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Upper Bounds on the Bisection Width of 3- and 4-regular Graphs*

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Abstract. We derive new upper bounds on the bisection width of graphs which have a regular vertex degree. We show that the bisection width of large 3-regular graphs with $|V|$ vertices is at most $\frac{1}{6}|V|$. For the bisection width of large 4-regular graphs we show an upper bound of $\frac{2}{5}|V|$.

Keywords: *graph partitioning; bisection width; regular graphs; local improvement;*

1 Introduction

There are graph-partitioning problems in a wide range of applications. The task is to divide the set of vertices of a graph equally into a given number of parts while keeping the number of crossing edges between vertices belonging to different parts, called the *cut size* of the partition, as small as possible. The special case of a partition of the graph into 2 parts is called a *bisection*, and the minimal cut size of all balanced bisections of a graph is called its *bisection width*. Its calculation is **NP**-complete for arbitrary graphs [11] and remains **NP**-complete for regular graphs [4].

There are several results on bounds on the bisection width of regular graphs (discussed below). Results for 3- and 4-regular graphs are of special interest because these are the lowest non-trivial degrees. Some previous results for small degrees have been generalized to results for larger degrees.

It is a general theoretical interest to improve previous upper bounds on 3- and 4-regular graphs. Moreover, there are some direct applications of these results. As a motivating example, upper bounds on the bisection width of 4-regular graphs have successfully been applied to the configuration of transputer systems [14].

Definitions and Previous Results Let $G = (V, E)$ be a simple undirected graph with vertex set V of cardinality $n := |V|$ and edge set E. A graph is d-regular if for all $v \in V$ it is $|\{w \in V; \{v,w\} \in E\}| = d$. Let $\pi : V \to \{0,1\}$ be a **bisection** of G. It distributes the vertices among parts V_0 and V_1. We focus on **balanced** bisections, i. e. the number of vertices in the parts differ by at most 1. Let $cut(\pi) := |\{\{v,w\} \in E; \pi(v) \neq \pi(w)\}|$ be the **cut size** of π. The **bisection width** of a graph G is $bw(G) := \min\{cut(\pi); \pi \text{ is a balanced bisection of G}\}$.

The bisection width is known for some graph classes with regular degree such as tori, cube-connected-cycles [19] or butterflies [3].

There are several results on bounds on the bisection width of arbitrary regular graphs. Clark and Entringer [6] present an upper bound of $\frac{n+138}{3}$ for the bisection width of 3-regular graphs. Kostochka and Melnikov improve this asymptotically and show an upper bound of $\frac{n}{4}+O(\sqrt{n}\log n)$ [16]. Recently, an upper bound of $0.198n + O(\log(n))$ has been proved in [25]. Hromkovic and Monien [14] proved an upper bound of $\frac{n}{2} + 1$ for the bisection width of 4-regular graphs with $n \geq 350$. A general upper bound of $\frac{n}{2} + 5$ for 4-regular graphs with any number of vertices is proven in [25]. The result of [16] for 3-regular graphs above is a corollary of an upper bound of $\frac{d-2}{4}n + O(d\sqrt{n}\log n)$ for the bisection width of d-regular graphs in the same paper. An upper bound of $\frac{d-2}{4}n + 1$ for $n \geq n_0(d)$ with some function $n_0(d)$ is shown in [21, 22] by generalizing

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the techniques of [14]. Alon [1] uses probabilistic arguments to show that the bisection width is at most $(\frac{d}{2} - \frac{3\sqrt{d}}{16\sqrt{2}})\frac{n}{2}$ for d-regular graphs with $n > 40d^9$.

Bollobas [2] shows that for $d \to \infty$ the bisection width of almost every d-regular graph is at least $(\frac{d}{2} - \sqrt{\ln(2) \cdot d})\frac{n}{2}$. For $d = 4$ he shows that almost all 4-regular graphs have a bisection width of at least $\frac{11}{50}n = 0.22n$. Furthermore, Kostochka and Melnikov show that almost every 3-regular graph has a bisection width of at least $\frac{1}{9.9}n \approx 0.101n$ [17].

There are some (slightly weaker) results for explicitly constructible infinite graph classes with high bisection width. The *Ramanujan Graphs* (see e. g. [5, 18, 20, 23]) have a regular degree d and a bisection width of at least $(\frac{d}{2} - \sqrt{d-1})\frac{n}{2}$. This value is derived by the use of the well-known spectral lower bound $\frac{\lambda_2 \cdot n}{4}$ with λ_2 being the second smallest eigenvalue of the Laplacian of the graph (cf. [10]). This implies lower bounds of $0.042n$ and $0.133n$ for the bisection widths of 3-regular and 4-regular Ramanujan graphs. The spectral lower bound has been improved in [8] to a lower bound of $0.082|V|$ for the bisection width of large 3-regular Ramanujan graphs and a lower bound of $0.176|V|$ for the bisection width of large 4-regular Ramanujan graphs.

There are many heuristics for graph partitioning which are successfully being used in applications. Furthermore, efficient software implementations of the most relevant methods are available by using software tools like e. g. CHACO [12], JOSTLE [27], METIS [15], SCOTCH [24] or PARTY [22]. These heuristics try to calculate a bisection with a small cut size. However, they do not guarantee an approximation of the bisection width. Recently, it has been shown that the bisection width can be approximated by a polynomial time algorithm within a factor of $O(\log^2(|V|))$ [9].

New Results and Outline of the Paper In this paper we improve previous upper bounds on the bisection width of large 3- and 4-regular graphs. In Section 2 we prove for any $\epsilon > 0$ an upper bound of $(\frac{1}{6} + \epsilon)n$ on the bisection width of large 3-regular graphs and in Section 3 we prove an upper bound of $(\frac{2}{5} + \epsilon)n$ on the bisection width of large 4-regular graphs. As discussed above, there are large 3-regular graphs with a bisection width of at least $0.101n$ and large 4-regular graphs with a bisection width of at least $0.22n$. Thus, our results are optimal up to a small constant factor.

Iterative Local Improvement with Helpful Sets The proofs in this paper are constructive and follow an iterative local improvement scheme. It starts with an arbitrary balanced bisection. If the cut size of it does not fulfill the stated upper bound, it performs two steps to improve the bisection as illustrated in Fig. 1. In the first step, a small set $S_0 \subset V_0$ is moved to V_1. S_0 is chosen such that this move decreases the cut size. In the second step, a set $S_1 \subset V_1 \cup S_0$ with $|S_1| = |S_0|$ is moved to V_0. S_1 is chosen such that the cut size does not increase too much, i. e. such that the increase is less than the decrease in the first step. Thus, the resulting bisection is balanced and has a smaller cut size. These steps are repeated until the cut size drops below the upper bound. The proofs in this paper ensure that there are sets S_0 and S_1 with the desired property as long as the cut size is higher than the stated upper bound.

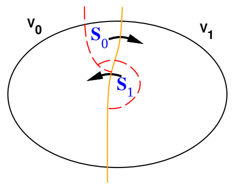


Fig. 1. One iteration of a local improvement.

This local improvement scheme has successfully been used to derive upper bounds on the bisection width of 4-regular graphs [14, 21]. Furthermore, it is the basis for the *Helpful-Set* heuristic which is able to calculate bisections with low cut sizes for very large graphs in a short time [7, 22]. An implementation of the Helpful-Set heuristic can be found in the software tool PARTY [26].

A move of a set of vertices from one part to the other changes the cut size of the bisection. The helpfulness of the set is defined to be the amount by which the cut size changes.

Definition 1. *Let π be a bisection of a graph $G = (V, E)$. For $S \subset V_p(\pi)$, $p \in \{0, 1\}$, let*

$$H(S) = |\{\{v, w\} \in E; v \in S, w \in V \backslash V_p(\pi)\}| - |\{\{v, w\} \in E; v \in S, w \in V_p(\pi) \backslash S\}|$$

be the **helpfulness** *of S. S is called $H(S)$-***helpful**.

