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Hamiltonian Cycles in Faulty Random Geometric Networks

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Abstract

In this paper we analyze the Hamiltonian properties of faulty random networks. This consideration is of interest when considering wireless broadcast networks. A random geometric network is a graph whose vertices correspond to points uniformly and independently distributed in the unit square, and whose edges connect any pair of vertices if their distance is below some specified bound. A faulty random geometric network is a random geometric network whose vertices or edges fail at random. Algorithms to find Hamiltonian cycles in faulty random geometric networks are presented.

1 Introduction

The use of distributed computing in wireless networks is a computational model that is gaining increasing importance in computer science and telecommunication. In this setting, the processors, scattered geographically, communicate through transmitters, effectively forming a *wireless broadcast network*. The following setting arises in applications of wireless broadcast networks: A set of stations are located in some geographical area. These stations can compute, send and receive messages in synchronous steps. All the transmitters have the same power, but their effective broadcast range is limited to some specified distance r, that is, two stations can only communicate if their distance is at most r. Unit disk graphs provide a convenient way to model this setting: A graph is a *unit disk graph* if each vertex can be mapped to a point in the plane in such a way that two vertices are adjacent if and only if their distance is at most some specified bound r. Several researchers have shown that some important problems on broadcast networks are, in fact, classic problems restricted to unit disk graphs (see [3]). However, as pointed by Clark, Colbourn and Johnson in [4], unit disk graphs assume that no interference from weather, mountains or other obstacles affects the communication between two stations. Also, this model does not take into account the possibility that individual stations go down because of problems with power supply, mechanical damages, sabotage, etc.

The advent of mobile computing and of cellular phones introduces an uncertainty with respect to the positions of the stations. Assuming that the stations are homogeneously distributed in the plane is a simplified way to cope with that changing environment. Random

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geometric graphs come at hand to model such a situation: A *random geometric graph* (RGG) is a unit disk graph whose vertices are points uniformly distributed in the unit square.

In this paper we present and analyze algorithms to find Hamiltonian paths on random geometric networks with *random faults*. This is an important issue, because if a network has this property it is possible to build a path to efficiently perform distributed computations based on *end-to-end communication protocols*, which allow distributed algorithms to treat an unreliable network as a reliable channel [8]. Both edge failures and vertex failures are taken into account, keeping in mind that, as said, edge failures can be interpreted as the inability to communicate between two stations because of the presence of some unexpected obstacle and vertex failures can be interpreted as the inability to perform computation in inoperative stations.

Recall that a *Hamiltonian cycle* is a cycle (or circuit) that visits once each vertex of a graph. If a graph has a Hamiltonian cycle, it is said that the graph is *Hamiltonian*. Deciding whether or not a given graph has a Hamiltonian cycle is a well-known **NP**-complete problem [6]. The problem is **NP**-complete even when restricted to unit disk graphs [7]. The question whether a uniform $\mathcal{G}_{n,m}$ or binomial $\mathcal{G}_{n,p}$ random graph is or not Hamiltonian is well solved; see Chapter VIII of [1] for an extensive account.

2 Preliminaries

In the following, a network is modeled by a graph, where processors correspond to vertices and communication channels correspond to edges. Let G be a graph and let F be a subgraph of G; G represents a fault-free graph and F represents the resulting graph after the occurrence of (edge or vertex) faults in G. In this paper we differentiate between edge faults and vertex faults: graphs with faulty edges are the result of removing edges from an original graph; graphs with faulty vertices are the result of removing vertices from an original graph together with the edges incident to the removed vertices. In graph theoretical terms, a graph with faulty vertices refers to a vertex induced subgraph of an original graph and a graph with faulty edges refers to a subgraph of an original graph.

Let $G = (V, E)$ be a graph and let $f \in (0, 1)$ denote its vertex failure probability. We assume that vertex faults happen independently and with a constant probability f. Let V' be a subset of V where for all $v \in V$, $\mathbf{Pr}[v \in V'] = 1 - f$, independently for all vertices in V. Then we say that the subgraph of G induced by V' is a graph with random faulty vertices. Given G and f, we denote this probability space as $\mathcal{FN}(G, f)$.

Analogously, let $G = (V, E)$ be a graph and let $f \in (0, 1)$ denote its edge failure probability. We assume that edge faults happen independently and with a constant probability f. Let E' be a subset of E where for all $e \in E$, $\mathbf{Pr}[e \in E'] = 1 - f$, independently for all edges in E. Then we say that the subgraph of G induced by E' is a graph with random faulty edges. Given G and f, we denote this probability space as $\mathcal{FE}(G, f)$.

A graph is a *unit disk graph* if each vertex can can be mapped to a point in the plane in such a way that two vertices are adjacent if and only if their distance is at most some specified bound r. Equivalently, a unit disk graph can be defined as the intersection graph of a set of disks with radius r [3].

