On the Robustness of Interconnections in Random Graphs: A Symbolic Approach*

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Abstract

Graphs are models of communication networks. This paper applies symbolic combinatorial techniques in order to characterize the interplay between two parameters of a random graph, namely its density (the number of edges in the graph) and its robustness to link failures. Here, robustness means multiple connectivity by short disjoint paths: a triple (G, s, t), where G is a graph and s, t are designated vertices, is called ℓ–robust if s and t are connected via at least two edge-disjoint paths of length at most ℓ. We determine the expected number of ways to get from s to t via two edge-disjoint paths of length ℓ in the classical random graph model $\mathcal{G}_{n,p}$ by means of "symbolic" combinatorial methods. We then derive bounds on related threshold probabilities as functions of ℓ and n.

Introduction

In recent years the development and use of communication networks has increased drastically. In such networks, basic physical architecture combined with traffic congestion or operating system decisions, result in a certain, dynamically changing geometry of the graph of interconnections. We adopt the random graph model of $\mathcal{G}_{n,p}$ (see [6, 7]) to capture link availability in networks: a graph of $\mathcal{G}_{n,p}$ has n nodes and any of the $\binom{n}{2}$ edges is present with probability p (independently for each edge). Even in such a simple network model, it is

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interesting to investigate the trade-off between *density* (the number of edges, which is $p\binom{n}{2}$ in the mean and close to this value with high probability) and *robustness* to link failures. Indeed, the existence of alternative paths in such graphs may model desired reliability and efficiency properties: an example is the ability to use alternative routes to guide packet flow in ATM networks or even improve the efficiency of searching robots on the World Wide Web, in the sense of an increased multiconnectivity of its hyperlink structure.

Given a triple (G, s, t), where G is a $\mathcal{G}_{n,p}$ random graph and s, t are two of its nodes, a natural notion of robustness is to require at least two edge-disjoint paths of short length (say, exactly ℓ or at most ℓ) between s and t, so that connectivity by short paths survives, even in the event of a link failure.

Definition 1 (ℓ–robustness) *A triple (G, s, t) with G a graph and s, t two nodes of G is ℓ–robust when there exist two edge-disjoint paths of length at most ℓ between s, t in G.*

In this work, we investigate the expected number $N_\ell(n, p)$ of such paths between two vertices of the random graph, as well as lower and upper bounds $P_L(n, \ell), P_U(n, \ell)$ for the threshold probability of the existence of such paths in the random graph $G \in \mathcal{G}_{n,p}$.

Although $\mathcal{G}_{n,p}$ has been extensively studied [2, 7, 22], some questions of existence of multiple paths, which are vertex- or edge-disjoint between specific vertices have not been investigated till recently. The theory of random graphs began with the celebrated work of Erdös and Rényi [11] in 1959 and nowadays researchers know a lot about the probable structure of these objects (see, e.g., the birth of the giant connected component in [18]). In this context we remark that, the question of existence of many *vertex-disjoint* paths of small length has been investigated by Nikoletseas et al in [20]; however the corresponding problem of the existence of *edge-disjoint* paths (which is more difficult to deal with, from the technical point of view) has remained untouched. Even the enumeration of paths among the vertices 1 and n that avoid all edges of the line graph $(1, 2 \ldots, n)$ but pass through all its vertices, is a non-trivial combinatorial task. In fact, such an enumeration corresponds to enumerating permutations $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ of $(1, 2, \ldots, n)$ where certain gaps $\sigma_{i+1} - \sigma_i$ are forbidden. In our case, $\sigma_{i+1} - \sigma_i$ must not be in the set $\{-1, 1\}$, and this basic problem resembles the classical *"ménage problem"* of combinatorial analysis [9, 25].

In this work, we provide a precise evaluation of the expected number of unordered pairs of paths in a random graph that connect a common source to a common destination, and have no edge in common, though they may share some nodes. In order to achieve this, we devise a finite-state mechanism that describes classes of permutations with free places and exceptions. The finite-state description allows for a direct construction of a multivariate generating function. The generating function is then subjected to an integral transform that implements an inclusion-exclusion argument from which an explicit enumeration derives; see Theorem 1 and Proposition 1. This enables us to quantify the trade-off between ℓ–robustness (as defined above) and the density of the graph (*i.e.*, the number of its edges). The originality of our approach consists in introducing in this range of problems methods of analytic combinatorics [14, 21] and recent research in automatic analysis based on symbolic computation [8, 13, 15, 23]. Additional threshold estimates regarding properties of multiple source-destination pairs are discussed in the last section of the paper.