2 Upper Bound on the Bisection Width of 3-regular Graphs

In this section we derive a new upper bound on the bisection width of 3-regular graphs. The proof is based on the iterative local improvement scheme described in the previous section. In the following, '$\log(x)$' denotes the logarithm of x to the basis 2.

Definition 2. *The vertices of V_0 (or V_1) are classified according to their distance to the cut. It is $V_0 = C \uplus D \uplus E$ with C vertices being at a distance of 1 to the cut, i. e. they are incident to a cut edge. D vertices are at a distance of 2 and E vertices at a distance of at least 3. D vertices are further classified with respect to the number of adjacent C vertices. I. e. $D = D_3 \uplus D_2 \uplus D_1$ and each D_x vertex is adjacent to x vertices in C. Overall,*

$$V_0 = C \uplus D_3 \uplus D_2 \uplus D_1 \uplus E \ .$$

The values c, d_3, d_2, d_1 and e specify the number of vertices of each type and the values $c(X)$, $d_3(X)$, $d_2(X)$, $d_1(X)$ and $e(X)$ denote the number of the according vertices in a set $X \subset V$.

Theorem 1. *For any $\epsilon > 0$ there is a value $n(\epsilon)$ such that the bisection width of any 3-regular graph $G = (V, E)$ with $|V| > n(\epsilon)$ is at most $(\frac{1}{6} + \epsilon)|V|$.*

Proof. We start with an arbitrary bisection and follow the iterative local improvement scheme described in Section 1. As long as the cut is above the bound, we repeatedly use Lemma 2 and 5 to calculate a new bisection with a lower cut. Thus, we can limit our focus on one iteration of the two lemmas. Let π_0 be a balanced bisection at the start of the iteration with $cut(\pi_0) > (\frac{1}{6} + \epsilon)|V|$.

Step 1: We construct a small helpful set $S \subset V_0$. Set $k = 4 \cdot \log(\frac{1}{\epsilon})$. The value of k is discussed below. We apply Lemma 2 several times. Each time we find an at least 1-helpful set. We proceed until we reach a total helpfulness of at least k, i. e. we apply the lemma k' times with $k' \leq k$. Let $S_i \subset V_0$, $1 \leq i \leq k'$, with $|S_i| = O(\frac{1}{\epsilon^2})$ be the sets constructed with Lemma 2. After a 1-helpful set S_i is constructed, it is moved from V_0 to V_1 and the next set S_{i+1} is constructed. Let $S = \uplus_{1 \leq i \leq k'} S_i$. It is $|S| = k' \cdot O(\frac{1}{\epsilon^2}) = k \cdot O(\frac{1}{\epsilon^2})$ and $H(S) \geq k$.
It remains to show that the requirement of Lemma 2 is fulfilled before each construction of a helpful set. Let $\bar{\epsilon} = \frac{\epsilon}{2}$. It is $cut(\pi_0) > (\frac{1}{3} + 2\bar{\epsilon})|V_0(\pi_0)| + \bar{\epsilon}|V|$ at the beginning. Let $n(\epsilon)$ be large enough such that $\bar{\epsilon}|V| \geq k$ for all $|V| > n(\epsilon)$. Thus, it is $cut(\pi_0) > (\frac{1}{3} + 2\bar{\epsilon})|V_0(\pi_0)| + k$. Each application of Lemma 2 decreases the size of the cut. We perform the lemma as long as $cut(\pi) > cut(\pi_0) - k > (\frac{1}{3} + 2\bar{\epsilon})|V_0(\pi_0)| \geq (\frac{1}{3} + 2\bar{\epsilon})|V_0(\pi)|$ with π being the current bisection. Thus, the condition $cut(\pi) > (\frac{1}{3} + 2\bar{\epsilon})|V_0(\pi)|$ is true before each application.
Let π_1 be the new bisection with $cut(\pi_1) = cut(\pi_0) - H(S)$.

Step 2: If $H(S) = k$, it is $cut(\pi_1) = cut(\pi_0) - k$. If $H(S) > k$, it is $cut(\pi_1) < cut(\pi_0) - k$ and we change π_1 by iteratively moving border vertices from V_1 to V_0 until we either get to $cut(\pi_1) = cut(\pi_0) - k$ or to a balanced bisection (in this case we are already finished). Each move of a single border vertex decreases the imbalance of the bisection and increases the cut by at most one.
Let $i := |V_1(\pi_1)| - \frac{n}{2}$ be the imbalance of π_1. It is $i \leq k \cdot O(\frac{1}{\epsilon^2})$. We use Lemma 5 to find a balancing set $\bar{S} \subset V_1(\pi_1)$ with $|\bar{S}| = i$.

Lemma 5 can only be applied if $|V_1(\pi_1)| < 3 \cdot cut(\pi_1)$. The fact $cut(\pi_0) > (\frac{1}{6}+\epsilon)n$ implies $|V_1(\pi_1)| = \frac{n}{2} + i < 3cut(\pi_0) - 3\epsilon \cdot n + i = 3cut(\pi_1) + 3k - 3\epsilon \cdot n + i \leq 3cut(\pi_1)$ if $3k + i \leq 3\epsilon \cdot n$. Clearly, there is a value $n(\epsilon)$ such that this equation holds for all graphs with $n > n(\epsilon)$.
We use Lemma 5 to get a set $\bar{S} \subset V_1(\pi_1)$ with $|\bar{S}| = i$ and $H(\bar{S}) \geq -1 - \log(i)$. The move of $\bar{S}$ from V_1 to V_0 results in a balanced bisection π_2 with $cut(\pi_2) \leq cut(\pi_1) + 1 + \log(i)$.

We need to ensure $cut(\pi_2) < cut(\pi_0)$ in order to show a decrease of the cut size. It is $cut(\pi_2) \leq cut(\pi_0) - k + 1 + \log(i)$ and $i \leq k \cdot x\frac{1}{\epsilon^2}$. for some constant x. Choosing $k = 4 \cdot \log(\frac{1}{\epsilon})$ fulfills $k > 1 + \log(k \cdot x\frac{1}{\epsilon^2})$ for $\frac{1}{\epsilon} \geq 2^8$ and $\frac{1}{\epsilon} \geq x$. □

Step 1 of Theorem 1 is based on Lemma 2. Lemma 2 itself makes use of Lemma 1 which is the main lemma of this section. We present it first because it may be of interest on its own.

Lemma 1. *Let $G = (V, E)$, $E = B \uplus R$, be a 3-regular graph with black edges B and red edges R. Let each vertex be adjacent to at least one black edge. Let $|R| > (\frac{1}{2}+\epsilon)|V|$ for an $\epsilon > 0$. Then there is a set $S \subset V$ of size $O(\frac{1}{\epsilon^2})$ such that the number of red edges between vertices of S is larger than the number of black edges between S and $V \backslash S$.*

Proof. Let e be the number of vertices which are adjacent to 3 black edges, d_1 be the number of vertices which are adjacent to 2 black edges and d_2 be the number of vertices which are adjacent to 1 black edge, i. e. $|V| = e + d_1 + d_2$ and $2|R| = 2d_2 + d_1$. It is

$$\frac{|R|}{|V|} = \frac{d_1 + 2d_2}{2(d_1 + d_2 + e)} \ . \tag{1}$$

The fact $|V| < 2|R|$ leads us to

$$e = |V| - d_2 - d_1 < 2|R| - d_2 - d_1 = d_2 \ . \tag{2}$$

Call a set $S \subset V$ **positive** if it has more internal red edges than external black edges, **negative** if it has more external black edges than internal red edges and **neutral** if the numbers are equal.