Observe that we have to fix which kind of norm is used to measure distances in the plane. Under the l_p norm ($p \geqslant 1$), the distance $\|x - y\|_p$ between two points $x = (x_1, y_1)$ and $y = (x_2, y_2)$ is $(|x_1 - x_2|^p + |y_1 - y_2|^p)^{1/p}$. Under the l_∞ norm, their distance is $\max\{|x_1 - x_2|, |y_1 - y_2|\}$. For sake of simplicity, we use the l_∞ norm all through this paper.

Let r be a positive number and let V be any finite set of points in the unit square

($[0,1]^2$). A *geometric graph* $\mathcal{G}(V;r)$ with vertex set V and radius r is the graph $G=(V,E)$ where $E=\{uv \ : \ u,v\in V \wedge 0<\|u-v\|\leqslant r\}$. By definition, a geometric graph is a unit disk graph. Notice that a geometric graph with faulty vertices is also a geometric graph, while a geometric graph with faulty edges is not.

Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of u.i.d. points in the unit square and let $(r_i)_{i\in\mathbb{N}}$ be a sequence of positive numbers. For all $n\in\mathbb{N}$, call $\mathcal{X}_n=\{X_1,\dots,X_n\}$. For any natural n, the graph $\mathcal{G}(\mathcal{X}_n;r_n)$ is a *random geometric graph with n vertices and radius r_n*.

All through this paper we restrict our attention to the particular case of random geometric graphs in l_∞ whose radius is of the form

$$r_n=\sqrt{\frac{a_n}{n}} \quad \text{where} \quad r_n\to 0 \quad \text{and} \quad a_n/\log n\to\infty.$$

In [9] it is shown that this choice guarantees almost surely the construction of sparse connected graphs—disconnected fault-free networks would not make much sense in our setting. In the following, we shall set $b_n=a_n/\log n$.

Finally, recall that a sequence of events $(E_n)_{n\in\mathbb{N}}$ occurs *with high probability* if $\mathbf{Pr}\left[E_n\right]\to 1$.

3 Hamiltonian cycles in RGGs with vertex faults

In this section we deal with the existence of Hamiltonian cycles in random geometric graphs with random vertex faults. The following definition and its subsequent lemma capture the property that vertices of a geometric graph are "nicely spread" on the unit square.

Definition 1 (Nice graphs). Consider any set V_n of n points in $[0,1]^2$, which together with a radius r_n, induce a geometric graph $G_n=\mathcal{G}(V_n;r_n)$. Dissect the unit square into $4\lceil 1/r_n\rceil^2$ boxes of size $s_n\times s_n$ with $s_n=1/2\lceil 1/r_n\rceil$, placed packed in $[0,1]^2$ starting at $(0,0)$. Given $\epsilon\in(0,1)$, let us say that G_n is ϵ-*nice* if every box of this dissection contains at least $(1-\epsilon)\frac{1}{4}a_n$ points and at most $(1+\epsilon)\frac{1}{4}a_n$ points.

Notice that, by construction, all the boxes in the above dissection exactly fit in the unit square and that $2s_n\leqslant r_n<3s_n$. So, two vertices in the same or in neighboring boxes are connected by an edge, and two vertices in boxes whose centers lie at a distance greater than $3s_n$ are not connected. All through the paper, when we speak about boxes we shall understand the above dissection. For any $i\in[\kappa_n]$, let $B(i)$ denote the i-th box in the dissection, according to some arbitrary but fixed order. Also, let $\alpha(i)$ denote the number of vertices of G_n in box $B(i)$. Our interest in nice graphs is that, with high probability, random geometric graphs are nice:

Lemma 1 ([5]). Let $\epsilon\in(0,\frac{1}{5})$. Then, $\lim_{n\to\infty}\mathbf{Pr}\left[\mathcal{G}(\mathcal{X}_n;r_n)\text{ is }\epsilon\text{-nice}\right]=1$.

The following definition expresses the fact that vertices of nice graphs fail "appropriately:"

Definition 2 (Friendly graphs). Let $\epsilon\in(0,\frac{1}{5})$ and $f\in[0,1)$ be two constants. Let G_n be an ϵ-nice geometric graph with n points and radius r_n and let F_n be a vertex induced subgraph of G_n. We say that F is (ϵ,f)-*friendly* if every box of this dissection contains at least $\frac{1}{4}(1-\epsilon)^2(1-f)a_n$ points of F_n and at most $\frac{1}{4}(1+\epsilon)^2(1-f)a_n$ points of F_n.

The following lemma states that, with high probability, nice geometric graphs with random faulty vertices are friendly.