Summary of results. From earlier known results [7, 20] and this paper, a picture of robustness under the $\mathcal{G}_{n,p}$ model emerges. (As is usual in random graph theory, various regimes for $p = p(n)$ are considered.) Start with an initially totally disconnected graph, corresponding to $p = 0$. As p increases, the graph becomes connected near the connectivity threshold $P_C(n) \simeq (\log n)/n$. Any fixed s, t pair (or equivalently a random s, t pair, given the invariance properties of $\mathcal{G}_{n,p}$) is likely to become ℓ–robust when p crosses the value

$$P_M(n, \ell) = 2^{\frac{1}{2\ell}}\ n^{-1+\frac{1}{\ell}}.$$

Here "likely" signifies that the *mean* number of edge-disjoint pairs is at least 1 when n grows to infinity, *cf.* Theorem 2 and Equation (13). Then, as long as $p \leq P_L(n, \ell)$, where

$$P_L(n, \ell) = n^{-1+\frac{1}{\ell}} \left(\log \frac{n^2}{\log n}\right)^{\frac{1}{\ell}}$$

we know, with high probability, the *existence* of s, t pairs that are *not* connected by short (of length at most ℓ) paths; see Theorem 3—the function $P_L(n)$ is in fact a threshold for diameter. However, we can prove that one only needs a tiny bit more edges, namely $p \geq P_U$ where

$$P_U(n, \ell) = 2\, n^{-1+\frac{1}{\ell}} \left(\log \left(n^2 \log n\right)\right)^{\frac{1}{\ell}}$$

to ensure that *almost all* s, t-pairs are ℓ–robust; see Theorem 4. In summary, interesting phase transitions take place when p is near to $n^{-1+1/\ell}$, meaning that the graph has about $n^{1+1/\ell}$ edges.

A preliminary presentation of our results has been given at the *IFIP International Conference on Theoretical Computer Science*; see [12]. Detailed supporting computations done with the symbolic manipulation system MAPLE are described in [8].

1 Avoiding permutations

The main problem treated in this paper is that of estimating the expected number of "avoiding pairs" of length ℓ between a random source and a random destination in a random graph G obeying the $\mathcal{G}_{n,p}$ model. (An avoiding pair of length ℓ means an unordered pair of paths, each of length ℓ, that connect a common source to a common destination, and have no edge in common though they may share some nodes). This problem necessitates the solution of enumeration problems that involve two major steps:

— Enumerate "avoiding permutations" (defined below) of size $n = \ell + 1$ that can be viewed as hamiltonian paths on the set of nodes $\{1, \ldots, \ell + 1\}$, connecting the source 1 and the destination $\ell + 1$, and having no edge of type $(i, i + 1)$ or $(i, i - 1)$.

— Enumerate "avoiding paths", that are simple paths allowed to contain outer nodes taken from outside the integer segment $[1, \ell + 1]$ and otherwise satisfy the constraints of avoiding permutations. This situation is close to the random graph problem since it allows nodes drawn from the pool of vertices available in the graph $G \in \mathcal{G}_{n,p}$.

The first problem is the object of this section. It is of independent combinatorial interest as it is equivalent to counting special cyclic permutations with restrictions on adjacent values. It then serves, in the next section, as a way to introduce the methods needed for the complete random graph problem that builds upon the enumeration of avoiding pairs. Both problems rely heavily on counting by generating functions (GF's) on which is grafted an analytic form of the inclusion-exclusion principle, a familiar tool from combinatorial analysis.

1.1 Symbolic enumeration methods

We use here a symbolic approach to combinatorial enumeration, according to which many general set-theoretic constructions have direct translations over generating functions. A specification language for elementary combinatorial objects is defined for this purpose. The problem of enumerating a class of combinatorial structures then simply reduces to finding a proper specification, a sort of a formal grammar, for the class in terms of basic constructions. The approach we take follows the exposition in [14, 21].

In this framework, classes of combinatorial structures are defined either iteratively or recursively in terms of simpler classes by means of a collection of elementary combinatorial constructions. The approach followed resembles the description of formal languages by means of context-free grammars, as well as the construction of structured data types in classical programming languages.

A path often taken in the literature consists in decomposing the structures to be enumerated into smaller structures either of the same type or of simpler types and then in extracting, from such a decomposition, the corresponding recurrence relations. The approach developed here is direct and "symbolic", as it relies on a precise specification language for combinatorial structures [13, 15]. It is based on so-called admissible constructions that have the important feature of admitting direct translations into generating functions.

