$R_n \ge \frac{(\alpha-2\epsilon)n}{\frac{1}{4}(1+\epsilon)a_n} = \frac{4(\alpha-2\epsilon)}{(1+\epsilon)r_n^2} \quad \text{and} \quad G_n \ge \frac{(1-\alpha-2\epsilon)n}{\frac{1}{4}(1+\epsilon)a_n} = \frac{4(1-\alpha-2\epsilon)}{(1+\epsilon)r_n^2}.$$

Let ∂F denote the number of pairs of neighbor boxes of opposite colors in F (with the modified coloring). By Lemma 7,

$$\partial F \ge 3\min\left\{\sqrt{R_n},\ \sqrt{G_n}\right\} \ge \frac{6}{r_n}\min\left\{\sqrt{\frac{\alpha-2\epsilon}{1+\epsilon}},\ \sqrt{\frac{1-\alpha-2\epsilon}{1+\epsilon}}\right\}.$$

By niceness and the definition of box coloring, each red box contains at least $(1-2\epsilon)\frac{1}{4}a_n$ red points, and each green box contains at least $(1-2\epsilon)\frac{1}{4}a_n$ green points. As a consequence,

$$\begin{aligned}\theta(i,\varphi,F) &\geq \partial F(1-\epsilon)^3(1-f)\tfrac{(1-2\epsilon)^2}{16}a_n^2 \\ &\geq (1-\epsilon)^3(1-f)\tfrac{3(1-2\epsilon)^2}{8\sqrt{1+\epsilon}}a_n^{3/2}n^{1/2}\min\left\{\sqrt{\alpha-2\epsilon},\ \sqrt{1-\alpha-2\epsilon}\right\} \\ &\geq (1-\epsilon)^3(1-f)\tfrac{3(1-5\epsilon)}{8}n^2r_n^3\min\left\{\sqrt{\alpha-2\epsilon},\ \sqrt{1-\alpha-2\epsilon}\right\}.\end{aligned}$$

This lower bound is smaller than the one for Case 1 and thus holds for both cases. □.

The following result presents our lower bounds for the layout problems, considered in this paper on (ϵ, f)-mixing graphs:

Lemma 10 (Lower bounds) *For any constants $\epsilon \in (0, \frac{1}{5})$ and $f \in (0,1)$ and for all n large enough, let G_n be a ϵ-homogeneous geometric graph with n nodes and radius r_n; let F_n be a (ϵ, f)-mixing subgraph of G_n. Then, the following lower bounds hold:*

$$\textsc{mineb}(F_n) \geq (1-\epsilon)^3(1-f)\tfrac{3(1-8\epsilon)}{8\sqrt{2}}\cdot n^2r_n^3, \tag{lb1}$$

$$\textsc{mincw}(F_n) \geq (1-\epsilon)^3(1-f)\tfrac{3(1-8\epsilon)}{8\sqrt{2}}\cdot n^2r_n^3, \tag{lb2}$$

$$\textsc{minla}(F_n) \geq (1-\epsilon)^3(1-f)\tfrac{(1-42\sqrt{\epsilon})}{4\sqrt{2}}\cdot n^3r_n^3. \tag{lb3}$$

Proof. The proofs of (lb1) and (lb2) are obtained from Lemma 9 by setting $i = \lfloor n/2 \rfloor$. To prove (lb3), take any layout φ of F_n. Then, by Lemma 9,

$$\begin{aligned}\textsc{la}(\varphi, F_n) = \sum_{i=1}^{n}\theta(i,\varphi,F_n) &\geq \sum_{2\epsilon n<i<(1-2\epsilon)n}\theta(i,\varphi,F_n) \\ &\geq (1-\epsilon)^3(1-f)\tfrac{3(1-5\epsilon)}{4}n^2r_n^3\sum_{2\epsilon n<i<n/2}\sqrt{i/n-2\epsilon}.\end{aligned}$$

Using the facts that $a>b$ implies $\sqrt{a-b}\geq\sqrt{a}-\sqrt{b}$ and that $\sum_{k=1}^{m}\sqrt{k}\geq\frac{2}{3}m^{3/2}+O\left(m^{1/2}\right)$ we obtain (lb3) by successive minorizations. □.