Lemma 2. Let $\epsilon \in (0, \frac{1}{5})$ and $f \in [0, 1)$ be two constants. For all $n \in \mathbb{N}$, let G_n be an ϵ-nice graph with n vertices and radius r_n. Then,

$$\lim_{n\to\infty} \mathbf{Pr}\left[\mathcal{FN}(G_n, f) \text{ is } (\epsilon, f)\text{-friendly}\right] = 1.$$

Proof. For any n, let F_n be drawn from $\mathcal{FN}(G_n, f)$. Choose a box in the dissection; let y_n be the number of points of G_n in this box and let Z_n be the random variable counting the number of points of F_n in this box. As G_n is ϵ-nice, we have that $(1-\epsilon)\frac{1}{4}a_n \leqslant y_n \leqslant (1+\epsilon)\frac{1}{4}a_n$. On the other hand, as Z_n is a sum of y_n Bernoulli variables with parameter $1-f$, $(1-\epsilon)\frac{1}{4}a_n(1-f) \leqslant \mathbf{E}[Z_n] = y_n(1-f) \leqslant (1+\epsilon)\frac{1}{4}a_n(1-f)$. The result is obtained using Chernoff's bounds and Boole's inequality. $\square$

We are ready to prove that, with high probability, friendly random geometric graphs with random vertex faults are Hamiltonian.

Lemma 3. Let $\epsilon \in (0, \frac{1}{5})$ and $f \in [0, 1)$ be two constants. Let G_n be any ϵ-nice geometric graph with n vertices and radius r_n and let F_n be an (ϵ, f)-friendly vertex induced subgraph of G_n. Then, F_n is Hamiltonian and there exists a $O(|E(F_n)|)$ algorithm to find it.

Proof. First of all, notice that any square grid with diagonal edges has a Hamiltonian cycle; see Figure 1. Also, remark that in an (ϵ, f)-friendly graph, any pair of points in the same box or in neighboring boxes are connected by an edge. For each box in the dissection, construct a path visiting all the points of F_n in the box. This is always possible, since vertices in a box form a clique. Then, following the order given by a Hamiltonian cycle in the $\kappa_n \times \kappa_n$ square grid, patch the last point of each path with the first point of the next path. This is always possible since the friendness of F_n ensures the existence of at least two vertices in each box. This construction yields a Hamiltonian cycle for F_n. It is clear that this algorithm can be done in time $O(|E(F_n)|)$. $\square$

The same argument implies that non-faulty random geometric graphs are also Hamiltonian. Therefore we have the following result:

Theorem 1. With high probability, random geometric graphs with n vertices and radius $r_n = \sqrt{a_n/n}$, where $r_n \to 0$ and $a_n/\log n \to \infty$, and random vertex faults with probability $f \in [0, 1)$ are Hamiltonian.

Observe that just taking $f = 0$ in the previous theorem, we get that random geometric graphs without faults are, with high probability, Hamiltonian.

4 Hamiltonian cycles in RGGs with edge faults

In this section we deal with the existence and search of Hamiltonian cycles in random geometric graphs with random edge faults.

Lemma 4. Let $\epsilon \in (0, \frac{1}{5})$ and $f \in (0, 1)$ be two constants. For all $n \in \mathbb{N}$, let G_n be an ϵ-nice geometric graph with n vertices and radius r_n. Then,

$$\lim_{n\to\infty} \mathbf{Pr}\left[\mathcal{FE}(G_n, f) \text{ is Hamiltonian}\right] = 1.$$

Proof. Let n be any natural. Let F_n be an edge faulty random geometric graph of G_n drawn from $\mathcal{FE}(G, f)$. For a box B and a vertex $u \in V(F_n)$ inside box B, let $d(u, B)$ denote the number of neighbors of u in F_n inside box B.

Let us compute the probability π that, for some box B and some vertex $u \in V(F_n)$ inside B, $d(u, B)$ is smaller than 2:

$$
\begin{aligned}
\pi &= \mathbf{Pr}\left[\vee_{i=1}^{\kappa_n} \vee_{u \in B(i)} d(u, B(i)) < 2\right] \\
&\leqslant \sum_{i=1}^{\kappa_n} \sum_{u \in B(i)} \mathbf{Pr}\left[d(u, B(i)) < 2\right] \\
&\leqslant \sum_{i=1}^{\kappa_n} \sum_{u \in B(i)} \left(\mathbf{Pr}\left[d(u, B(i)) = 0\right] + \mathbf{Pr}\left[d(u, B(i)) = 1\right]\right) \\
&\leqslant 4\left\lceil 1/r_n \right\rceil^2 \cdot (1+\epsilon)\tfrac{1}{4}a_n \cdot \left(f^{(1-\epsilon)\frac{1}{4}a_n - 1} + (1-f)f^{(1-\epsilon)\frac{1}{4}a_n - 2}\right) \\
&= \left\lceil 1/r_n \right\rceil^2 (1+\epsilon) a_n f^{(1-\epsilon)\frac{1}{4}a_n - 2} \\
&\leqslant (1+\epsilon)^3 n f^{(1-\epsilon)\frac{1}{4}b_n \log n - 2} \\
&\leqslant (1+\epsilon)^3 n f^{(1-\epsilon)\frac{1}{5}b_n \log n}.
\end{aligned}
$$

Set $t = -3/\log f$; then,

$$
\pi \leqslant (1+\epsilon)^3 n f^{t \log n} = (1+\epsilon)^3 n^{1 + t \log f} = (1+\epsilon)^3 n^{-2}. \tag{1}
$$

Observe that inside each box $B(i)$, there is a $\mathcal{G}_{\alpha(i),1-f}$ binomial random graph. The probability that such a binomial random graph has a Hamiltonian cycle is the same that the probability that each of its vertices has at least degree 2 (see Theorem VIII.11 of [1]). Therefore, with probability greater than $1 - (1+\epsilon)^3 n^{-2}$, each box of the dissection contains a Hamiltonian cycle.