Let $\mathcal{A}$ be a class of combinatorial objects with an associated notion of size. We let $\mathcal{A}_n$ denote[1] the subset of objects in $\mathcal{A}$ that have size n and write A_n for

[1]Throughout the paper, we make use of the convention of denoting a combinatorial class ($\mathcal{A}$ or simply A), its counting sequence ($\{A_n\}$), and its generating function ($A(z)$) by similar groups of letters.

the corresponding cardinality. The ordinary generating function (OGF) of the sequence $\{A_n\}$ (or equivalently of the class $\mathcal{A}$) is then defined as

$$A(z) = \sum_{n=0}^{\infty} A_n z^n.$$

Next, consider a binary construction Φ that associates to two classes of combinatorial structures $\mathcal{B}$ and $\mathcal{C}$ a new class

$$\mathcal{A} = \Phi(\mathcal{B}, \mathcal{C})$$

in some finite way. The Φ is admissible iff the counting sequence $\{A_n\}$ of $\mathcal{A}$ is a function of the counting sequences $\{B_n\}$ and $\{C_n\}$ of $\mathcal{B}$ and $\mathcal{C}$ *only*:

$$\{A_n\} = \Xi[\{B_n\}, \{C_n\}].$$

In that case, there exists a well defined operator Ψ relating the corresponding ordinary generating functions.

$$A(z) = \Psi[B(z), C(z)]$$

(The notion generalizes to unary, ternary, etc, constructions in an obvious way.) In this work, we will basically use three important constructions: union, product and sequence, which we describe below.

(i) *Union Construction.* The disjoint union $\mathcal{A}$ of two classes $\mathcal{B}, \mathcal{C}$, written $\mathcal{A} = \mathcal{B} + \mathcal{C}$, is the union (in the standard set-theoretic sense) of two disjoint copies, $\mathcal{B}^o$ and $\mathcal{C}^o$, of $\mathcal{B}$ and $\mathcal{C}$. (Formally, we can introduce two distinct "markers" ϵ_1 and ϵ_2, each of size zero, and define the (disjoint) union of $\mathcal{B}, \mathcal{C}$ by $\mathcal{B} + \mathcal{C} = (\{\epsilon_1\} \times \mathcal{B}) \cup (\{\epsilon_2\} \times \mathcal{C})$.) Then one has $A_n = B_n + C_n$ so that the ordinary generating function is

$$A(z) = B(z) + C(z).$$

(ii) Product Construction. If $\mathcal{A}$ is the cartesian product of two classes $\mathcal{B}$ and $\mathcal{C}$ $, written \mathcal{A} = \mathcal{B} \times \mathcal{C}$, then, considering all possibilities, the counting sequences corresponding to $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are related by the convolution relation:

$$A_n = \sum_{k=0}^{n} B_k \cdot C_{n-k}$$

and the ordinary generating function satisfies accordingly

$$A(z) = B(z) \cdot C(z).$$

(iii) *Sequence Construction.* If $\mathcal{C}$ is a class of combinatorial structures then the sequence class $\mathcal{A} = \mathfrak{S}\{\mathcal{C}\}$ is defined as the infinite sum

$$\mathfrak{S}\{\mathcal{C}\} = \{\epsilon\} + \mathcal{C} + (\mathcal{C} \times \mathcal{C}) + (\mathcal{C} \times \mathcal{C} \times \mathcal{C}) + \cdots$$

with ϵ being a "null structure", meaning a structure of size 0. (The null structure plays a rôle similar to that of the empty word in formal language theory while the sequence construction is analogous to the Kleene star operation, $\mathcal{C}^\star$.) By the two previous rules, the ordinary generating function of the sequences is given by

$$A(z) = 1 + C(z) + C^2(z) + C^3(z) + \cdots = \frac{1}{1 - C(z)},$$

where the geometric sum converges in the sense of formal power series provided[2] $[z^0]C(z) = 0$.

In the sequel, we represent the constructions of disjoint union, product, and sequence by

Union, Prod, Sequence.

Various combinatorial objects are specified in terms of them, and by the discussion above, each such specification is automatically translated into generating function equations. Our naming conventions are consistent with those of the MAPLE library Combstruct, that itself implements the ideas of [13, 15]. As a matter of fact, Combstruct is used heavily in order to support and check the necessary calculations; see [8].