Consider the graph consisting of black edges only and let F be the family of its connected components. Clearly, the elements of F are neutral or positive. As a simple example, a positive set $I \in F$ of size $O(\frac{1}{\epsilon^2})$ fulfills the lemma. In the following we manipulate the sets in F. However, the number of sets in F remains constant and the sets in F remain neutral or positive.

The number of vertices of degree 1 in a connected component exceeds the vertices of degree 3 by at most 2. Thus, for each $I \in F$ it is $e(I) \geq d_2(I) - 2$. More precisely, it is $e(I) = d_2(I) + 2r(I) - 2$ with $r(I)$ being the number of edges which can be removed from I without splitting it into disconnected components. Let $r = \sum_{I \in F} r(I)$. It is

$$e = d_2 + 2r - 2|F| \ . \tag{3}$$

Let $\delta > 0$ be a constant. The value of δ will be assigned below. A set $I \in F$ is called **small** if $|I| \leq \frac{1}{\delta}$ and **large** otherwise. Denote with $\alpha(S)$, $S \subset V$, the number of red edges between vertices of S and vertices of small sets. Call a black path P a **thin path** if it has the following property. P contains only vertices with black degree of 2, P has a maximal path length, i. e. both end vertices are adjacent via a black edge to a vertex of black degree 1 or 3 (or, as described below, to a marked vertex of degree 2) and it is $|P| \leq \frac{\alpha(P)}{\delta}$. Call a black path P a **thick path** if it has all these properties except the last one.

Let $B(I)$, $I \in F$, be the family of disconnected components of I if all red edges and thin paths of I are removed. Let $\beta(B) = |B(I)|$. Let $s(I)$ be the size of the union of I and all small sets which are connected to I via a red edge. A set $I \in F$ is called **thin** if $s(I) < \frac{\beta(I)-1}{\delta}$ and **thick** otherwise.

The outline of the proof is the following. We state an algorithm below which manipulates the elements of F. Especially, we will disconnect certain thin paths from the rest of their set. The disconnection of a thin path P will shade it and mark both vertices previously connected to the ends of P as well as mark the small sets connected to P via a red edge. We disconnect thin paths in each set $I \in F$ until there is no thin path in any cycle of I. For such a set $I \in F$ we construct the tree $T(I) = (X, Y)$ with the vertex set X being the disconnected components from $B(I)$ and with the edges connecting the components if there is an unshaded thin path between them. It is $\beta(I) = |X| = |Y| + 1$. Let the weight $w(v)$, $v \in X$, be the size of the subset represented by v.

The algorithm may terminate with a positive set of size $O(\frac{1}{\delta^2})$. Since we will set later $\delta = \frac{\epsilon}{6(1+2\epsilon)}$, we can say that any positive set of size at most $O(\frac{1}{\delta^2})$ fulfills the lemma. Otherwise, the algorithm terminates with a family F consisting of sets of the following types only.

(i) $I \in F$ is large.
(ii) $I \in F$ is small and has at least one red edge to a thick set or to a thick path.
(iii) $I \in F$ is small and has at least 2 vertex marks or there is a cycle of black edges in I.

We will finish the proof by showing that if there are only sets of these types, we get a contradiction. Thus, the algorithm was successful in finding a set fulfilling the lemma.

The following algorithm manipulates the elements of F.

Step 1: If there is a positive small set $I \in F$, then I fulfills the lemma. If there is a red edge between two different small sets $I_a, I_b \in F$, then $I_a \cup I_b$ fulfills the lemma because I_a and I_b are neutral themselves and the union has an additional internal red edge. Both kind of fulfilling sets have a size of $O(\frac{1}{\delta})$.

Step 2: If there is a thin path P with $\alpha(P) \geq 3$, we show that there is a set fulfilling the lemma. We will take 3 small sets which are connected to P via a red edge and unify them with a subpath of P such that the subpath connects the 3 small sets. This union is a positive set. We use Lemma 3 to show that there are 3 small sets such that the connecting subpath is not too long, i. e. such that the size of the union is at most $O(\frac{1}{\delta})$.

P can be divided into subpaths by deleting the vertices in P which are connected to a small set via a red edge. There are $\alpha(P)$ such vertices, i. e. $\alpha(P)+1$ such (possibly empty) subpaths P_i, $1 \leq i \leq \alpha(P)+1$. Let $x_i = |P_i|$. It is $|P| = \sum_{i=1}^{\alpha(P)+1} x_i + \alpha(P)$. In the following we do not consider the subpaths P_1 and $P_{\alpha(P)+1}$ on both ends of the path. Lemma 3 ensures that there is an i, $2 \leq i < \alpha(P)$ such that $x_i + x_{i+1} \leq 3\frac{1}{\alpha(P)-1}\sum_{j=1}^{\alpha(P)-1} x_j$. Thus, it is $x_i + x_{i+1} \leq 3\frac{|P|}{\alpha(P)-1} \leq 3\frac{\alpha(P)}{(\alpha(P)-1)\delta} \leq 3\frac{3}{2\delta} = O(\frac{1}{\delta})$. The union of P_i, P_{i+1}, the three vertices connecting the paths P_{i-1} with P_i, P_i with P_{i+1} and P_{i+1} with P_{i+2} and the three small sets connected to these vertices is a positive set with a size of $O(\frac{1}{\delta})$.

From now on it is $\alpha(P) \leq 2$ for each thin path P.

Step 3: If there is a thin path P in a large set $I \in F$ and there is a cycle in I which includes P, remove the edges connecting P with $I \backslash P$ and shade P. The cycle in I ensures that both vertices which are connected to the ends of P had a black degree of 3 or (as we will see below) they had a degree of 2 and were marked once.

It is $\alpha(P) \leq 2$ due to step 2. If $\alpha(P) = 1$, let $S_1 \in F$ be the small set connected to P via a red edge and assign 2 marks to S_1. If $\alpha(P) = 2$, let $S_1, S_2 \in F$ be the small sets connected to P via a red edge and assign 1 mark to S_1 and 1 mark to S_2. The vertices which were connected to both ends of P get a mark, too. Their black degree is reduced by one due to the removal of the connecting edges. Although a vertex with black degree of 2 may be generated, such a vertex is not to be taken as part of a thin or thick path.

The removal of the connecting edges changes the graph. However, only two black edges are removed which are internal to I, i. e. I remains neutral or positive. Although P is disconnected from the rest of I and shaded, we still consider it to be a part of I. Therefore, neither the value $s(I)$ nor the number of elements in F does change.

The changes of the graph in this step come into account again when a positive set S of size $O(\frac{1}{\delta})$ is found in a future part of the algorithm. S is only positive with respect to the graph with removed edges. As we will see in step 4, S may include a subset of $I \backslash P$. Especially, S may include a marked vertex v which is connected to the end of P. However, this external edge can be compensated by enlarging S with P, S_1 and, if existing, S_2. Thus, the black edge between v and P is now internal and the possible external black edge between P and the vertex connected to the other end of P is compensated by the internal edge between P and S_1. Thus, the enlarged set S is positive after adding the formerly removed edges. S is getting enlarged for each vertex mark of a vertex in S. Each enlargement adds at most $|P|+|S_1|+|S_2| \leq \frac{4}{\delta}$ vertices. All enlargements together lead to a positive set S of size $O(\frac{1}{\delta^2})$ fulfilling the lemma. Repeat step 3 until there are no such thin paths.