In order to get a Hamiltonian cycle for F_n, we shall patch the Hamiltonian cycles inside each box. We proceed as shown in Figure 2: The cycle will be made in a snake-like way, removing two edges from each Hamiltonian cycle inside each box and joining the paths with the small Hamiltonian cycles of the previous and next boxes. The probability that this construction cannot be done is bounded above by

$$
\left(\left(f^2\right)^{\alpha(1)\alpha(2)}\right) \prod_{i=2}^{\kappa_n - 1} \left(f^2\right)^{(\alpha(i)-1)\alpha(i+1)} \leqslant \left(f^{2(1-\epsilon)\frac{1}{4}a_n}\right)\left(f^{2(1-\epsilon)^2 \frac{1}{16}a_n^2}\right)^{4\lceil 1/r_n \rceil^2 - 1}. \tag{2}
$$

Therefore, the probability that F_n does not have a Hamiltonian cycle is smaller than the sum of the probabilities (1) and (2), which tends to zero as n tends to infinity. $\square$

The previous lemma shows, with high probability, the existence of a Hamiltonian cycle in a random geometric graph with edge faults. Still, it would be desirable to have a polynomial time algorithm to search for such a Hamiltonian cycle. In the remaining of this section we present and analyze an algorithm that, given a geometric graph with edge faults, either returns a Hamiltonian cycle or reports that none is found.

Algorithm FGeo-HAM. Let G_n be a geometric graph with n vertices and radius r_n. Let F_n be an edge induced subgraph of G_n. Given a realization of F_n and r_n, the following algorithm computes a Hamiltonian cycle of F_n or fails.

Dissect the unit square into $\kappa_n = 4\lceil 1/r \rceil^2$ boxes of side $s_n = 1/2\lceil 1/r_n \rceil$. For all $i \in [\kappa_n]$, let $B(i)$ be the i-the box of this dissection, according to the snake-like ordering.

According to the proof of Lemma 4, to obtain a Hamiltonian cycle for F, we have to

- find a Hamiltonian cycle for the vertices in each box, and
- patch the cycles in an snake-like way.

In order to perform the first step, we resort to the HAM algorithm of Bollobás, Fenner and Frieze [2]. For each box $i \in [\kappa_n]$, the HAM algorithm finds a Hamiltonian cycle $\langle u_{i,0}, u_{i,1}, \dots, u_{i,\alpha(i)-1}, u_{i,0}\rangle$ with edges in $V(F_n)$ for the vertices in $B(i)$ or fails. If HAM does not find a Hamiltonian cycle for some box $B(i)$, FGeo-HAM reports that no Hamiltonian cycle for F_n can be found.

To simplify notation, in the following, a subindex (i,j) must be understood as $(i, j \bmod \alpha(i))$.

In order to perform the second step, for each box $B(i)$, the algorithm needs to:

1. Decide in which direction will be the cycle traversed: $\langle u_{i,0}, u_{i,1}, \dots, u_{i,\alpha(i)-1}\rangle$ or $\langle u_{i,\alpha(i)-1}, \dots, u_{i,1}, u_{i,0}\rangle$.
2. Decide which edge of the cycle will be removed to patch with the cycle in box $B(i-1)$ (unless $i = 1$).
3. Decide which edge of the cycle will be removed to patch with the cycle in box $B(i+1)$ (unless $i = \kappa_n$).

Of course, these three decisions must agree with the edges in $E(F_n)$. In order to take the above decisions, we use a directed multistage flow network. This network will contain $2\kappa_n$ stages, where stages $2i-1$ and $2i$ are defined by the box $B(i)$:

- Stage 1 contains a source node s and stage $2\kappa_n$ contains a target node t.
- For all $j \in [\alpha(1)]$, the network contains two nodes $w^+_{1,j}$ and $w^-_{1,j}$ at stage 2.
- For all $j \in [\alpha(\kappa_n)]$, the network contains two nodes $v^+_{\kappa_n,j}$ and $v^-_{\kappa_n,j}$ at stage $2\kappa_n - 1$.
- For all $i \in \{2, \dots, \kappa_n - 1\}$ and all $j \in [\alpha(i)]$, the network contains two nodes $v^+_{i,j}$ and $v^-_{i,j}$ at stage $2i-1$ and two nodes $w^+_{i,j}$ and $w^-_{i,j}$ at stage $2i$.