1.2 Enumeration of avoiding permutations

In this subsection, we discuss a toy problem of intrinsic combinatorial interest that shows in the small all the essential features of what is needed for the complete random graph problem: *In how many ways can a kangaroo jump from 1 to n by visiting all the nodes $\{1, \ldots, n\}$ once and only once, while making jumps (in number $\ell = n - 1$) that always avoid nearest neighbours?* A more serious definition is as follows:

Definition 2 *An* avoiding permutation *of size n is a sequence $\tau = [\tau_1, \tau_2, \ldots, \tau_n]$ that is a permutation of $[1, \ldots, n]$ satisfying the conditions: $\tau_1 = 1$, $\tau_n = n$, and $\tau_{i+1} - \tau_i \neq \pm 1$ for all i such that $1 \leq i < n$.*

Clearly, such a permutation encodes a simple path from node 1 to node n ,

$$1 = \tau_1 \to \tau_2 \to \cdots \to \tau_{n-1} \to \tau_n = n,$$

that has no edge in common with the line graph $1 \to 2 \to \cdots \to n$. We shall principally operate with such a *graphical interpretation* of arrays $[\tau_1, \ldots \tau_n]$. In this graphical representation, for a path, we reserve the term *size* for its number of distinct nodes and the term *length* for the number of its edges. Naturally, in the case of a simple path (*i.e.*, there are no repeated nodes) the length ℓ and the size n are related by $\ell = n - 1$.

[2] We use the well-established notation $[z^n]f(z)$ to represent the coefficient of z^n in the power series $f(z)$.

There are no avoiding permutations for sizes $2, 3, 4, 5$. Surprisingly, the first nontrivial configurations occur at size 6, where the 2 possibilities are $[1, 4, 2, 5, 3, 6]$ and $[1, 3, 5, 2, 4, 6]$, while for size 7, there appear to be 10 possibilities:

$$[1,3,5,2,6,4,7],[1,3,6,4,2,5,7],[1,4,2,6,3,5,7],[1,4,6,2,5,3,7],[1,4,6,3,5,2,7],$$
$$[1,5,2,4,6,3,7],[1,5,3,6,2,4,7],[1,5,3,6,4,2,7],[1,6,3,5,2,4,7],[1,6,4,2,5,3,7].$$

The goal in this subsection is to determine the number Q_n of avoiding permutations of size n. The generating function to be obtained is expressible in terms of the basic quantity

$$\mathbf{F}(z) := \sum_{n=0}^{\infty} n!\, z^n,$$

that is the (divergent) OGF of permutations and factorial numbers. This divergent series is actually a particular hypergeometric series (corresponding to ${}_2F_0[1, 1; z]$; see [10]) that was studied analytically already by Euler, and in the MAPLE language it is expressed as 'hypergeom $([1, 1], [\ ], z)$'.

Theorem 1 *Avoiding permutations have ordinary generating function*

$$Q(z) := \sum_n Q_n z^n = \frac{z}{1+z} + \frac{z}{(1+z)^2}\mathbf{F}\left(z\frac{1-z}{1+z}\right),$$

where $\mathbf{F}$ *is the divergent OGF of all permutations. Equivalently, the number of avoiding permutations* Q_n *is a double binomial sum:*

$$\begin{aligned} Q_{n+2} &= (-1)^{n-1} + \sum_{k_2=0}^{n} \sum_{k_1=0}^{n-k_2} (-1)^{k_1+k_2} \times \\ &\quad \times (n-k_1-k_2)! \binom{n-k_1-k_2}{k_1} \binom{n+1-k_1}{k_2}. \end{aligned}$$

Proof. By the inclusion-exclusion principle (see, *e.g.*, the formulation in [16]), we need to determine the number of permutations with "at least" j exceptions, where an exception is defined as a succession of values of the form $\tau_{i+1} - \tau_i = \pm 1$. More precisely, we let $P_n^{\langle j \rangle}$ be the number of permutations $[\tau_1 = 1, \tau_2, \ldots, \tau_{n-1}, \tau_n = n]$ with j exceptions *distinguished*. The number of permutations with no exception is then, by inclusion-exclusion:

$$Q_n = \sum_{j=0}^{n-1} (-1)^j P_n^{\langle j \rangle}. \tag{1}$$

Under the graphical interpretation, a permutation with distinguished exceptions can itself be regarded as including a subcollection of "exceptional" edges that belong to the graph $1 \to 2 \to \cdots \to n$. For instance, one of the elements counted by $P_{13}^{\langle 7 \rangle}$ is (only *some* of the exceptions need be distinguished)

$$\boxed{1 \to 2 \to 3} \to 4 \to \boxed{5 \to 6} \to 7 \to \boxed{11 \to 10 \to 9 \to 8} \to \boxed{12 \to 13}.$$