Step 4: Let $I \in F$ be a thin set. Let $T(I)=(X,Y)$ be the graph of I with edges Y representing the unshaded thin paths in I as described above. $T(I)$ is a tree due to step 3. For each leaf l in $T(I)$ let $L(l)$ be the union of the vertices represented by l, all adjacent (possibly shaded) thin paths and all small sets which are adjacent to these vertices via a red edge. Clearly, such a set $L(l)$ is neutral or positive. We defined the weight $w(v)$ of a vertex $v \in T$ above to be the number of vertices represented by v. We redefine the weight $w(l)$ for each leaf l in $T(I)$ to be $|L(l)|$. Call a leaf l *small* if $w(l) < \frac{1}{\delta}$. If there is a shaded thin path between two small leaves l_1 and l_2, then $L(l_1) \cup L(l_2)$ is positive of size $O(\frac{1}{\delta})$ and the lemma is fulfilled. In the remainder we can assume that there is no such path and, therefore, for the new definition of the weights it holds $\sum_{v \in X} w(v) \leq s(I)$.

If there is a thin set $I \in F$ with $w(l) \geq \frac{1}{\delta}$ for each leaf l in $T(I)$, we can fulfill the lemma. It is $w(l) \geq \frac{1}{\delta} \geq \frac{s(I)}{\beta(I)-1} \geq \frac{\sum_{v \in X} w(v)}{|X|-1}$. $T(I)$ fulfills the requirements of Lemma 4. Therefore, there are adjacent vertices $u, v \in T$ with

$$w(u)+w(v) \leq 2 \cdot (\sum\nolimits_{v \in X} w(v))/(|X|-1) \leq 2 \cdot \frac{s(I)}{\beta(I)-1} \leq \frac{2}{\delta}\,.$$

The union S of the vertices represented by u, v and the edge between u and v has a size of $O(\frac{1}{\delta})$. However, we have to insert the edges which were removed in step 3 and 4. For each removed edge of step 4 (see below) and for each vertex mark of step 3 we enlarge S with the adjacent thin path and a small set connected to this path via a red edge. Each enlargement increases S by at most $O(\frac{1}{\delta})$ and compensates an external black edge as discussed in step 3. All enlargements together result in a positive set S of size $O(\frac{1}{\delta^2})$ and the lemma is fulfilled.

If we could not fulfill the lemma, consider all thin sets $I \in F$ which have a leaf l in $T(I)$ with $w(l) < \frac{1}{\delta}$. We will manipulate two elements of F. Let P be the thin path which connects l with its neighbor v in the tree and let S be the small set connected to P. Remove I and S from F and construct some new sets I_1 and I_2. Let I_1 be the union of the vertices represented by l, the path P and the small set S. Let I_2 be the set I reduced by P and the vertices represented by l. If $|I_2| \leq \frac{1}{\delta}$, then the union $L(l) \cup I$ fulfills the lemma. Otherwise, remove the edge which connects P with v. Now I_1 and I_2 are disconnected and both are neutral or positive with I_1 being a small set and I_2 being a large set. Unlike before, let the value $|L(l)|$ be the size of I_1. To keep our notations correctly we exchange the red color of the edge between P and S and the black color of the edge between P and v. This does not change the neutral status of I_1. We now add I_1 and I_2 to F. Thus, this step does not change the number of elements in F.

Note that, like in step 3, we will find a positive set of size $O(\frac{1}{\delta})$ in a future part of the algorithm. And, again, we have to insert the removed edges and we compensate the external edge with enlarging the set by the path P and the small set S. Thus, each single enlargement will increase the size of the set by at most $O(\frac{1}{\delta})$ leading to a positive set of size $O(\frac{1}{\delta^2})$ fulfilling the lemma. Go back to step 1 of the algorithm if any new small set was generated in step 4.

In the remaining we show the contradiction if only sets of types (i)-(iii) remain. Let z be the number of sets of type (i) or (ii). Let z_2 be the number of sets of type (iii) with at least two vertex marks and let z_1 be the number of sets of type (iii) with an internal black cycle.

It is $e = d_2 + 2r - 2|F|$ due to equation (3). At most r thin paths were shaded, leading to at most $2(r-z_2)$ vertex marks. Notice that only thin paths in large sets were shaded. It is $2(z_1+z_2) \leq 2r$. It follows $e = d_2 + 2r - 2z - 2(z_1+z_2) \geq d_2 - 2z$.

Each set of type (ii) is adjacent to a set of type (i) via a red edge. Let $I \in F$ be a set of type (i). We will show that $s(I)$ is large enough to reserve an average size of $\frac{1}{6\delta}$ for I itself and for all small sets of type (ii) connected to thick paths and unshaded thin paths of I. Each thick path P is connected to $\alpha(P)$ sets of type (ii) and has a size of at least $\frac{\alpha(P)}{\delta}$. We reserve a value of $\frac{1}{2\delta}$ for each set of type (ii) which is connected to a thick path. It remains a value of at least $\frac{s(I)}{2}$ for I and all small sets connected to unshaded thin paths of I. There are at most $\beta(I)-1$ unshaded thin paths in I, the others were shaded in step 3. Thus, there are at most $2\beta(I)-2$ small sets of type (ii) which are adjacent to unshaded thin paths of I via a red edge. Together with I these are at most $2\beta(I)-1$ sets of type (i) or (ii). If $\beta(I)=1$ it is $\frac{s(I)}{2} \geq \frac{1}{2\delta}$. If $\beta(I) \geq 2$ it is $\frac{s(I)}{2} \geq \frac{\beta(I)-1}{2\delta} \geq \frac{2\beta(I)-1}{6\delta}$. Thus, we can reserve at least $\frac{1}{6\delta}$ for each set of type (i) and (ii) and it is $e+d_1+d_2 \geq \frac{z}{6\delta}$. With $e<d_2$ (equation (2)) we get $z \leq 6\delta(d_1+2d_2)$.

With the use of $e \geq d_2 - 2z$ and $z \leq 6\delta(d_1+2d_2)$ the equation (1) leads us to

$$\frac{|R|}{|V|} = \frac{d_1+2d_2}{2(d_1+d_2+e)} \leq \frac{d_1+2d_2}{2(d_1+2d_2-2z)} \leq \frac{d_1+2d_2}{2(d_1+2d_2-12\delta(d_1+2d_2))} = \frac{1}{2(1-12\delta)}.$$

However, it is $\frac{|R|}{|V|} > \frac{1}{2}+\epsilon$. This leads to contradiction for $\delta \leq \frac{\epsilon}{6(1+2\epsilon)}$. This shows that during the algorithm we have found a set fulfilling the lemma. □

Lemma 2. *Let π be a bisection of a 3-regular graph $G=(V,E)$ with $V=V_0 \uplus V_1$. If $cut(\pi) > (\frac{1}{3}+2\epsilon)|V_0|$, $\epsilon>0$, then there is an at least 1-helpful set of size $O(\frac{1}{\epsilon^2})$ in V_0.*

Proof. We focus on part V_0 of the bisection only. Let $m := |V_0|$.

There are some structures which would directly lead to small 1-helpful sets.

(i) If a C vertex is incident to two or three cut edges, it is an at least 1-helpful set by itself.
(ii) A set of three connected C vertices is 1-helpful.
(iii) If there are two adjacent C vertices and one of them is adjacent to a D_3 vertex, then the union of the adjacent C vertices, the D_3 vertex and its two other adjacent C vertices form a 1-helpful set of size 5.
(iv) Another 1-helpful set can be formed if two D_3 vertices are adjacent to a common C vertices. Then, the union of both D_3 vertices and their adjacent C vertices is a 1-helpful set of size 7.
(v) Let v be a vertex which is adjacent to a C vertex which itself is adjacent to another C vertex or a D_3 vertex. If both other neighbors of v are from $C \cup D_2 \cup D_3$, the union of the mentioned vertices and their adjacent C vertices forms an at least 1-helpful set of size at most 11.

In the remainder we can assume that these types of structures do not exist. Especially, since (i) is excluded, it is $c = cut(\pi)$. We derive the helpful set in three steps.