The connections in this network are given by the following rules, defined from stage $2i-1$ to stage $2i$ by a box $B(i)$ and from stage $2i$ to stage $2i+1$ by boxes $B(i)$ and $B(i+1)$ ($x \to y$ means connect node x towards node y):

- For all $i \in \{2, \dots, \kappa_n - 1\}$ and all $j, k \in [\alpha(i)]$, add $v^+_{i,j} \to w^+_{i,k}$ provided $k \neq j+1$.
- For all $i \in \{2, \dots, \kappa_n - 1\}$ and all $j, k \in [\alpha(i)]$, add $v^-_{i,j} \to w^-_{i,k}$ provided $k \neq j-1$.
- The source s is connected towards all nodes in stage 2, and all nodes in stage $2\kappa_n - 1$ are connected towards the target t.
- For all $i \in [\kappa_n - 1]$, and for all $j \in [\alpha(i)]$ and all $k \in [\alpha(i+1)]$, do the following connections ($p \sim q$ means $pq \in E(F_n)$):
  - If $u_{i,j} \sim u_{i+1,k}$ and $u_{i,j+1} \sim u_{i+1,k-1}$, then add $w^+_{i,j} \to v^+_{i+1,k}$.
  - If $u_{i,j} \sim u_{i+1,k-1}$ and $u_{i,j+1} \sim u_{i+1,k}$, then add $w^+_{i,j} \to v^-_{i+1,k}$.
  - If $u_{i,j+1} \sim u_{i+1,k}$ and $u_{i,j} \sim u_{i+1,k-1}$, then add $w^-_{i,j} \to v^+_{i+1,k}$.
  - If $u_{i,j+1} \sim u_{i+1,k-1}$ and $u_{i,j} \sim u_{i+1,k}$, then add $w^-_{i,j} \to v^-_{i+1,k}$.

Let $\sigma_h \in \{+,-\}$ for $h \in [\kappa_n]$. By construction, the above network has the following property: If

$$\Big\langle\ s,\ \ w^{\sigma_1}_{1,j_1}, \dots,\ v^{\sigma_h}_{h,i_h},\ w^{\sigma_h}_{h,j_h},\ v^{\sigma_{h+1}}_{h+1,i_{h+1}},\ w^{\sigma_{h+1}}_{h+1,j_{h+1}},\ \dots,\ v^{\sigma_{\kappa_n}}_{\kappa_n,i_{\kappa_n}},\ t\ \Big\rangle,$$

is a valid path of nodes in the network, then F contains the Hamiltonian cycle determined by

$$
\begin{aligned}
\langle \quad & \overrightarrow{\pi}\left(s,\ w^{\sigma_1}_{1,j_1}\right), \\
& \quad \ldots,\ \overrightarrow{\pi}\left(v^{\sigma_h}_{h,i_h},\ w^{\sigma_h}_{h,j_h}\right), \overrightarrow{\pi}\left(v^{\sigma_{h+1}}_{h+1,i_{h+1}},\ w^{\sigma_{h+1}}_{h+1,j_{h+1}}\right), \ldots, \\
& \overrightarrow{\pi}\left(v^{\sigma_{\kappa_n}}_{\kappa_n,j_{\kappa_n}},\ t\right), \\
& \quad \ldots, \overleftarrow{\pi}\left(v^{\sigma_{h+1}}_{h+1,i_{h+1}},\ w^{\sigma_{h+1}}_{h+1,j_{h+1}}\right), \overleftarrow{\pi}\left(v^{\sigma_h}_{h,i_h},\ w^{\sigma_h}_{h,j_h}\right), \ldots, \\
& \overleftarrow{\pi}\left(s,\ w^{\sigma_1}_{1,j_1}\right) \\
\rangle &
\end{aligned}
$$

where $\overrightarrow{\pi}$ and $\overleftarrow{\pi}$ are the following paths (illustrated in Figure 6):

$$
\begin{aligned}
\overrightarrow{\pi}(s,\ w^{+}_{1,j_1}) &= \langle u_{1,j_1} \rangle \\
\overrightarrow{\pi}(s,\ w^{-}_{1,j_1}) &= \langle u_{1,j_1+1} \rangle \\
\overrightarrow{\pi}(v^{+}_{h,i_h},\ w^{+}_{h,j_h}) &= \langle u_{h,i_h}, u_{h,i_h+1}, \ldots, u_{h,j_h} \rangle \\
\overrightarrow{\pi}(v^{-}_{h,i_h},\ w^{-}_{h,j_h}) &= \langle u_{h,i_h-1}, u_{h,i_h-2}, \ldots, u_{h,j_h+1} \rangle \\
\overrightarrow{\pi}(v^{+}_{\kappa_n,i_{\kappa_n}},\ t) &= \langle u_{\kappa_n,i_{\kappa_n}}, u_{\kappa_n,i_{\kappa_n}+1}, \ldots, u_{\kappa_n,i_{\kappa_n}-1} \rangle \\
\overrightarrow{\pi}(v^{-}_{\kappa_n,i_{\kappa_n}},\ t) &= \langle u_{\kappa_n,i_{\kappa_n}-1}, u_{\kappa_n,i_{\kappa_n}-2}, \ldots, u_{\kappa_n,i_{\kappa_n}} \rangle \\
\overleftarrow{\pi}(v^{+}_{h,i_h},\ w^{+}_{h,j_h}) &= \langle u_{h,j_{h+1}}, u_{h,j_{h+2}}, \ldots, u_{h,i_{h-1}} \rangle \\
\overleftarrow{\pi}(v^{-}_{h,i_h},\ w^{-}_{h,j_h}) &= \langle u_{h,j_h}, u_{h,j_{h-1}}, \ldots, u_{h,i_h} \rangle \\
\overleftarrow{\pi}(s,\ w^{+}_{1,j_1}) &= \langle u_{1,j_1+1}, u_{1,j_1+2}, \ldots, u_{1,j_1} \rangle \\
\overleftarrow{\pi}(s,\ w^{-}_{1,j_1}) &= \langle u_{1,j_1}, u_{1,j_1-1}, \ldots, u_{1,j_1+1} \rangle
\end{aligned}
$$

