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# Communication Tree Problems *

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**Abstract**

In this paper, we consider random communication requirements and several cost measures for a particular model of tree routing on a complete network. First we show that a random tree does not give any approximation. Then give approximation algorithms for the case for two random models of requirements.

## 1 Introduction

The general *routing tree* problem consists in: given a weighted *communication net* with a subset of distinguished nodes, the *terminals*, and given communication requirements between each pair of terminals, decide whether there is a spanning tree, minimizing some communication parameter. Problems in which a routing tree has to be constructed arise in many applications. In phone communication, it is usual to have $n$ locations together with an expected number of phone calls between each pair of locations. In this case the goal is to design a network to handle these calls in an optimal way. In distributed or mobile computing, there are shared resources as disks, input, output devices, etc., and system requirements that force to establish an optimal point-to-point communication. In tree-structured computations, the computational activity is limited to the leaves of the tree. In such a case it is important not only to distribute evenly the tasks among the leaves but to build an adequate computation tree taking into account communication parameters.

Hu [10] considered the problem of constructing a routing tree of minimum communication cost among all such trees. The cost of communication between a pair of nodes, with respect to a spanning tree is the product of their communication requirement and the weighted length of the path between the two nodes. In this case the set of terminals is the complete set of nodes. He shows that the problem can be solved in polynomial time when all the weights are equal. Notice that there is no restriction on the degree of the routing tree and that the communication net is a complete graph. Polynomial time algorithms for two particular cases of this problem were shown in [1]. In the first case, the tree must contain specified terminals as leaves, and in the second, the tree must contain a specified set of edges from the communication net.

Johnson et al. [11] show that finding a spanning tree of minimum communication cost in general weighted communication nets is NP-hard, even if all requirements are

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equal. A polynomial time approximation scheme for this problem is given in [14]. Notice again that the resulting routing tree may not have bounded degree.

In this paper we are interested in the complexity of finding routing trees minimizing some measures. We consider an unweighted complete graph as the communication net. We require that the routing tree has internal nodes of degree 3, and all the terminals must be the leaves of the routing tree. Furthermore, the communication requirements between terminals is 0 or 1.

The particular measures that we will minimize are *congestion*, *dilation* and *total communication cost* (see definitions below). We will refer to these problems as the *routing tree* problems.

Some results are known for these problems. The minimum congestion routing tree (minimum *carving-width*) was introduced in [13]. There it is shown that obtaining a routing tree of minimum congestion is NP-hard, and that the problem is solvable in polynomial time when the communication requirements form a planar graph. In [9] it is shown that there is a logarithmic gap between the minimum congestion and the minimum dilation of a given graph. The minimum is taken over all routing trees (*tree arrangements*) with internal nodes of degree 3. To the best of our knowledge no complexity results are known for the other two problems.

Our first question is whether a randomly selected routing tree can provide a good approximation to any of the problems. We answer this on the negative by showing that for some of the problems the average routing cost, for a given graph, is far away from the optimum cost.

Our next question concerns the approximability of the tree routing problems when the communication requirements are obtained randomly, that is by a given random graph. We deal with two models, the canonical class $\mathcal{G}_{n,p}$ [4] and *random geometric graphs*, denoted by $\mathcal{G}(n;r)$ [6], which are a probabilistic distribution of disk graphs where the $n$ vertices correspond to $n$ points uniformly distributed on the unit square and the radius is $r$.

Our results show that for any of the three considered measures, we can produce a routing tree that with high probability has cost within a constant of the optimum when the graph is drawn at random. For the $\mathcal{G}_{n,p}$ model we show that any *balanced* routing tree will have cost within a constant of the optimum with high probability. For the $\mathcal{G}(n;r)$ model, an adequate balanced routing tree provides, with high probability a constant approximation. In order to get this last result we will give also deterministic constant approximation algorithms for square meshes.

## 2 Preliminaries

Given an undirected graph $G$, a *routing tree* of $G$ is a tree $T$ whose leaves are the nodes of $G$ and whose internal nodes have degree 3. Given a routing tree $T$ of $G$ and an edge $uv$ in $G$, let $\lambda(uv,T,G)$ be the distance from $u$ to $v$ in $T$. We say that $\lambda(uv,T,G)$ is the *dilation* of edge $uv$ of $G$ in $T$. Given a routing tree $T$ of $G$ and an edge $xy$ in $T$, let $\theta(xy,T,G)$ be the number of edges $uv$ in $G$ such that the path from $u$ to $v$ in $T$ traverses $xy$. We say that $\theta(xy,T,G)$ is the *congestion* of edge $xy$ in $T$ for $G$. Given a routing tree $T$ of $G$ and an edge $xy$ in $T$, let $\delta(xy,T,G)$ be the number of vertices $u$ in $G$ that are connected to some other vertex $v$ in $G$ such that the the path from $u$ to $v$ in $T$ traverses $xy$. We say that $\delta(xy,T,G)$ is the *separation* of edge $xy$ in $T$ for $G$.

The problems we address are:

- Minimum Tree Dilation (MinTD): Given a graph $G$, find $\textsc{mintd}(G) = \min_T \textsc{td}(T,G)$ where $\textsc{td}(T,G) = \max_{uv\in E(G)} \lambda(uv,T,G)$.
- Minimum Tree Congestion (MinTC): Given a graph $G$, find $\textsc{mintc}(G) = \min_T \textsc{tc}(T,G)$ where $\textsc{tc}(T,G) = \max_{xy\in E(T)} \theta(xy,T,G)$.
- Minimum Tree Length (MinTL): Given a graph $G$, find $\textsc{mintl}(G) = \min_T \textsc{tl}(T,G)$ where $\textsc{tl}(T,G) = \sum_{uv\in E(G)} \lambda(uv,T,G) = \sum_{xy\in E(T)} \theta(xy,T,G)$.
- Minimum Tree Separation (MinTS): Given a graph $G$, find $\textsc{mints}(G) = \min_T \textsc{ts}(T,G)$ where $\textsc{ts}(T,G) = \max_{xy\in E(T)} \delta(xy,T,G)$.

The following basic upper bounds on the cost of a routing tree will prove to be useful. Recall that the diameter of a tree is the longest distance between any two leaves.