To obtain upper bounds, we make use of the *projection algorithm* which creates a layout by ordering the nodes according to their projection onto the x-axis. Another way to see this algorithm is to sweep a vertical line starting from $x=0$ to $x=1$, numbering vertices in the order the line touches them. The expected running time of the projection algorithm to compute the projected layout of a faulty random geometric graph is linear, as it requires only the ranking of n numbers distributed uniformly.

We can restate a result of [9] on upper bounds on layout problems, on geometric graphs. These bounds were obtained analyzing the behavior of the projection algorithm. Taking into account that by deletion of an edge any of the consider layout measures will never increase, we get:

Lemma 11 *For any constants $\epsilon \in (0, \frac{1}{5})$ and $f \in (0,1)$, let G_n be a ϵ-homogeneous geometric graph with n points and radius r_n; let F_n be a (ϵ, f)-mixing subgraph of G_n. Then with high probability, the following upper bounds on the cost of the projected layout π of F_n hold:*

$$\begin{aligned}\textsc{cw}(\pi, F_n) &\le (1+3\epsilon)^5 \cdot n^2 r_n^3,\\ \textsc{eb}(\pi, F_n) &\le (1+3\epsilon)^5 \cdot n^2 r_n^3,\\ \textsc{la}(\pi, F_n) &\le (1+3\epsilon)^5 \cdot n^3 r_n^3.\end{aligned}$$

Joining all the previous results, we get:

Theorem 3 *With high probability, the Projection algorithm is a constant approximation algorithm for the Minimum Linear Arrangement, Minimum Cutwidth and Bisection problems on random geometric graphs with n nodes and radius $r_n = \sqrt{a_n/n}$, $r_n \to 0$ and $a_n = b_n \log n$ for any function $b_n \to \infty$ and $b_n = o(n/\log n)$, and random edge faults, provided the failure probability is constant.*

In the case of random geometric graphs with node faults, the resulting subgraph is also a random geometric graph. It is straightforward to see that a random geometric graph with vertex faults is ϵ-homogeneous with a slightly modification of the constant, just to incorporate the failure probability f. Therefore all the results in [9] concerning layout measures are also true for random geometric graphs with random node faults. In particular it holds:

Theorem 4 *With high probability, the Projection algorithm is a constant approximation algorithm for the Minimum Linear Arrangement, Minimum Cutwidth and Bisection problems on random geometric graphs with n nodes and radius $r_n = \sqrt{a_n/n}$, $r_n \to 0$ and $a_n = b_n \log n$ for any function $b_n \to \infty$ and $b_n = o(n/\log n)$, and random node faults, provided the failure probability is constant.*

6. Hamiltonian cycles in random faulty geometric graphs

In this section we deal with Hamiltonian cycles in faulty random geometric networks. Using friendness (see Definition 2), it is fairly easy to prove that, with high probability, random geometric networks with faulty nodes do have a Hamiltonian cycle.

In the following, we consider random geometric networks with faulty edges.

Lemma 12 *For any constants $\epsilon \in (0,1)$ and $f \in (0,1)$ and for all n, let G_n be a ϵ-spreading geometric graph with n nodes and radius r_n. Then,*

$$\lim_{n\to\infty} \mathbf{Pr}\left[\mathcal{FE}(G_n, f) \text{ has a Hamiltonian cycle}\right] = 1.$$

Proof. Let n be any natural. Let G denote G_n and let $F = F_n$ be an edge faulty random geometric graph of G drawn from $\mathcal{FE}(G, f)$. Consider the dissection of the unit square into $\kappa_n = 9\lceil 1/r_n\rceil^2$ boxes of side $w_n = 1/(3\lceil 1/r_n\rceil)$. Let $\{B_1, \ldots, B_{\kappa_n}\}$ be the set of boxes of this dissection. For a box B_i and a node $u \in V(F)$ inside box B_i, let $d(u, B_i)$ denote the number of neighbors of u in F inside box B_i.