In order to discover whether there is a path from s to t in the multistage network, we use a depth first search algorithm. If t is reachable from s, then FGeo-HAM returns the corresponding Hamiltonian cycle; otherwise, FGeo-HAM reports that no Hamiltonian cycle for F_n can be found.

The following result characterizes the behavior of the FGeo-HAM algorithm on nice random geometric with random edge faults.

Lemma 5. Let $\epsilon \in (0, \frac{1}{5})$ and $f \in (0, 1)$ be two constants. For all $n \in \mathbb{N}$, let G_n be an ϵ-nice geometric graph with n vertices and radius r_n. Let T_n be the random variable that measures the cost of applying algorithm FGeo-HAM to $\mathcal{FE}(G_n, f)$. Let H_n be the 0/1 random variable that indicates if algorithm FGeo-HAM returns a Hamiltonian cycle in $\mathcal{FE}(G_n, f)$. Then,

$$
\lim_{n\to\infty} \mathbf{Pr}\left[T_n \leqslant \gamma n a_n^{3+\epsilon}\right] = 1 \qquad \text{and} \qquad \lim_{n\to\infty} \mathbf{Pr}\left[H_n = 1\right] = 1,
$$

for some constant $\gamma > 0$.

Proof. Let F_n be drawn from $\mathcal{FE}(G_n, f)$.

We first analyze the cost of the FGeo-HAM algorithm: As G_n is nice, we have $\max_{i\in[\kappa_n]} \alpha(i) = O(a_n)$. The cost of finding (or not) a small Hamiltonian cycle in a box with HAM is $O(a_n^{4+\epsilon})$ [2]. As there are $O(n/a_n)$ boxes, each with $O(a_n)$ vertices, we need time $O(na_n^{3+\epsilon})$ to compute all the small Hamiltonian cycles. Then, we have to find a path in a multistage graph with $O(n/a_n)$ stages and $O(a_n)$ nodes per stage, which can be done in $O(a_n \cdot n/a_n) = O(n)$ time. Thus, the total cost is $O(na_n^{3+\epsilon} + n) = O(na_n^{3+\epsilon})$.

We now analyze the failure probability of FGeo-HAM. There are two reasons for failure: no small Hamiltonian cycle is found for some box, or no path exists from s to t. The probability that HAM does not find a Hamiltonian cycle on a $\mathcal{G}_{n,p}$ binomial random graph is $o(2^{-n})$ [2]. So, the probability of not finding a small Hamiltonian cycle for some of the $O(n/a_n)$ boxes is $o(2^{-a_n} \cdot n/a_n) = o(n^{1-b_n}/b_n \log n)$, which tends to zero because b_n tends to infinity. The probability that no path exists from s to t is given by Equation (2), which tends to 0. Therefore, the probability that FGeo-HAM returns a Hamiltonian cycle for a graph drawn from $\mathcal{FE}(G_n, f)$ tends to 1 as n tends to infinity. $\square$

As a consequence of Lemmas 1 and 5, we have:

Theorem 2. With high probability, the FGeo-HAM algorithm returns in polynomial time a Hamiltonian cycle on random geometric graphs with n vertices and radius $r_n = \sqrt{a_n/n}$, where $r_n \to 0$ and $a_n/\log n \to \infty$, and random edge faults, provided the failure probability is constant.

5 Open problems

In this paper we have analyzed random geometric networks with either edge faults or either vertex faults. The case where both edge and vertex faults can occur can possibly be dealt using the same techniques. On the other hand, the algorithms we have presented are sequential and centralized, but the setting we have considered seems to call for distributed solutions. We leave as future work the research of distributed algorithms to find Hamiltonian cycles in faulty random geometric networks.