**Lemma 1.** *Let $G$ be any graph with $n$ nodes and $m$ edges. Let $T$ be any routing tree of $G$ of diameter $d$. Then, $\textsc{tc}(T,G) \leq m$, $\textsc{ts}(T,G) \leq n$, $\textsc{td}(T,G) \leq d$, and $\textsc{tl}(T,G) \leq dm$.*

So as we always have a routing tree with $n$ leaves and diameter $\log n + 1$ we have that $\textsc{td}(T,G) \leq \log n + 1$, and $\textsc{tl}(T,G) \leq m \log n + m$. In contrast $\textsc{tc}(T,G)$ can be $\Theta(n^2)$ and $\textsc{ts}(T,G)$ can be $\Theta(n)$, for example when $G$ is a complete graph on $n$ vertices.

We say that an edge of a routing tree $T$ is a *$s$-splitter* edge if its removal splits $T$ in two components, each one with at least $s$ leaves. Observe that for any routing tree there always is a $\lfloor n/3 \rfloor$-splitter edge.

The following result on trees is easy to prove.

**Lemma 2.** *Let $\alpha, \beta \in (0,1)$. Let $T$ be a routing tree with $n$ leaves (and $n$ big enough). For any node $u$ in $T$ and any integer $i$, let $L_{>i}(T,u)$ denote the set of leaves of $T$ at distance greater than $i$ from $u$. Then, for all node $u$ in $T$, it holds that $L_{>\alpha \log n}(T,u) \geq \beta n$.*

*Proof.* En Rafel tenia alguna esmena aquí. Consider a breadth first search process in $T$ starting at $u$. At iteration $i$, all nodes at distance $i$ from $u$ have been marked and there can be at most $3 \cdot 2^{i-1}$ such nodes (at the first iteration, at most three nodes can be reached; in subsequent iterations, the number of reached nodes can at most double). Therefore,

$$L_{>\alpha \log n}(u) \geq n - \sum_{i=0}^{\alpha \log n} 3 \cdot 2^{i-1} \geq n - 3n^{\alpha} + \tfrac{3}{2} \geq \beta n$$

by the assumption that $n$ is large enough. □

Finally, recall that a sequence of events $(\mathcal{E}_n)_{n\geq 1}$ is said to occur *with high probability* if $\lim_{n\to\infty} \mathbf{Pr}[\mathcal{E}_n] = 1$, and that in the case $\mathbf{Pr}[\mathcal{E}_n] \geq 1 - 2^{-\Omega(n)}$ for all $n$ big enough, we say that $(\mathcal{E}_n)_{n\geq 1}$ occurs *with overwhelming probability.*

# 3 Average

In this section we seek for the average costs of the MinTD and MinTL problems over all possible routing trees for a fixed graph.

Notice that the routing trees we are using to define our problems are non-rooted, commutative trees, with $n$ labeled leaves (the labeled vertices of $G$) and such that each of its $n-2$ internal nodes has degree 3. Let us denote such trees as $n$-CLN trees. Let $n$-CLR denote the set of commutative, rooted binary trees with $n$ labeled leaves. Finally define the $n$-Catalan trees, as the non-commutative and non-labeled, rooted binary trees with $n$ internal nodes.

**Lemma 3.** The number of different $n$-CLN trees that can be placed on a graph with $n$ vertices is

$$\binom{2(n-2)}{n-2}\frac{(n-1)!}{2^{n-2}}.$$

*Proof.* Let us define the following isomorphism between the $n$-CLN trees and the $(n-1)$-CLR trees: given a $n$-CLN tree, suppress the leaf with label $n$ and make its father the root of the new $(n-1)$-CLR tree. Two plane representations of $(n-1)$-CLR trees are equivalent iff one can be obtained from the other by a finite number of rotations of internal nodes. Then, the number of $(n-1)$-CLR trees are the number of $(n-2)$-Catalan trees multiplied by all possible $(n-1)!$ permutations of the labels divided by the $2^{n-2}$ equivalent $(n-1)$-CLR trees, which gives the statement of the lemma. □

**Lemma 4.** Given a characteristic function on a tree, which is invariant under commutation (rotation of internal nodes), the average value of the function is the same on the CLR trees and on the Catalan trees.

*Proof.* Let $\mathcal{B}_n$ denote the set of all $n$-Catalan trees, $\mathcal{E}_n$ the set of all $(n+1)$-CLR trees and $\mathcal{C}_n$ the set of all commutative rooted non-labeled trees with $n$ internal nodes. Also, let $f(T)$ be a characteristic function of a tree $T$ that is invariant under commutation, i.e. all trees that are equivalent under commutation have the same value $f(T)$. Do the following decompositions,

$$\mathcal{B}_n = \bigcup_{T\in\mathcal{C}_n} \mathcal{B}_T \quad \text{and} \quad \mathcal{E}_n = \bigcup_{T\in\mathcal{C}_n \mathcal{E}_T},$$

where $\mathcal{B}_T$ is the set of Catalan trees $T'$ that are equivalent to $T$ under commutation and where $\mathcal{E}_T$ is the set of CLR trees $T''$ such that we could obtain $T$ from $T''$ by successive erasing of labels.

Let us define the *commutative characteristic* of a binary tree $T$, $k(T)$, to be the number of internal nodes in $T$ for which the left son is equivalent to the right son under commutation (rotation of the node). Then for any given non-labeled, commutative tree $T$, let $e(T)$ be the number of CLR trees that could be obtained from $T$, and let $c(T)$ be the number of Catalan trees that could be obtained from $T$. Then

$$c(T) = |\mathcal{B}_T| = 2^{|T|-k(T)} \quad \text{and } e(T) = |\mathcal{E}_T| = \frac{(|T|+1)!}{2^{k(T)}}.$$

Therefore,

$$\sum_{T'\in\mathcal{B}_n} f(T') = \sum_{T\in\mathcal{C}_n}\sum_{T'\in\mathcal{B}_T} f(T') = \sum_{T\in\mathcal{C}_n} f(T)c(T)$$
$$\sum_{T''\in\mathcal{E}_n} f(T'') = \sum_{T\in\mathcal{C}_n}\sum_{T''\in\mathcal{E}_T} f(T'') = \sum_{T\in\mathcal{C}_n} f(T)e(T).$$