1. Perform several transformations on the edges between vertices of V_0 to avoid certain structures in the graph. Let $\tilde{G}=(V,\tilde{E})$ be the transformed graph.
2. Derive a 1-helpful set $\tilde{H}$ of size $O(\frac{1}{\epsilon^2})$ in $\tilde{G}$.
3. Perform the transformations in the reverse direction. The helpfulness of the set might decrease. In this case, we include some further vertices to the set during the reverse transformations and derive a 1-helpful set H of G with $|H| = O(|\tilde{H}|)$.

There are two types of transformation, each of which avoids a certain structure in the graph. The transformation type 1 is performed until there are no adjacent C vertices. Then, transformation type 2 is performed until there are no D_3 vertices. The transformations are illustrated in Fig. 2. Each transformation deletes two edges $\{\alpha,\beta\}$ and $\{\gamma,\delta\}$ and includes two edges $\{\alpha,\gamma\}$ and $\{\beta,\delta\}$. Thus, the graph remains 3-regular.

Transformation type 1: Avoid adjacent C vertices (illustrated in Fig. 2, left). It is $\alpha,\beta \in C$, $\gamma \in D_1 \cup D_2$ (due to (ii) and (iii) above) and there is a $\delta \in D_1 \cup E$ (due to (ii) and (v) above).
Transformation type 2: Avoid D_3 vertices (illustrated in Fig. 2, right). It is $\alpha \in D_3$, $\beta \in C$, $\gamma \in D_1 \cup D_2$ (due to (iii) and (iv) above) and there is a $\delta \in D_1 \cup E$ (due to (v) above).

Notice that the transformations neither generate a new pair of adjacent C vertices nor a new D_3 vertex.

The graph $\tilde{G}$ is the result after all transformations and we will show how to derive a 1-helpful set in $\tilde{G}$ below. We first show how a 1-helpful set of $\tilde{G}$ can be transformed to a 1-helpful set in G. We perform the transformations in the reverse order. Consider a transformation of the graph and assume there is a 1-helpful set $\tilde{H}$ of the graph after this transformation. A 1-helpful set H of the graph after the reverse transformation is constructed in the following way.

Fig. 2. The transformations of the edges. Left: Adjacent C vertices. Right: A D_3 vertex.

Reverse of transformation type 2:
 (i) If $\alpha, \gamma \in \tilde{H}$, $\beta, \delta \notin \tilde{H}$: Let $H = \tilde{H} \cup \{\beta\}$.
 (ii) If $\beta, \delta \in \tilde{H}$, $\alpha, \gamma \notin \tilde{H}$: Let $H = \tilde{H} \cup \{\alpha, \gamma\} \cup \{v \in C; \{v, \alpha\} \in E\}$.
 (iii) In all other cases the helpfulness of $\tilde{H}$ does not change, i. e. let $H = \tilde{H}$.

Reverse of transformation type 1:
 (i) If $\alpha, \gamma \in \tilde{H}$, $\beta, \delta \notin \tilde{H}$: Let $H = \tilde{H} \cup \{\beta\}$.
 (ii) If $\beta, \delta \in \tilde{H}$, $\alpha, \gamma \notin \tilde{H}$: Let $H = \tilde{H} \cup \{\alpha, \gamma\}$.
 (iii) In all other cases the helpfulness of $\tilde{H}$ does not change, i. e. let $H = \tilde{H}$.

We have to ensure that a reverse transformation does not increase the size of the set too much. Each reverse transformation enlarges the set by at most a constant number of vertices. We show that the number of reverse transformations of type 1 is at most the initial number of C vertices in the set. Each time the set is enlarged by a reverse of type 1, there is one C vertex in the set which was previously not adjacent to another C vertex in the set and now becomes adjacent to one. Notice that this property already holds true for any new C vertex included in a reverse transformation of type 1. Furthermore, a C vertex cannot become adjacent to a second C vertex because this would lead to a structure (ii) discussed above. A similar argument can be used to show that the number of reverse transformations of type 2 is also bounded by the number of initial vertices in the set. Consequently, the reverse transformations increase the size of the set by not more than a constant factor.

It is left to show that there is a small 1-helpful set of size $O(\frac{1}{\epsilon^2})$ in $\tilde{G}$.

There are no D_3 vertices and each C vertex is connected to two D vertices, i. e. $d_3 = 0$ and $2c = 2d_2 + d_1$. Furthermore, because of $m < 3c$ it is (similar to equation (2))

$$e = m - c - d_2 - d_1 < 2c - d_2 - d_1 = 2d_2 + d_1 - d_2 - d_1 = d_2 \ . \tag{4}$$

Construct a new graph K consisting of the D and E vertices of $\tilde{G}$, i. e. $K = (U, F)$ with $U = D \uplus E$. Let $F = B \uplus R$ with black edges B and red edges R. The black edges are the edges between the D and E vertices as in $\tilde{G}$. Furthermore, there is a red edge between two vertices if they are adjacent to a common C vertex in $\tilde{G}$, i. e. $|R| = c$. Thus, K is 3-regular with a maximum red degree of 2, due to the fact that there are no D_3 vertices. It is

$$|R| = c > (\frac{1}{3} + 2\epsilon)(c + d_2 + d_1 + e) > (\frac{1}{3} + 2\epsilon)\frac{3}{2}(d_2 + d_1 + e) = (\frac{1}{2} + 3\epsilon)|U| \ .$$

Thus, K fulfills the requirements of Lemma 1 for $\bar{\epsilon} = 3\epsilon$. We use Lemma 1 to derive a set S of D and E vertices with size $O(\frac{1}{\bar{\epsilon}^2}) = O(\frac{1}{\epsilon^2})$.

The number b_{ext} of external black edges of S with respect to K is equal to the number of edges between S and other D and E vertices in V_0. The number r_{int} of internal red edges of S with respect to K is equal to the number of C vertices in V_0 which are connected to two vertices in S. Lemma 1 ensures $r_{int} > b_{ext}$.

Let $\bar{S}$ be the union of S with all adjacent C-vertices. It is $|\bar{S}| = O(\frac{1}{\epsilon^2})$. Each external black edge connects $\bar{S}$ with $V_0 \backslash \bar{S}$. The external red edges are neutral, because they connect S via a C vertex to $V_0 \backslash \bar{S}$. Thus, such a C vertex has one edge to V_1 and one edge to $V_0 \backslash \bar{S}$. Each internal red edge is a C vertex which is connected to two vertices in S. Thus, such a vertex has one edge to V_1 and no edges to $V_0 \backslash \bar{S}$. Overall, there are r_{int} edges between $\bar{S}$ and V_1 and b_{ext} edges between $\bar{S}$ and $V_0 \backslash \bar{S}$. This leads to $H(\bar{S}) = r_{int} - b_{ext} > 0$ and $\bar{S}$ fulfills the lemma. □

Lemma 3. *Let $r \geq 2$ and let $x_1, \ldots x_r, A \in \mathbf{R}$ with $\sum_{i=1}^{r} x_i = rA$. Then there is an i, $1 \leq i < r$ with $x_i + x_{i+1} \leq 3A$.*

Proof. Assume it is $x_i + x_{i+1} > 3A$ for all $1 \leq i < r$. It follows $2rA = 2\sum_{i=1}^{r} x_i = \sum_{i=1}^{r-1}(x_i + x_{i+1}) + x_1 + x_r > (r-1)3A + x_1 + x_r$. This leads to the contradiction $3A > rA + x_1 + x_r$. □

Lemma 4. *Let $T = (V, E)$ be a tree with weights $w : V \to \mathbf{N}$. Let $A = (\sum_{v \in V} w(v))/(|V| - 1)$. Let $w(l) \geq A$ for all leaves l. Then there are adjacent vertices u and v with $w(u) + w(v) \leq 2 \cdot A$.*

Proof. Consider an arbitrary edge $e \in E$. The removal of e divides T into trees T_0 and T_1. For at least one $i \in \{0, 1\}$ it is

$$\sum\nolimits_{v \in V(T_i)} w(v) < A \cdot |V(T_i)| \ . \tag{5}$$

W.l.o.g let $i = 0$. Let $V_0 = V(T_0)$. Let $u_0 \in e \cap V_0$ be the vertex in V_0 which is incident to e. T_0 can be viewed as weighted tree with root u_0. For each $u \in V_0$ let $T(u)$ be the sub-tree of T_0 with root u. Let $V(u)$ be the set of vertices of $T(u)$. We can now presume that for all $u \in V(u_0)\backslash\{u_0\}$ it is

$$\sum\nolimits_{v \in V(u)} w(v) \geq A \cdot |V(u)| \ . \tag{6}$$

If this is false, then go in T_0 down to a lowest vertex which fulfills equation (6). Re-assign u_0 to this vertex. Clearly, u_0 is not a leaf, due to the assumption.