Let us compute the probability that, for some box B_i and some vertex $u \in V(F)$ inside B_i, $d(u, B_i)$ is smaller than 2:

$$\begin{aligned}
\pi &= \mathbf{Pr}\left[\vee_{i=1}^{\kappa_n} \vee_{u \in B_i} d(u, B_i) < 2\right] \leq \sum_{i=1}^{\kappa_n} \sum_{u \in B_i} \mathbf{Pr}\left[d(u, B_i) < 2\right] \\
&= \sum_{i=1}^{\kappa_n} \sum_{u \in B_i} \left(\mathbf{Pr}\left[d(u, B_i) = 0\right] + \mathbf{Pr}\left[d(u, B_i) = 1\right]\right) \\
&\leq 9 \left\lceil 1/r_n \right\rceil^2 \cdot (1+\epsilon)\tfrac{1}{9} a_n \cdot \left(f^{(1+\epsilon)\frac{1}{9}a_n - 1} + (1-f) f^{(1+\epsilon)\frac{1}{9}a_n - 2}\right) \\
&= \left\lceil 1/r_n \right\rceil^2 (1+\epsilon) a_n f^{(1+\epsilon)\frac{1}{9}a_n - 2} \leq (1+\epsilon)^3 n f^{(1+\epsilon)\frac{1}{9}b_n \log n - 2}.
\end{aligned}$$

Taking t such that $t < -2/\log f$ and $t < (1+\epsilon)\frac{1}{10} b_n$, we get

$$\mathbf{Pr}\left[\vee_{i=1}^{\kappa_n} \vee_{u \in B_i} d(u, B_i) < 2\right] \leq (1+\epsilon)^3 n f^{t \log n} \leq (1+\epsilon)^3 n^{1 + t \log f}. \qquad (1)$$

Call N_i the number of nodes in box B_i. Observe that inside each box B_i, there is a random graph in the $\mathcal{G}_{N_i, 1-f}$ model ($\mathcal{G}_{n,p}$). By [5], the probability that such a random graph has a Hamiltonian cycle is the same than the probability that each of its nodes has at least degree two. Therefore, with probability greater than $1 - (1+\epsilon)^3 n^{1+t\log f}$, each box of the dissection contains a Hamiltonian cycle.

In order to sew a Hamiltonian cycle for F using the Hamiltonian cycles inside each box, we proceed as shown in Figure 1: The cycle will be made in a snake-like way, removing two edges from the small Hamiltonian cycles inside each box and joining the paths with the small Hamiltonian cycles of the previous and next boxes. The probability that this construction cannot be done is bounded above by

$$\left(\left(f^2\right)^{N_1 N_2}\right) \prod_{i=2}^{\kappa_n - 1} \left(f^2\right)^{(N_i - 1)N_{i+1}} \leq \left(f^{2(1+\epsilon)\frac{1}{9}a_n}\right) \left(f^{2(1+\epsilon)^2 \frac{1}{81} a_n^2}\right)^{9\lceil 1/r_n \rceil^2 - 1}. \qquad (2)$$

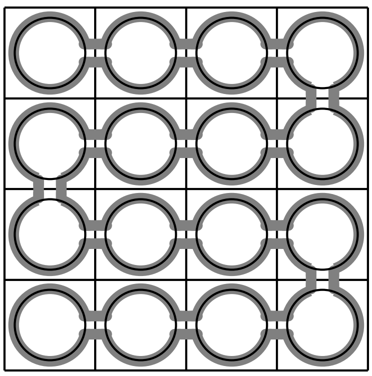

Figure 1: How to sew a big Hamiltonian cycle for the faulty graph using the small Hamiltonian cycles inside each box of the dissection.

Therefore, the probability that F does not have a Hamiltonian cycle is smaller than the sum of the probabilities (1) and (2), which tends to zero as n tends to infinity. □.

As a consequence of Lemmas 12 and 4, we have:

Theorem 5 *With high probability, random geometric networks, with n nodes and radius $r_n = \sqrt{a_n/n}$, with $r_n \to 0$ and $a_n = b_n \log n$ for any function $b_n \to \infty$ and $b_n = o(n/\log n)$, have a Hamiltonian cycle, even in the presence of random edge or node faults, provided the failure probability is constant.*

Notice that our proof, combined with the algorithm of [6], shows how to find, in polynomial time and with high probability, the Hamiltonian cycle of a random geometric network with edge faults, which can be useful in message passing through our broadcast network using end-to-end communication protocols.

7. Conclusions and open problems

In this paper we have considered random faults in an important family of random geometric networks. Our work states that, in this family, several network parameters are affected only by a constant, provided the failure probability is constant. Other parameters that have been analyzed in the non-faulty version remain open, for instance, k-connectivity or chromatic number [2,3,4,18].

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