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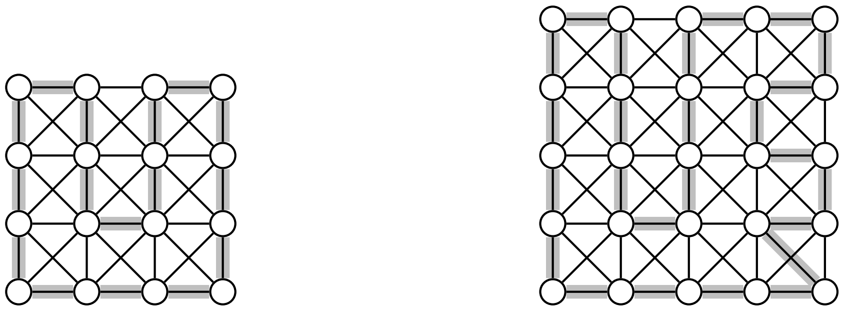


(a) Even number of vertices

(b) Odd number of vertices

Figure 1: Hamiltonian cycles in grid graphs with diagonal edges.

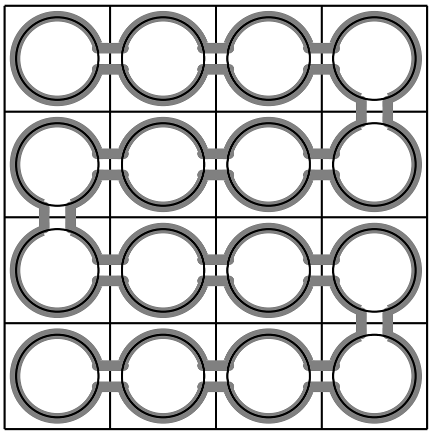


Figure 2: How to patch a big Hamiltonian cycle for the faulty graph using the small Hamiltonian cycles inside each box of the dissection.

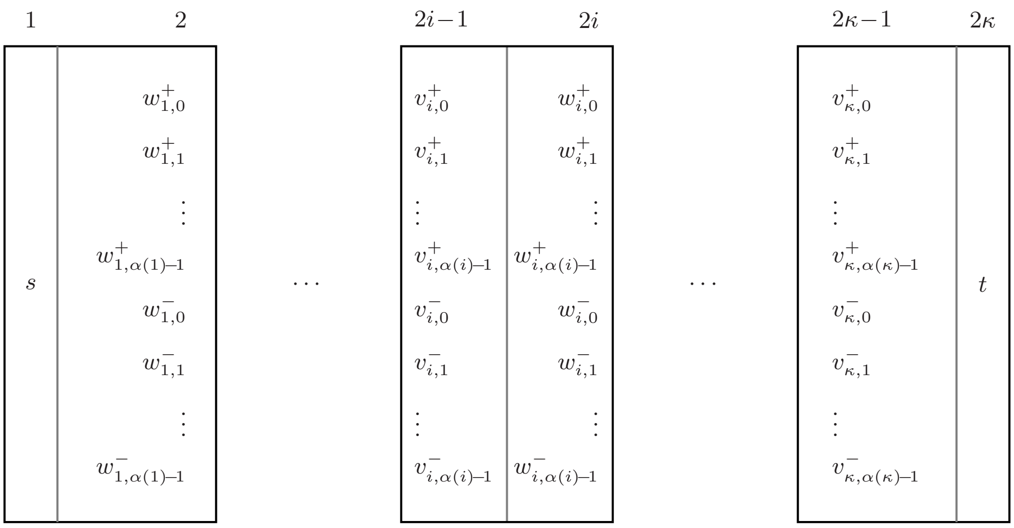


Figure 3: Boxes in the multistage network build by the FGeo-HAM algorithm.

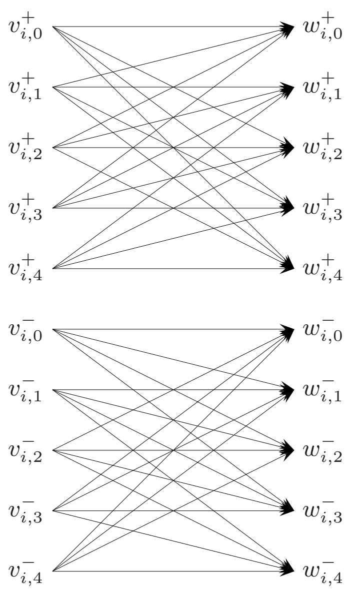
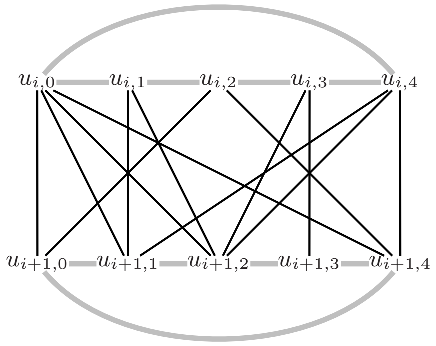
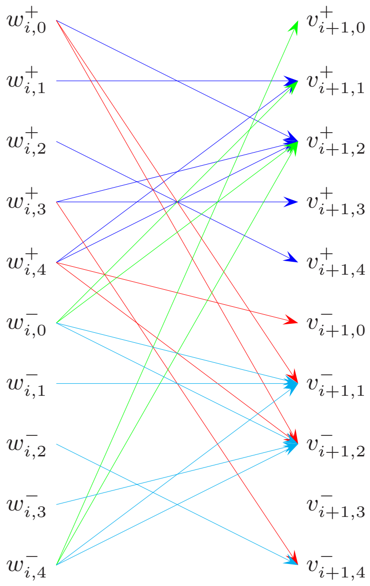


Figure 4: Connections in the multistage network build by the FGeo-HAM algorithm in a box $B(i)$. (Part 1)



(a) Thin drak lines represent edges between $B(i)$ and $B(i+1)$ and bold light lines represent the cycle in the two boxes



(b) Corresponding connections from stage $2i$ to stage $2i+1$. The four colors represent the four kinds of arcs.