Let $N(u_0)$ be the set of neighbors of u_0 in $T(u_0)$. For all $v \in N(u_0)$ it holds equation (6). It is $V(u_0)\backslash \bigcup_{v \in N(u_0)} V(v) = u_0$ and, thus, it is

$$w(u_0) \leq A \cdot |V(u_0)| - A \cdot \sum\nolimits_{v \in N(u_0)} |V(v)| = A \ . \tag{7}$$

We distinguish between the following two cases.

(i) All vertices in $N(u_0)$ are leaves, i. e. $V(u_0) = \{u_0\} \cup N(u_0)$.
It is $\sum_{v \in N(u_0)} w(v) < A|N(u_0)| + A - w(u_0)$, due to equation (5). Let $v_0 \in N(u_0)$ be a vertex with the lowest weight. Using equation (7) it follows

$$w(u_0) + w(v_0) < w(u_0) + (A|N(u_0)| + A - w(u_0))/|N(u_0)| \leq 2A \ .$$

(ii) At least one vertex in $N(u_0)$ is not a leaf, i. e. $|V(u_0)| > |\{u_0\} \cup N(u_0)|$.
Let $u' \in V(u_0)$, $u' \notin \{u_0\} \cup N(u_0)$ be a vertex of distance 2 from u_0, i. e. there is a $v' \in N(u_0)$ with $\{v', u'\} \in E$. The equations (5) and (6) lead us to

$$w(u_0) + w(v') \leq A|V(u_0)| - \sum\nolimits_{v \in N(u_0)\backslash\{v'\}} w(v) - \sum\nolimits_{v \in N(v')} w(v) \leq 2A \ . \square$$

Lemma 5. *Let $G = (V, E)$ be a connected 3-regular graph and let π be a bisection of G. If $|V_1(\pi)| < 3 \cdot cut(\pi)$ and $0 < x < |V_1(\pi)|$, then there is a set $S \subset V_1(\pi)$ with $|S| = x$ and $H(S) \geq -1 - \log(|S|)$.*

Proof. We first discuss the following cases.

(i) If $x \leq 2$, any set S of x single border vertices has the desired property $H(S) \geq -1 - \log(|S|)$.
(ii) If we find a set $Z \subset V_1(\pi)$ with $|Z| \leq x$ and $H(Z) \geq 0$, we can move Z from V_1 to V_0 without increasing the cut size. It remains to apply the lemma again with $\bar{x} = x - |Z|$.
Notice that in the case $H(Z) > 0$ the move may result in $|V_1| \geq 3 \cdot cut$. In this case single border vertices can be moved from V_1 to V_0 until it holds $|V_1| < 3 \cdot cut$ again and the lemma is applied again.
(iii) If we find a set $Z \subset V_1(\pi)$ with $\frac{x}{2} \leq |Z| \leq x$ and $H(Z) \geq -1$, we can move Z from V_1 to V_0 with increasing the cut size by at most 1. It remains to apply the lemma again with $\bar{x} = x - |Z| < \frac{x}{2}$. This will construct a set $\bar{S}$ with $|\bar{S}| = x - |Z|$ and $H(\bar{S}) \geq -1 - \log(|\bar{S}|)$, and a unified set $S = Z \cup \bar{S}$ with $|S| = x$ and $H(S) \geq -1 - 1 - \log(|\bar{S}|) \geq -1 - \log(|S|)$.

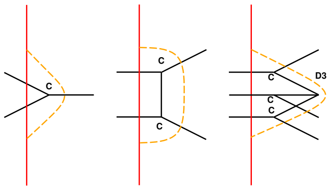


Fig. 3. Sets Z with $H(Z) \geq 0$.

In the following we can exclude the existence of certain small 0-helpful sets as illustrated in Fig. 3. One example are C vertices incident to two or three cut edges and any set of two adjacent C vertices. A D_3 vertex, together with its adjacent C vertices, also forms a 0-helpful set.

In the remainder there are no such sets, i. e. it is $cut = c$, $d_3 = 0$ and $2c = 2d_2 + d_1$. Because of $|V_1| < 3c$ it holds equation (4).

In the following we consider the graph induced by the vertex set $D \cup E$ and its connected components. Let F be the family of these components. Furthermore, for a set $I \subset V_1(\pi)$ define the enlarged set $Z(I) = I \cup \{v \in C; \exists w \in I \text{ with } \{v, w\} \in E\}$ which includes the adjacent C-vertices. Clearly, each set $Z(I)$ for an $I \in F$ is at least 0-helpful. If there is a set $Z(I)$ with $I \in F$ and $|Z(I)| \leq x$, we proceed as discussed above in case (ii).

Consider a connected component $I \in F$ and let $K = (I, J)$ be the subgraph of G induced by I. The E vertices in K have degree 3, D_1 vertices have degree 2 and D_2 vertices have degree 1. It is easy to see that $e(I) \geq d_2(I)$ iff K contains a cycle and $e(I) = d_2(I) - 2$ otherwise. Because of equation (4) there is an $I \in F$ for which the induced subgraph is a tree.

Let $I \in F$ be a connected component with the induced subgraph $T = (I, J)$ being a tree. Assign a weight $w(v)$ to each vertex v in the tree with $w(v) = |Z(\{v\})|$. For each vertex v this is one higher than the number of C vertices adjacent to v. Thus, each leaf has a weight of 3, each vertex of degree 2 has a weight of 2 and each vertex with a degree of 3 has a weight of 1. It is $\sum_{v \in L} w(v) = |Z(L)|$ for an $L \subset I$ if there are no C vertices which are connected to two vertices of L. It is $\sum_{v \in L} w(v) > |Z(L)|$ for an $L \subset I$ if there is at least one such vertex.

With $|Z(I)| > x$ it is $\sum_{v \in T} w(v) > x$. Clearly, for this type of weight distribution there is an edge in T which separates T into T_1 and T_2 with $\frac{x}{2} \leq \sum_{v \in T_1} w(v) \leq x$. If $|Z(T_1)| < \sum_{v \in T_1} w(v)$, it is $|Z(T_1)| < x$ and $H(Z(T_1)) \geq 0$ and we proceed with case (ii) above. If $|Z(T_1)| = \sum_{v \in T_1} w(v)$, it is $\frac{x}{2} \leq |Z(T_1)| \leq x$ and $H(Z(T_1)) \geq -1$. We proceed with case (iii) above. $\square$

3 Upper Bound on the Bisection Width of 4-regular Graphs

In this section we are presenting a new upper bound on the bisection width of 4-regular graphs. The proof is based on the iterative local improvement scheme described in section 1.

We categorize the vertices of a 4-regular graph according to their number of external edges in the bisection as shown in Fig. 4. Notice that these categories differ from the ones in section 2.