So, we get

$$\sum_{T\in\mathcal{C}_n} f(T)e(T) = \sum_{T\in\mathcal{C}_n} f(T)c(T)\frac{e(T)}{c(T)} = \frac{(n+1)!}{2^n}\sum_{T\in\mathcal{C}_n} f(T)c(T),$$

and therefore $\sum_{T''\in\mathcal{E}_n} f(T'') = \frac{(n+1)!}{2^n}\sum_{T'\in\mathcal{B}_n} f(T')$. As a consequence,

$$\frac{\sum_{T''\in\mathcal{E}_n} f(T'')}{\sum_{T''\in\mathcal{E}_n} 1} = \frac{\sum_{T'\in\mathcal{B}_n} f(T')}{\sum_{T'\in\mathcal{B}_n} 1},$$

and we can conclude that the average value of of $f(T'')$ for $T''\in\mathcal{E}_n$ is the same that the average value of $f(T')$ for $T'\in\mathcal{B}_n$. □

From the previous lemmas, together with the known fact that for a $n$-Catalan tree, the average distance between any two leaves is $\sqrt{n\pi}+o(\sqrt{n})$, we get the following result:

**Theorem 1.** Given a graph $G=(V,E)$ with $|V|=n$ and $|E|=m\geq 1$ the average length for $G$ is $\Theta(m\sqrt{n})$, and the average dilation for $G$ is $\Theta(\sqrt{n})$; the average being taken over all possible $n$-CLN trees.

The previous theorem says that using a random CLN as routing tree, will provide communication costs far away from the optimal ones, as by Lemma 1, selecting a routing tree of logarithmic diameter will do better than a randomly selected routing tree. Note however that a routing tree with logarithmic diameter not always provides the optimum, in particular when the graph $G$ is a line or a cycle, a worm (caterpillar with hair length 1) gives the optimum.

# 4 $\mathcal{G}_{n,p}$ graphs

In this section we show that, with overwhelming probability, all of our routing tree problems are approximable within a constant for random graphs drawn from the classical $\mathcal{G}_{n,p_n}$ model provided that $C_0/n\leq p_n\leq 1$ for some properly characterized parameter $C_0>1$. In fact, our results establish that the cost of any *balanced* routing tree of such a random graph is within a constant of the optimal cost.

Let us recall the definition of the class of random graphs [2, 4]: Let $n$ be a positive integer and $p$ a probability. The class $\mathcal{G}_{n,p}$, is a probability space over the set of undirected graphs $G=(V,E)$ on the vertex set $V=\{1,\ldots,n\}$ determined by $\mathbf{Pr}[uv\in E]=p$ with these events mutually independent.

We introduce now a class of graphs that captures the properties we need to bound our routing tree costs on uniform random graphs.

**Definition 1 (Mixing graphs).** Let $\epsilon\in(0,\frac{1}{9})$, $\gamma\in(0,1)$ and define $C_{\epsilon,\gamma}=3(1+\ln 3)(\epsilon\gamma)^{-2}$. Consider a sequence $(c_n)_{n\geq 1}$ such that $C_{\epsilon,\gamma}\leq c_n\leq n$ for all $n\geq n_0$ for some natural $n_0$. A graph $G=(V,E)$ with $|V|=n$ and $|E|=m$ is said to be $(\epsilon,\gamma,c_n)$-*mixing* if $m\leq(1+\gamma)\frac{1}{2}nc_n$ and for any two disjoint subsets $A,B\subset V$ such that $|A|\geq\epsilon n$ and $|B|\geq\epsilon n$, it holds that

$$1-\gamma\leq \left.\frac{\theta(A,B)}{|A||B|}\right/\frac{c_n}{n}\leq 1+\gamma,$$

where $\theta(A,B)$ denotes the number of edges in $E$ having one endpoint in $A$ and another in $B$.

Our interest in mixing graphs is motivated by the fact that, with overwhelming probability, uniform random graphs are mixing:

**Lemma 5 ([8]).** Let $\epsilon \in (0, \frac{1}{9})$, $\gamma \in (0,1)$ and define $C_{\epsilon,\gamma} = 3(1+\ln 3)(\epsilon\gamma)^{-2}$. Consider a sequence $(c_n)_{n\geq 1}$ such that $C_{\epsilon,\gamma} \leq c_n \leq n$ for all $n \geq n_0$ for some natural $n_0$. Then, for all $n \geq n_0$, random graphs drawn from $\mathcal{G}_{n,p_n}$ with $p_n = c_n/n$ are $(\epsilon, \gamma, c_n)$-mixing with probability at least $1 - 2^{-\Omega(n)}$.

Using a balanced tree, it is possible to approximate the considered problems on mixing graphs up to a constant:

**Lemma 6 (Lower bounds).** Let $\epsilon \in (0, \frac{1}{6})$, $\gamma \in (0,1)$. Consider a sequence $(c_n)_{n\geq 1}$ such that $C_{\epsilon,\gamma} \leq c_n \leq n$ for all $n \geq n_0$ for some natural $n_0$. Let $G$ be any $(\epsilon, \gamma, c_n)$-mixing graph with $n$ nodes where $n$ is large enough. Let $T_b$ be a balanced routing tree of $G$. Then,

$$\frac{\textsc{tc}(T_b, G)}{\textsc{mintc}(G)} \leq \frac{2(1-\gamma)\epsilon^2}{1+\gamma}, \qquad \frac{\textsc{td}(T_b, G)}{\textsc{mintd}(G)} \leq \frac{(1-\gamma)^2}{1+\gamma},$$
$$\frac{\textsc{tl}(T_b, G)}{\textsc{mintl}(G)} \leq \frac{(1-\gamma)^3 2\epsilon^2}{(1+\gamma)^2}, \qquad \frac{\textsc{ts}(T_b, G)}{\textsc{mints}(G)} \leq \frac{3}{2(1-7\epsilon^2)}.$$

*Proof.* To prove this result, we present lower and upper bounds to the considered problems.

Let us start with a lower bound for $\textsc{mintc}(G)$. Consider any routing tree $T$ of $G$. Let $uv$ be a $n\sqrt{\epsilon}$-separator edge of $T$ that separates $T$ into two binary trees $T_u$ and $T_v$ rooted at $u$ and $v$ respectively (it is clear that such an edge must exist). Let $\alpha, \beta \in (0,1)$ be two parameters to be determined latter. By Lemma 2, there exists a set of leaves $L_u$ of $T_u$ such that for all $x \in L_u$, the distance between $x$ and $u$ in $T_u$ is greater or equal than $\alpha \log(n\sqrt{\epsilon})$ and $|L_u| \geq \beta n\sqrt{\epsilon}$. Also, there exists a set of leaves $L_v$ of $T_v$ such that for all $y \in L_v$, the distance between $y$ and $v$ in $T_v$ is greater or equal than $\alpha \log(n\sqrt{\epsilon})$ and $|L_v| \geq \beta n\sqrt{\epsilon}$. Setting $\beta = \sqrt{\epsilon}$, we have $|L_u| \geq \epsilon n$ and $|L_v| \geq \epsilon n$. As $G$ is $(\epsilon, \gamma, c_n)$-mixing, we have $\theta(L_u, L_v) \geq (1-\gamma)|L_u||L_v|c_n/n \geq (1-\gamma)\epsilon^2 n c_n$. Thus, the congestion of edge $uv$ is at least $(1-\gamma)\epsilon^2 n c_n$. Therefore, $\textsc{tc}(T, G) \geq (1-\gamma)\epsilon^2 n c_n$ and thus $\textsc{mintc}(G) \geq (1-\gamma)\epsilon^2 n c_n$ because $T$ is arbitrary.