Figure 5: Connections in the multistage network built by the FGeo-HAM algorithm for boxes $B(i)$ and $B(i+1)$. (Part 2)

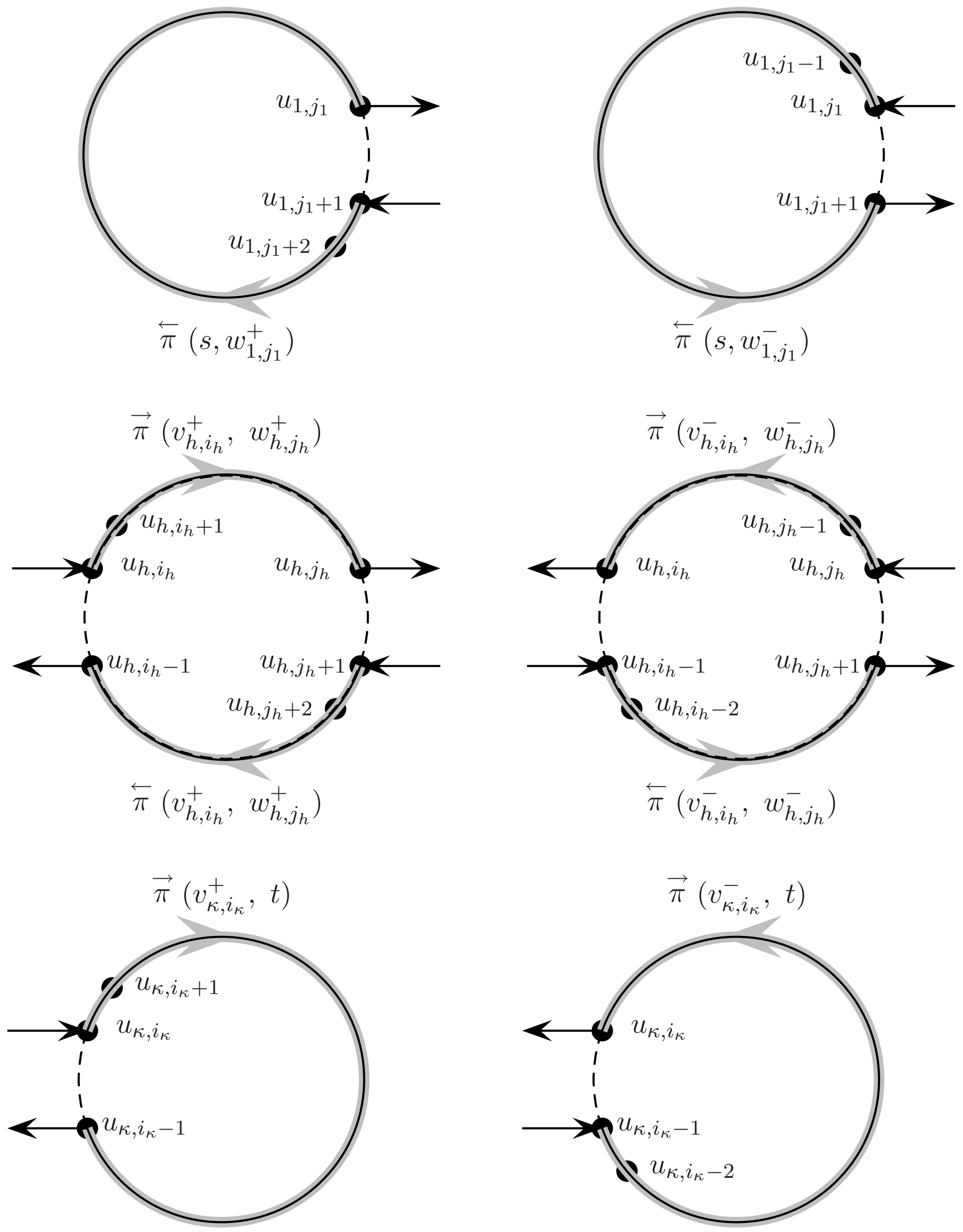


Figure 6: Illustration of the paths. Original cycles are drawn clockwise with a thin line, dashed at the edges that must be deleted. The bold gray line with an arrow shows the paths $\overrightarrow{\pi}$ and $\overleftarrow{\pi}$, together with their direction. The thin arrows show the points and direction where to patch the paths with the ones of the previous or following box. The dashed lines are the edges that are removed from the cycles.