Definition 3. *Let $G = (V, E)$ be a 4-regular graph and let $\pi(G)$ be a bisection of V. A-vertices are incident to 3 or 4 cut edges, B-vertices are incident to 2 cut edges, C-vertices are incident to 1 cut edge and D-vertices are not incident to any cut edge.*

Whenever this categorization is used, only one part of the bisection is considered and the values $a = |A|$, $b = |B|$, $c = |C|$ and $d = |D|$ denote the cardinalities of the sets in the according part.

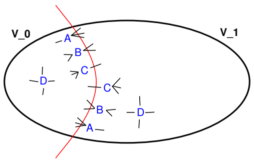
The following theorem improves the upper bound for large 4-regular graphs. It is an extension and improvement of the local improvement Helpful-Set approach of [14].

Theorem 2. *For any $\delta > 0$ there is a value $n(\delta)$ for which the bisection width of any 4-regular graph $G = (V, E)$ with $|V| > n(\delta)$ and $|V| \equiv 0 \bmod 4$ is at most $(0.4 + \delta)|V|$.*

Proof. Start with a balanced bisection π of G with $cut(\pi) \leq \frac{n}{2}$. It has been shown in [14] that the bisection width is at most $\frac{n}{2} + 1$ for $n \geq 350$. Obviously, the cut of any bisection of a 4-regular graph has to be even. Thus, in our case of $n \equiv 0 \ mod\ 4$, the bisection width is at most $\frac{n}{2}$ for $n \geq 350$. Additionally, the upper bound of $(0.4 + \delta)n$ with $\delta \geq \frac{1}{10}$ is already solved. Thus, the only thing left for consideration is the case $0 < \delta < \frac{1}{10}$.

As long as there are A vertices in both parts, exchange a pair of one of each side. The cut will be decreased by at least 2. W.l.o.g. assume that $V_0(\pi)$ does not contain any further A vertices. As long as $cut(\pi)$ is larger than $(0.4 + \delta)n$, find a small at least 4-helpful set $S_0 \subset V_0(\pi)$. Balance this set with an at least (-2)-helpful set $S_1 \subset V_1(\pi)$ of the same size as shown in Fig. 1.

Lemma 7 shows that there is an at least 4-helpful set of size $O(\frac{1}{\delta}(\log(n))^2)$ in $V_0(\pi)$ for large graphs with $n \geq 8\frac{4+10\delta}{(0.4+\delta)25\delta}$ and $cut(\pi) > (0.4 + \delta)n$. Therefore, set $n(\delta) := 8\frac{4+10\delta}{(0.4+\delta)25\delta}$. If the set is moved to $V_1(\pi)$, the resulting bisection π_1 will have $cut(\pi_1) = cut(\pi) - 4$ and $|V_1(\pi_1)| = \frac{n}{2}+O(\frac{1}{\delta}(\log(n))^2)$. According to lemma 6, this bisection can be re-balanced while increasing the cut by at most 2 if $|V_1(\pi_1)| \leq \frac{5}{4}cut(\pi_1)$. This holds for large graphs, i. e. when $O(\frac{1}{\delta}(\log(n))^2) \leq \frac{5}{4}\delta n - 5$. Increase $n(\delta)$ such that this equation holds true for $n \geq n(\delta)$. The exchange of sets will be repeated until the desired cut is achieved. Overall, the cut will be decreased in each iteration by at least 2 until a cut of at most $(0.4 + \delta)n$ is achieved. □



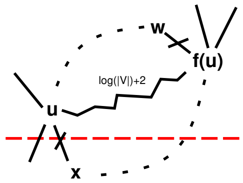


Fig. 4. Left: A, B, C and D vertices. Right: Transform $u \in B$ and $f(u) \in D$ to C-vertices.

Hromkovic and Monien [14] proved the following lemma for re-balancing a bisection of a 4-regular graph without increasing the cut size too much.

Lemma 6 ([14]). *Let π be a bisection of a 4-regular graph $G = (V, E)$ with $|V_0(\pi)| < |V_1(\pi)| \leq \frac{5}{4}cut(\pi)$ and $cut(\pi) \geq 4$. Then a bisection $\bar{\pi}$ with $|V_0(\bar{\pi})| = |V_1(\bar{\pi})|$ and $cut(\bar{\pi}) \leq cut(\pi) + 2$ can be constructed.*

Lemma 7. *Let π be a bisection of a 4-regular graph $G = (V, E)$ with $|V_0(\pi)| = |V_1(\pi)|$ and $n \geq 8\frac{4+10\delta}{(0.4+\delta)25\delta}$. If $(0.4 + \delta)n < cut(\pi) \leq \frac{n}{2}$ for $0 < \delta < \frac{1}{10}$ and $V_0(\pi)$ has no A vertices, there will be an at least 4-helpful set of size $O(\frac{1}{\delta}(\log(n))^2)$ in $V_0(\pi)$.*

Proof. Consider only the vertices of part $V_0(\pi)$ throughout this proof. Lemma 8 shows that there is either a 4-helpful set S of size $O(\log(n))$ or there is an injective function $f : B \to D$ which assigns each vertex of B to a different vertex of D, connected with a path of length $O(\log(n))$ along B or C vertices. The proof is finished if the first case holds true.

Now construct a new graph $\bar{G} = (\bar{V}, \bar{E})$, which is a copy of G with some transformations. The vertices $u \in B$ and $f(u) \in D$ are transformed to two C vertices as shown in Fig. 4. Choose any neighbor x of u in part $V_1(\pi)$ and any neighbor w of $f(u)$ with $w \neq u$ and $\{u, w\} \notin E$. There is such a vertex w, because $f(u)$ has got 4 and u has only got 2 internal neighbors. Now delete the edges $\{u, x\}$ and $\{f(u), w\}$ and include the edges $\{u, w\}$ and $\{x, f(u)\}$. This is performed for any $u \in B$. The transformations can be performed all in one step, due to the function being injective. The involved B and D vertices are becoming C vertices and the result is a graph $\bar{G}$ with only C and D vertices.

Only C vertices are incident to cut edges, i. e. $c = cut(\pi)$. The number of edges in the new graph $\bar{G}$ between C and D vertices cannot exceed $4d$ and, therefore, the number of edges between

C vertices is at least $\frac{3c-4d}{2} = \frac{3c-4(\frac{n}{2}-c)}{2} = \frac{7}{2}c - n$. Furthermore, consider the graph $H = (U, F)$ which consists of all C vertices and edges between C vertices. It holds $|U| = c = cut(\pi)$ and $|F| \geq \frac{7}{2}|U| - n > \frac{7}{2}|U| - \frac{cut(\pi)}{0.4+\delta} = \frac{7}{2}|U| - \frac{|U|}{0.4+\delta} = (1+\beta)|U|$ with $\beta := \frac{25\delta}{4+10\delta}$. It follows from $0 < \delta < \frac{1}{10}$ that $0 < \beta < \frac{1}{2}$ and since $n \geq \frac{8}{(0.4+\delta)\beta}$, it is $|U| = cut(\pi) \geq (0.4+\delta)n \geq \frac{8}{\beta}$. Lemma 9 ensures that there is a set $S \subset U$ of size $O(\frac{1}{\delta}\log(|U|))$ with at least $|S|+2$ internal edges. With respect to graph $\bar{G}$, the set S of C vertices with $|S|+2$ edges is at least 4-helpful.

S is at least 4-helpful with respect to $\bar{G}$, but we need a 4-helpful set with respect to G. We perform the transformations from G to $\bar{G}$ in the reverse direction. Again, the function being injective allows the performance of all reverse transformations in one step. In each reverse transformation we possibly enlarge the set S such that it remains 4-helpful. The result is a 4-helpful set S with respect to G. In each reverse transformation, a pair of vertices u and $f(u)$ is transformed back to B and D vertices and the edges are set back to the old constellation.