In order to get lower bounds for $\textsc{mintd}(G)$ and $\textsc{mintl}(G)$, observe that the distance from $x$ to $y$ in $T$ is at least $2\alpha \log(n\sqrt{\epsilon}) + 1$ for all $x \in L_u$ and all $y \in L_v$. Setting $\alpha = 1 - \gamma$, we have

$$\textsc{td}(T, G) \geq 2\alpha \log(n\sqrt{\epsilon}) + 1 \geq (1-\gamma)^2 2 \log n$$

and

$$\textsc{tl}(T, G) \geq (1-\gamma)\epsilon^2 n c_n (2\alpha \log(n\sqrt{\epsilon}) + 1) \geq (1-\gamma)^3 2\epsilon^2 c_n n \log n.$$

Therefore, $\textsc{mintd}(G) \geq (1-\gamma)^2 2 \log n$ and $\textsc{mintl}(G) \geq 2(1-\gamma)^3 \epsilon^2 c_n n \log n$ because $T$ is arbitrary.

To obtain a lower bound for $\textsc{mints}(G)$, let us say that a graph with $n$ nodes satisfes the *dispersion* property if, for any two disjoint subsets $A$ and $B$ of $V(G)$ with $|A| \geq \epsilon n$ and $|B| \geq \epsilon n$, it is the case that there is at least one edge between $A$ and $B$. Mixing graphs satisfy the dispersion property: From Definition 1 we get $\theta(A, B) \geq (1-\gamma)\epsilon^2 n^2$, which implies $\theta(A, B) \geq 1$ as $n$ is large enough.

Let $xy$ be a $\lfloor n/3 \rfloor$-separator edge of $T$ separating $T$ into two binary trees $T_x$ and $T_y$ rooted at $x$ and $y$ respectively. Let $L_x$ and $L_y$ denote the leaves of $T_x$ and $T_y$ respectively. As $n$ is large enough, $|L_x| \geq (1-\epsilon)\frac{1}{3}n$ and $|L_y| \geq (1-\epsilon)\frac{1}{3}n$. Let $L_x^{(1)}$ be a subset of size $\lceil \epsilon n \rceil$ of $L_x$ and let $L_y^{(1)}$ be a subset of the same size of $L_y$. Becase of dispersion, there must be at least one edge in $E(G)$ connecting a node from $L_x^{(1)}$ to a node in $L_y^{(1)}$. Let $u_x^{(1)}$ and $u_y^{(1)}$ be the endpoints of one such edge.

Let $1 < i \leq (1-\epsilon)\frac{1}{3}n - (1+\epsilon)\epsilon n$. Let $v_x^{(i)}$ be a node in $L_x \setminus L_x^{(i)} \setminus \{u_x^{(j)} \mid 1 \leq j < i\}$ (this node must exist) and let $L_x^{(i)} = L_x^{(i-1)} \setminus \{u_x^{(i-1)}\} \cup \{v_x^{(i)}\}$. Likewise, let $v_y^{(i)}$ be a node in $L_y \setminus L_y^{(i)} \setminus \{u_y^{(j)} \mid 1 \leq j < i\}$ and let $L_y^{(i)} = L_y^{(i-1)} \setminus \{u_y^{(i-1)}\} \cup \{v_y^{(i)}\}$. As, $L_x^{(i)} = L_x^{(i-1)} = L_y^{(i)} = L_y^{(i-1)} = \lceil \epsilon n \rceil$, by dispersion, there must be at least one edge in $E(G)$ connecting a node from $L_x^{(i)}$ to a node in $L_y^{(i)}$. Call $u_x^{(i)}$ and $u_y^{(i)}$ the endpoints of one such edge. By finite induction, we have that all nodes in $\{u_x^{(i)} \mid 1 \leq i \leq (1-\epsilon)\frac{1}{3}n - (1+\epsilon)\epsilon n\}$ are connected in $G$ to some node in $L_y$ and, likewise, all nodes in $\{u_y^{(i)} \mid 1 \leq i \leq (1-\epsilon)\frac{1}{3}n - (1+\epsilon)\epsilon n\}$ are connected in $G$ to some node in $L_x$. Therefore, $\text{TS}(T,G) \geq 2 \cdot ((1-\epsilon)\frac{1}{3}n - (1+\epsilon)\epsilon n) \geq (1-7\epsilon^2)\frac{2}{3}n$ and thus we have $\text{MINTS}(G) \geq (1-7\epsilon^2)\frac{2}{3}n$ because $T$ is arbitrary.

We consider now the upper bounds. Let $m$ denote the number of edges in $G$. By Lemma 1, we have $\text{TS}(T_b, G) \leq n$. Moreover, as $G$ is mixing, we also obtain $\text{TC}(T_b, G) \leq m \leq (1+\gamma)\frac{1}{2}nc_n$. As $T_b$ is a balanced tree of $G$, its height is at most $\lceil \log n \rceil \leq (1+\gamma)\log n$. Therefore, we have $\text{TD}(T_b, G) \leq 2(1+\gamma)\log n$ and $\text{TL}(T_b, G) \leq m(1+\gamma)\log n \leq (1+\gamma)^2 nc_n \log n$. $\square$

As a consequence of the two previous lemmas, we get:

**Theorem 2.** Let $\epsilon \in (0, \frac{1}{9})$, $\gamma \in (0,1)$ and define $C_{\epsilon,\gamma} = 3(1+\ln 3)(\epsilon\gamma)^{-2}$. Consider a sequence $(c_n)_{n\geq 1}$ such that $C_{\epsilon,\gamma} \leq c_n \leq n$ for all $n \geq n_0$ for some natural $n_0$ and let $p_n = c_n/n$. Then, with overwhelming probability, the problems MinTC, MinTD, MinTL and MinTS can be approximated within a constant on random graphs $\mathcal{G}_{n,p_n}$ using a balanced routing tree. Moreover, in the case of the MinTD, the approximation factor can be as small as desired.