It is easy to see that only in the case $f(u) \in S$ and $u, w \notin S$ the helpfulness may decrease by the reverse transformation. In this case the helpfulness decreases by two and we include the path from u to $f(u)$ into S. The enlargement of S increases the helpfulness by at least 2, resulting in an at least 0-helpful set S. The length of each path is $O(\log(n))$. Thus, it is $|S| = O(\frac{1}{\delta}(\log(n))^2)$. □

Lemma 8. *Let π be a bisection of a 4-regular graph $G = (V, E)$. If $|V_0(\pi)| = |V_1(\pi)|$ and $cut(\pi) \leq \frac{n}{2}$, there is an at least 4-helpful set S_0 of size $\leq 6(\log(n)+2)+2$ in $V_0(\pi)$ or an injective function $f : B \rightarrow D$ with the property: for each $u \in B$ there is a path of a length of at most $\log(n)+2$ from u to $f(u)$ along B or C vertices.*

Proof. Perform a breath-first search which starts from B vertices along C vertices. Proceed until a 4-helpful set or 2 D vertices are found. A 4-helpful set occurs if there are either (i) 2 cycle-edges or (ii) 2 further B vertices or (iii) one of each kind in the search tree. In this case the set which consists of the cycles (or B vertices) and the paths between the cycles (or B vertices) and the initial B vertex is 4-helpful and of size at most $4(\log(n)+2)$.

If there is no 4-helpful set to be found, the length of the path to the second D vertex is at most $\log(n)+3$, because otherwise the search tree exceeds the total number of vertices. Thus, the depth of the breath-first search is at most $\log(n)+3$. The worst case for the distance is when the first D vertex and the only possible cycle-edge or B vertex occurs at the start of the search (Fig. 5).

Now construct a bipartite graph which consists of the B and D vertices, such that a vertex B will be connected to a vertex D if it was reached in the search. B vertices have a degree of 2 in this graph. If a D-vertex has a degree of 4, it and the 4 paths to the B vertices form a 4-helpful set S with $|S| \leq 4(\log(n)+2)+1$. If there are two D vertices of degree 3, they form a 4-helpful set S together with their paths to B vertices of size $|S| \leq 6(\log(n)+2)+2$. Thus, only one D vertex in the bipartite graph may have a degree of 3 and all other vertices have a lower degree. For each vertex $w \in D$ of degree 1 and its neighbor $w \in B$ choose $f(w) := u$ and delete both vertices from the bipartite graph. Now there are no D vertices of degree 1 left. This implies that there is no D vertex of degree 3, because otherwise it would be the only vertex with an odd degree. This is impossible. Thus, the bipartite graph now consists of cycles of pairwise connected B and D vertices. This structure implies a neighboring assignment of the function f within the cycles. □

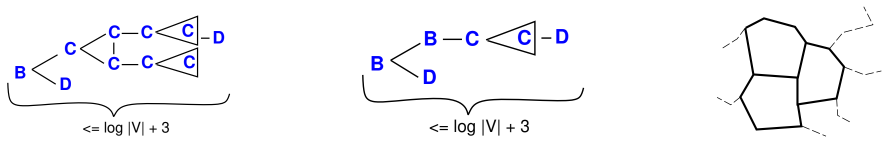


Fig. 5. Left/Center: The path between B and the second D vertex will be no longer than $\log(n)+3$ vertices, if no 4-helpful set is found. Right: Repeatedly delete all vertices of degree 1 (doted paths) until there are only vertices of degree 2 and 3 (cycles) remaining.

Lemma 9. *Let $G = (V, E)$ be a graph with a maximum degree of 3, $e = |E| \geq (1+\beta)n$, $\frac{1}{2} \geq \beta > 0$ and $n \geq \frac{8}{\beta}$. There is a set $S \subset V$ of size $|S| = O(\frac{\log(n)}{\beta})$ with at least $|S| + 2$ internal edges.*

Proof. Let x, y and z be the number of vertices of degree 3, 2 and 1. Then $x + y + z = n$ and $3x + 2y + z = 2e$. It holds

$$x - z = 3x + 2y + z - 2n = 2e - 2n \geq 2\beta n \quad \text{and} \quad y = n - x - z \leq (1 - 2\beta)n - 2z \ .$$

In a first step, repeatedly delete all vertices of degree 1. Adjacent vertices of degree 2 become vertices of degree 1. Adjacent vertices of degree 3 become vertices of degree 2 and, possibly, vertices of degree 1 later on. These new vertices of degree 1 are also deleted until only vertices of degree 2 and 3 are left as shown in Fig. 5. Thus, all connected components without any cycles and all induced subtrees will be deleted. For each initial vertex of degree 1, the vertices along the path to the closest vertex of degree 3 are deleted and that vertex of degree 3 becomes a vertex of degree 2. Therefore, for each initial vertex of degree 1 the number of vertices of degree 3 decreases by at most 1. Although many vertices of degree 2 may be deleted, in the worst case a vertex of degree 1 is directly connected to a vertex of degree 3 and, thus, for each initial vertex of degree 1 the number of vertices of degree 2 may increase by at most 1. Let x_1 and y_1 be the number of vertices with degree 3 and 2 after step one. Now it holds

$$x_1 \geq x - z \geq 2\beta n \qquad \text{and} \qquad y_1 \leq y + z \leq (1 - 2\beta)n - 2z + z \leq (1 - 2\beta)n \ .$$

In the second step delete long paths of vertices of degree 2. The average length of the paths is

$$\frac{y_1}{\frac{3}{2}x_1} \leq \frac{1 - 2\beta}{3\beta} =: \lambda \ .$$

Now delete all paths of a length of at least $4\lambda + 2$ one after another. Let r be the number of such paths. The number of deleted vertices of degree 2 is at least $r(4\lambda + 2)$. Each deletion of a path transforms two vertices of degree 3 to vertices of degree 2. The original number of vertices of degree 2 is y_1. However, this number can increase by 2 for each deleted path. In general, at most $y_1 + 2r$ vertices of degree 2 can be deleted, thus

$$4r\frac{1 - 2\beta}{3\beta} + 2r = r(4\lambda + 2) \leq (1 - 2\beta)n + 2r \ .$$

This leads to $r \leq \frac{3}{4}\beta n$. The above implies that the number x_2 of vertices of degree 3 after step two fulfills

$$x_2 \geq x_1 - 2r \geq (2 - \frac{3}{2})\beta n = \frac{1}{2}\beta n \geq 4 \ .$$

Now consider a graph that consists of the x_2 vertices with degree 3 and edges between them if they are connected via a path along vertices of degree 2. The length of a shortest cycle in a 3-regular graph with x_2 vertices is at most $2\log(\frac{x_2}{2} + 1)$ (using the bounds on cubic cages, see e. g. [13]). Now perform a breath-first search which starts from the cycle vertices until a path of new vertices between two cycle vertices is found. It is easy to see that this path consists of at most $2\log(\frac{x_2}{2} + 1)$ vertices. The new set S that consists of the cycle and the path has $|S| \leq 4\log(\frac{x_2}{2} + 1)$ and $|S| + 1$ internal edges. A second breath-first search from the vertices of S finds another path between two vertices of S consisting of at most $2\log(\frac{x_2}{2} + 1)$ additional vertices, which is also included to S. Now, $|S| \leq 6\log(\frac{x_2}{2} + 1)$ and S has $|S| + 2$ internal edges.

After transforming S back to the original graph, the paths between the vertices of degree 3 consist of at most $(\lceil 4\lambda \rceil + 1)(6\log(\frac{n}{2} + 1) + 2)$ vertices of degree 2. Thus, $|S| = O(\frac{\log(n)}{\beta})$ and S has $|S| + 2$ internal edges. □

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