# 5 Square meshes

In this section we study our routing tree problems on square meshes. This is intended as an intermediate step to treat random geometric graphs on the next section. In the following we will denote an $n \times n$ mesh by $L_n$: $V(L_n) = \{1, \ldots, n\}^2$ and $E(L_n) = \{uv \,:\, u \in V(L_n) \wedge v \in V(L_n) \wedge \|u - v\|_2 = 1\}$.

The following result presents a lower bound of the cost of a mesh.

**Lemma 7 (Lower bounds).** Let $n$ be a large enough natural. Then,

$$\text{MINTC}(L_n) \geq \tfrac{1}{2}n, \qquad \text{MINTD}(L_n) \geq \log n,$$
$$\text{MINTL}(L_n) \geq 6n^2 - 8n + 1, \qquad \text{MINTS}(L_n) \geq \tfrac{\sqrt{6}}{3}n.$$

*Proof.* Let $(A, B)$ be a partition of $V(L_n)$. We claim that $\theta(A,B) \geq \min\left\{\sqrt{|A|}, \sqrt{|B|}\right\}$: If $A$ includes an entire row of nodes, and $B$ includes an entire row of nodes, then each column includes an edge with one endpoint in $A$ and the other in $B$, which contributes 1 to $\theta(A, B)$, so that $\theta(A, B) \geq n$. If $B$ contains no entire row or column, and at least as many rows as columns have non-empty intersection with $B$, then there are at least $\sqrt{B}$ such rows, and each contains a cutting edge which contributes 1 to $\theta(A, B)$, so that $\theta(A, B) \geq \sqrt{B}$. Applying similar arguments to the other possible cases, we have

$$\theta(A, B) \geq \min\left\{\sqrt{|A|},\ \sqrt{|B|},\ n\right\}$$

but this minimum is always achieved at $\sqrt{|A|}$ or at $\sqrt{|B|}$.

Let $T$ be any routing tree of $L_n$. Let $uv$ be a $\lfloor n^2/3 \rfloor$-splitter edge of $T$. As $uv$ determines a partition $(A, B)$ of $L_n$ with $|A|, |B| \geq \lfloor n^2/3 \rfloor$, the congestion of edge $uv$ is at least equal to $\min\left\{\sqrt{|A|},\ \sqrt{|B|}\right\} \geq \sqrt{\lfloor n^2/3 \rfloor}$. Therefore, $\textsc{tc}(T, L_n) \geq \sqrt{\lfloor n^2/3 \rfloor} \geq \frac{1}{2}n$. Now the MINTC result follows because $T$ is arbitrary.

A numbering $\varphi$ of a $n \times n$ mesh is a one-to-one function that maps the nodes of the mesh to $\{1, \ldots, n^2\}$. For any $i \in \{1, \ldots, n^2\}$, let $\partial(i, \varphi, L_n)$ denote the number of vertices $u \in V(L_n)$ with $\varphi(u) \leq i$ that are connected in $E(L_n)$ to some other node $v \in V(L_n)$ with $\varphi(v) > i$. Let $\varphi_D$ denote the "diagonal numbering" of the mesh (see Figure 1). As a special case of [5, Corollary 9], we have that for any numbering $\varphi$ on $L_n$ and any $k \in \{1, \ldots, n^2\}$, it is the case that $\partial(k, \varphi, L_n) \geq \partial(k, \varphi_D, L_n)$. This means that $\partial(\frac{1}{3}n^2, \varphi, L_n) \geq q_n$ where $q_n$ is the smallest positive integer such that $\sum_{i=1}^{q_n} i \geq \frac{1}{3}n^2$. A simple computation shows that $q_n = \frac{1}{6}\sqrt{9 + 24n^2} - \frac{1}{2}$.

Again, let $T$ be any routing tree of $L_n$ and let $uv$ be a $\lfloor n^2/3 \rfloor$-splitter edge of $T$. As there are at least $q_n$ leaves from one subtree connected in $L_n$ to at least one other leave in the other subtree, we have that we have that $\textsc{td}(T, L_n) \geq \log q_n + 2$ and $\textsc{ts}(T, G) \geq \partial(\frac{1}{3}n^2, \varphi, L_n)$. As $T$ is arbitrary, assuming $n$ large enough, we get $\textsc{mintd}(L_n) \geq \log q_n + 2 \geq \log n$ and $\textsc{mints}(L_n) \geq \frac{\sqrt{6}}{3}n$.

We finally prove the MINTL result. Let $G$ by any graph with $t$ nodes. Observe that in any routing tree of $G$ no edge can have length 0 nor 1. Also, observe that, at most, only $t/2$ edges can have length 2 (it is the case of the balanced tree) and that, at most, only $t - 1$ edges can have length 3 (this is the case of the worm). Finally, observe that all not yet counted edges must have, at least, length 4. In the case of $L_n$ with $t = n^2$ nodes and $m = 2n^2 - 2n$ edges, we get

$$\textsc{mintl}(L_n) \geq 2(n^2/2) + 3(n^2 - 1) + 4(m - (n^2/2) - (n^2 - 1)) = 6n^2 - 8n + 1,$$

which proves the lemma. □

In order to get upper bounds, we shall analyze a recursive algorithm to produce a routing tree of a $n \times n$ mesh in the case $n$ is a power of two.

**Definition 2 (The recursive algorithm).** Let $L_n$ be a $n \times n$ mesh with $n = 2^k$ for some integer $k \geq 1$. The *recursive algorithm* generates a routing tree of $L_n$ according to the following two rules:

- If $k = 1$: form a routing tree by joining the four nodes of the mesh as shown in Figure 2(a).
- If $k > 1$: divide the mesh in four $L_{n/2}$ sub-meshes (top/left, bottom/left, top/right and bottom/left); recursively create a routing tree for each one of the sub-meshes; join the four routing trees in one routing tree as shown in Figure 2(b).

Figure 3 illustrates the routing tree and problem costs produced by the recursive algorithm on $L_2$, $L_4$ and $L_8$ meshes. Observe that the recursive algorithm generates balanced routing trees and produces a $(2^{k-1})$-splitting edge, which we call the *top* edge. The following lemma states the costs computed by the recursive algorithm.

**Lemma 8.** Let $L_{2^k}$ be a $2^k \times 2^k$ mesh with $k \geq 1$. Let $T_{2^k}$ be the routing tree of $L_{2^k}$ computed by the recursive algorithm. Then,

$$\begin{aligned} \textsc{tc}(T_{2^k}, L_{2^k}) &= 2^k, & \textsc{td}(T_{2^k}, L_{2^k}) &= 4k - 1, \\ \textsc{ts}(T_{2^k}, L_{2^k}) &= 2^{k+1}, & \textsc{tl}(T_{2^k}, L_{2^k}) &= 14 \cdot 4^k - 8 \cdot 2^k k - 15 \cdot 2^k. \end{aligned}$$

*Proof.* The proof for TC and TS is straightforward because the maximal congestion and the maximal separation are reached at the top edge.

In order to compute $\textsc{td}(T_{2^k}, L_{2^k})$, observe that $T_{2^k}$ is made of four routing trees $T_{2^{k-1}}$ for which the distance from any leaf to their respective top edge is $2^k - 3$. But one node is inserted in these top edges to construct $T_{2^k}$ and three edges are added to connect a left sub-mesh with a right sub-mesh. So, $\textsc{td}(T_{2^k}, L_{2^k}) = 2((2^k - 3) + 1) + 3 = 4k - 1$.

In the following, let $f(k) = \textsc{tl}(T, L_{2^k})$. Counting on one's fingers, it is immediate to establish that $f(1) = 10$. In order to compute $f(k)$ for $k \geq 2$, observe that $T_{2^k}$ is made of four routing trees $T_{2^{k-1}}$ which contain $2^{k-1} \cdot 2^{k-1}$ nodes each one and whose height from the top edge is $2k - 3$. We can then establish the following recurrence:

$$\begin{cases} f(1) = 10, \\ f(k) = 4f(k-1) + 4 \cdot 2^{k-1} + 2 \cdot 2^{k-1}(2(2k-3)+4) + 2 \cdot 2^{k-1}(2(2k-3)+5). \end{cases}$$

The first term comes from the cost of the four recursive routing trees; the second term comes from the lengthening of the four recursive routing trees due to the addition of a new node on its top edge; the third term comes from the cost of the length of the horizontal edges between the two top trees and the two bottom trees; the fourth term comes from the cost of the length of the vertical edges between the two left trees and the two right trees.

The resolution of the recurrence yields the claimed result. □

We now generalize the algorithm to handle $n \times n$ meshes when $n$ is not a power of two.

**Definition 3 (The generalized recursive algorithm).** Let $L_n$ be a $n \times n$ mesh. Let $k$ be the integer such that $n \leq 2^k < 2n$ and let $T_{2^k}$ be the tree computed by the recursive algorithm on $L_{2^k}$. The *generalized recursive algorithm* generates a routing tree $T_n$ of $L_n$ applying iteratively the following transformation for all node $u \in V(L_{2^k}) \setminus V(L_n)$: let $p_1$ be the parent of $u$, let $v$ be the sibling of $u$ and let $p_2$ be the parent of $p_1$; remove the nodes $u$ and $p_1$ from $T$ together with its three incident edges; add the edge $p_2 v$ to $T$.

The following theorem states that the generalized recursive algorithm is a constant approximation algorithm for our routing tree problems on meshes:

**Theorem 3.** For all $n$ big enough, let $L_n$ be a $n \times n$ mesh and let $T_n$ be its routing tree computed by the generalized recursive algorithm. Then,

$$\frac{\text{TC}(T_n, L_n)}{\text{MINTC}(L_n)} < 4, \qquad \frac{\text{TD}(T_n, L_n)}{\text{MINTD}(L_n)} < 4,$$
$$\frac{\text{TS}(T_n, L_n)}{\text{MINTS}(L_n)} < 4, \qquad \frac{\text{TL}(T_n, L_n)}{\text{MINTL}(L_n)} < 10.$$

*Proof.* Let $k$ be the integer such that $n \leq 2^k < 2n$ and let $T_{2^k}$ be the tree computed by the recursive algorithm on $L_{2^k}$. Observe that the iterative deletion of a leaf by the generalized recursive algorithm cannot increase the congestion or separation at an edge of the routing tree. Also, the iterative deletion of a leave by the generalized recursive algorithm cannot increase the length of a graph edge in the routing tree. Therefore, using Lemma 8, we get

$$\begin{aligned}
\text{TC}(T_n, L_n) &\leq \text{TC}(T_{2^k}, L_{2^k}) \leq 2^k < 2n, \\
\text{TS}(T_n, L_n) &\leq \text{TS}(T_{2^k}, L_{2^k}) \leq 2^k < 2n, \\
\text{TL}(T_n, L_n) &\leq \text{TL}(T_{2^k}, L_{2^k}) \leq 14 \cdot 4^k - 8 \cdot 2^k \cdot k - 15 \cdot 2^k \leq 14 \cdot 4^k \leq 56n^2, \\
\text{TD}(T_n, L_n) &\leq \text{TD}(T_{2^k}, L_{2^k}) \leq 4k - 1 < 4 \log n - 3.
\end{aligned}$$

Using the lower bounds of Lemma 7, the statement of the theorem follows. □

# 6 Random geometric graphs

Let $r$ be a positive number and let $V$ be any set of $n$ points in the unit square ($[0, 1]^2$). A *geometric graph* $G(V, r)$ with vertex set $V$ and radius $r$ is the graph $G = (V, E)$ where $E = \{uv \mid u, v \in V \wedge 0 < \|u - v\| \leq r\}$. In the following, $\|\cdot\|$ denotes the $l_\infty$ norm, but similar results can be obtained with any other $l_p$ norm, $p > 0$.

Let $(r_i)_{i\geq 1}$ be a sequence of positive numbers and let $X = (X_i)_{i\geq 1}$ be a sequence of independently and uniformly distributed random points in $[0, 1]^2$. For any natural $n$, we write $\mathcal{X}_n = \{X_1, \ldots, X_n\}$ and call $G(\mathcal{X}_n, r_n)$ a *random geometric graph* of $n$ nodes on $X$. We denote by $\mathcal{G}(n; r_n)$ the class of random geometric graphs with $n$ nodes and radius $r_n$.

In the remainder of this section we restrict our attention to the particular case where the radius is of the form

$$r_n = \sqrt{\frac{a_n}{n}} \quad \text{where} \quad r_n \to 0 \quad \text{and} \quad a_n / \log n \to \infty.$$

It is important to remark that through this choice, the construction of sparse but connected graphs is guaranteed: By defining the connectivity distance $\rho_n$ as the smallest radius $r_n$ such that a random geometric graph is connected, it is known that $\rho_n \sqrt{n/\log n}$ converges to $\frac{1}{2}$ almost surely [3].

An easy adaptation of the proofs of [7, Lemma 5.2] and [7, Lemma 5.4] suffice to prove the following lower bounds:

**Lemma 9 (Lower bounds).** Let $G_n$ denote a random geometric graph with $n$ nodes drawn from the $\mathcal{G}(n; r_n)$ model. Then, with high probability,

$$\text{MINTC}(G_n) = \Omega(n^2 r_n^3), \qquad \text{MINTD}(G_n) = \Omega(\log n),$$
$$\text{MINTS}(G_n) = \Omega(n r_n), \qquad \text{MINTL}(G_n) = \Omega(n^2 r_n^2 \log n).$$

We introduce now a class of geometric graphs that captures the properties we need to bound our routing tree costs on random geometric graphs.

**Definition 4 (Well behaved graphs).** Consider any set $\mathcal{X}_n$ of $n$ points in $[0,1]^2$, which together with a radius $r_n$, induce a geometric graph $G = G(\mathcal{X}_n, r_n)$. Dissect the unit square into $4\lfloor 1/r_n \rfloor^2$ boxes of size $1/(2\lfloor 1/r_n \rfloor) \times 1/(2\lfloor 1/r_n \rfloor)$ placed packed in $[0,1]^2$ starting at $(0,0)$. By construction, all the boxes exactly fit in the unit square, and any two points of $\mathcal{X}_n$ connected by an edge in $G$ will be in the same or neighboring boxes (including diagonals) because $1/(2\lfloor 1/r_n \rfloor) \geq r_n/2$. Given $\epsilon \in (0,1)$, let us say that $G$ is $\epsilon$-*well behaved* if every box of this dissection contains at least $(1-\epsilon)\frac{1}{4}a_n$ points and at most $(1+\epsilon)\frac{1}{4}a_n$ points.

Our interest in well behaved graphs is motivated by the fact that, with high probability, random geometric graphs are well behaved:

**Lemma 10.** Let $\epsilon \in (0, \frac{1}{5})$. Then, with high probability, random geometric graphs drawn from $\mathcal{G}(n; r_n)$ are $\epsilon$-well behaved.

*Proof.* Choose a box in the dissection and let $Y$ be the random variable counting the number of points of $\mathcal{X}_n$ in this box. As the points in $\mathcal{X}_n$ are i.u.d.,

$$\mathbf{E}[Y] = n/\left(4\lfloor 1/r_n \rfloor^2\right) \sim \tfrac{1}{4} n r_n^2 = \tfrac{1}{4} a_n.$$

Let $b_n = a_n/\log n$; by hypothesis, we have $b_n \to \infty$. Using Chernoff's bounds [12], we obtain

$$\begin{aligned}\mathbf{Pr}\left[Y \geq (1+\epsilon)\tfrac{1}{4}a_n\right] &\leq \mathbf{Pr}\left[Y \geq (1+\tfrac{1}{2}\epsilon)\mathbf{E}[Y]\right] \leq \exp\left(-\left(\tfrac{1}{2}\epsilon\right)^2 \mathbf{E}[Y]/3\right) \\ &\leq \exp\left(-\tfrac{1}{13}\epsilon^2 \tfrac{1}{4} a_n\right) = n^{-\epsilon^2 b_n/52}\end{aligned}$$

and

$$\begin{aligned}\mathbf{Pr}\left[Y \leq (1-\epsilon)\tfrac{1}{4}a_n\right] &\leq \mathbf{Pr}\left[Y \leq (1-\tfrac{1}{2}\epsilon)\mathbf{E}[Y]\right] \leq \exp\left(-\left(\tfrac{1}{2}\epsilon\right)^2 \mathbf{E}[Y]/2\right) \\ &\leq \exp\left(-\tfrac{1}{9}\epsilon^2 \tfrac{1}{4} a_n\right) = n^{-\epsilon^2 b_n/36} \leq n^{-\epsilon^2 b_n/52}.\end{aligned}$$

The number of boxes is certainly smaller than $n$, so by Boole's inequality, the probability that for some box the number of points in the box is less than $(1-\epsilon)\frac{1}{4}a_n$ or bigger than $(1+\epsilon)\frac{1}{4}a_n$, is bounded by $2n^{1-b_n\epsilon^2/52}$, which goes to 0 as $n \to \infty$. □

We present now a modification to the recursive algorithm to handle geometric graphs.

**Definition 5 (The boxed recursive algorithm).** Let $G$ be a geometric graph with $n$ nodes and radius $r_n$. Dissect the unit square into $4\lfloor 1/r_n \rfloor^2$ boxes of size $1/(2\lfloor 1/r_n \rfloor) \times 1/(2\lfloor 1/r_n \rfloor)$ placed packed in $[0,1]^2$ starting at $(0,0)$. The *boxed recursive algorithm* generates a routing tree $T$ of $G$ in the following way:

- All points in the same box are the leaves of a balanced routing tree.
- The generalized recursive routing tree is used to form a routing tree for all the graph, taking as its leaves a node that is inserted at the top edge of each of the balanced trees for each box.

The following lemma presents upper bounds on the cost of routing tree problems on well behaved graphs that match the lower bounds. Its proof uses the boxed recursive algorithm.

**Lemma 11 (Upper bounds).** Let $\epsilon \in (0,1)$ and $n$ large enough. Let $G_n$ denote any $\epsilon$-well behaved geometric graph with $n$ nodes and radius $r_n$ and let $T_n$ be the routing tree computed by the boxed recursive algorithm for $G_n$. Then,

$$\begin{aligned} \textsc{tc}(T_n, G_n) &= O(n^2 r_n^3), & \textsc{td}(T_n, G_n) &= O(\log n), \\ \textsc{ts}(T_n, G_n) &= O(n r_n), & \textsc{tl}(T_n, G_n) &= O(n^2 r_n^2 \log n). \end{aligned}$$

*Proof.* As in the case of the mesh, it is easy to see that that the maximal congestion and separation is located at the top of $T_n$. In this place we have an edge which hosts the edges of two rows of $\sqrt{n/a_n}$ boxes, each with at most $(1+\epsilon)a_n$ points and connected to at most 3 neighbors. So, we have

$$\textsc{tc}(T_n, G_n) \le 3 \cdot (1+\epsilon)a_n^2 \cdot \sqrt{n/a_n} = O\left(a_n \sqrt{a_n n}\right) = O(n^2 r_n^3)$$

and

$$\textsc{ts}(T_n, G_n) \le 2 \cdot \sqrt{n/a_n} = O(n r_n).$$

The diameter of the routing tree $T$ obtained by the boxed recursive algorithm is upper bounded by $\lceil \log((1+\epsilon)a_n \rceil + 1 + \lceil \log(4 \lfloor 1/r_n^2 \rfloor) \rceil = O(\log n)$. So, applying Lemma 1, we get that $\textsc{td}(T_n, G_n) = O(\log n)$.

According to the boxed recursive algorithm, we can analyze the cost of the edges that appear at each level of the mesh-like construction. At level 0, we consider all the edges that form a clique in each of the boxes. The total number of levels is $l = \log \sqrt{4 \lfloor 1/r \rfloor^2}$. Let us define $h_i$ as the height of the subtree at level $i$. We have $h_0 = \log((1+\epsilon)\frac{1}{4}a_n)$ and $h_{i+1} = h_i + 2$. Let $t_i$ be the contribution of the edges taken into account in level $i$. We have $t_0 = ((1+\epsilon)\frac{1}{4}a_n)^2\, h_0\, 4 \lfloor 1/r \rfloor^2$ and $t_{i+1} = 48\ 2^i\, ((1+\epsilon)\frac{1}{4}a_n)^2\, h_{i+1}\, 4 \lfloor 1/r \rfloor^2\, 4^{1-i}$. Adding $\sum_{i=1}^{l} t_i$, we get the claimed result. $\square$

The combination of lemmas 9, 10 and 11 leads to state our main result on routing trees for random geometric graphs:

**Theorem 4.** With high probability, the problems MinTC, MinTD, MinTL and MinTS can be approximated within a constant on random geometric graphs $\mathcal{G}(n; r_n)$ using the routing tree computed by the boxed recursive algorithm when $r_n = \sqrt{a_n/n}$, $r_n = o(1)$ and $a_n = \omega(\log n)$.

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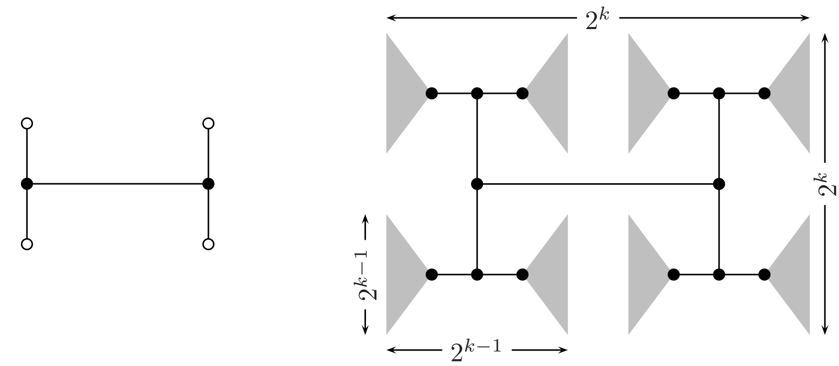
| | | | | | | |
|---|---|---|---|---|---|---|
| | | | | | $n^2-1$ | $n^2$ |
| | | | | | | $n^2-2$ |
| $\vdots$ | | | | | | |
| 10 | | | | | | |
| 6 | 9 | $\ddots$ | | | | |
| 3 | 5 | 8 | | | | |
| 1 | 2 | 4 | 7 | $\cdots$ | | |

| | | | | | | |
|---|---|---|---|---|---|---|
| $n$ | $n$ | $n-1$ | $\cdots$ | 4 | 3 | 0 |
| $n-1$ | $n$ | $n$ | $n-1$ | $\ddots$ | 4 | 2 |
| $\vdots$ | $n-1$ | $n$ | $n$ | $n-1$ | | 3 |
| 4 | | $n-1$ | $n$ | $n$ | $n-1$ | $\vdots$ |
| 3 | 4 | $\ddots$ | $n-1$ | $n$ | $n$ | $n-2$ |
| 2 | 3 | 4 | | $n-1$ | $n$ | $n-1$ |
| 1 | 2 | 3 | 4 | $\cdots$ | $n-1$ | $n$ |

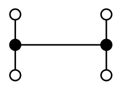
Figure 1: At left, diagonal ordering of the $n \times n$ mesh: at each node $u$, $\varphi(u)$ is shown. At right, vertex separation induced by the diagonal ordering: at each node $u$, $\partial(\varphi_D(u), \varphi_D, L_n)$ is shown.



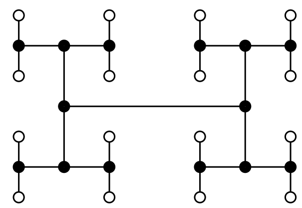

(a) Base case: $k = 1$

(b) Recursive case

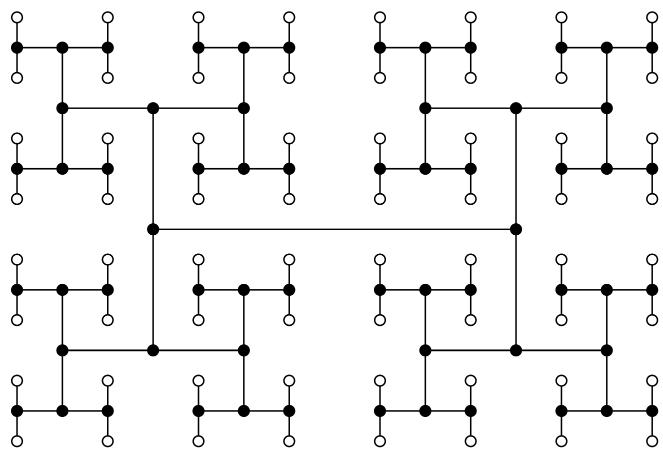
Figure 2: Recursive algorithm to build a routing tree for a $L_{2^k}$ mesh.



(a) $L_2$: TC = 2, TD = 3, TL = 10, TS = 4.



(b) $L_4$: TC = 4, TD = 7, TL = 100, TS = 8.



(c) $L_8$: TC = 8, TD = 11, TL = 584, TS = 16.

Figure 3: Illustration of routing trees computed by the recursive algorithm.