If we scan the integer line from left to right and group such exceptions into maximal contiguous blocks, we obtain a *template*. A template thus represents a possible pattern of exceptional edges and in general it describes many permutations. For instance, the template of the example permutation is

$$\boxed{1 \to 2 \to 3}\ ,\ 4,\ \boxed{5 \to 6}\ ,\ 7,\ \boxed{8 \leftarrow 9 \leftarrow 10 \leftarrow 11}\ ,\ \boxed{12 \to 13}\ .$$

and it will correspond to any permutation that has exceptional edges (in the cycle traversal order)

$$(1,2)\,;\ (2,3)\,;\ (5,6)\,;\ (9,8)\ ;\ (10,9)\,;\ (11,10)\,;\ (12,13).$$

At this stage, the proof strategy can be enunciated: (A) describe symbolically templates; (B) effect the enumeration by GF's of templates from their symbolic description; (C) relate the counting problems for templates and for permutations with distinguished exceptions (this is achieved by a specific transform over GF's); (D) conclude about the enumeration of avoiding permutations. We now carry out this programme.

A. Symbolic description of templates. From the definition, a template can be defined directly as made of blocks that are either: (i) isolated points (P); (ii) maximal blocks of contiguous unit intervals oriented left to right (LR); (iii) maximal blocks of contiguous unit intervals oriented right to left (RL). There is the additional constraint that the first and last blocks cannot be of type RL (one starts from 1 "pointing East" and arrives at n "from the West").

First, the three types of blocks in a template are described by the following rules[3],

$$\begin{aligned} P &= Z; \\ LR &= \mathrm{Prod}(\beta_{LR}, Z, \mathrm{Sequence}(\mathrm{Prod}(\mu, Z), \mathrm{card} \geq 1)); \\ RL &= \mathrm{Prod}(\beta_{RL}, Z, \mathrm{Sequence}(\mathrm{Prod}(\mu, Z), \mathrm{card} \geq 1)); \end{aligned}$$

corresponding to isolated points (P), LR blocks and RL blocks respectively. By convention, Z represents an "atom" of size 1 meant to specify an arbitrary node in the graphical representation of templates and permutations. The symbols β_{LR}, β_{RL} mark the beginning of each LR or RL block; μ serves as an additional marker for measuring length (*i.e.*, the number of edges) of LR/RL-blocks. (Clearly, LR and RL are combinatorially isomorphic.) Here, the markers are taken to have size 0 and they will serve in the later application of the inclusion-exclusion argument.

Next, let $\{a, b\}$ be a binary alphabet. The collection of strings beginning and ending with a letter a is specified as follows:

$$S_0 = \mathrm{Prod}(\mathrm{Sequence}(\mathrm{Prod}(a, \mathrm{Sequence}(b))), a) \tag{2}$$

[3] $\mathrm{Sequence}(A, \mathrm{card} \geq k_0)$ is a "macro" that denotes sequences with at least k_0 components.

(It suffices to decompose according to each occurrence of the letter a). Then, the grammar of templates is completed by substituting into S_0

$$a = \mathrm{Union}(P, LR), \quad b = RL.$$

B. Template enumeration. Let $T_{n,k,j}$ be the number of templates with size n (the number of nodes), k blocks of type either LR or RL in total, and j exceptional edges (that is, the cumulated lengths of the LR and RL blocks). Here, we determine the trivariate GF,

$$T(z,u,v) = \sum_{n,k,\ell} T_{n,k,\ell}\, z^n u^k v^l.$$

The generating function equations for templates can be obtained mechanically from the translation rules from constructions to GF's, as detailed in Section 1.1. (We briefly sketch the translation as a pedagogical aside.) First, the set of words made of a's and b's that start and end with an a is described symbolically by S_0 above and the GF is

$$S_0(a,b) = \frac{1}{1 - a \cdot \frac{1}{1-b}} \cdot a\,.$$

This is because $(1-f)^{-1} = 1 + f + f^2 + f^3 + \cdots$ generates symbolically all sequences of objects of type f. Thus, $S_0(a,b)$ enumerates sequences of objects of type $\frac{a}{1-b}$ that end with an a. On the other hand, $a/1-b$ represents an a prepended to a sequence of objects of type b. Therefore, globally, $S_0(a,b)$ represents all sequences described by the combinatorial type S_0 of (2).

Take next the three types of blocks: isolated (P), LR, and RL. The GF's are, respectively, z, $LR(z) = z^2/(1-z)$, $RL(z) = z^2/(1-z)$. This is because isolated points are always of size 1 (and the specification Z translates to the GF z), while LR and RL objects must be of size at least 2 (we have thus to multiply with z^2). Since the first and the last blocks can only be isolated points or LR blocks, the univariate GF for blocks is obtained by substituting a by $z + LR$ (isolated point or LR block) and b by RL in S_0. This gives here

$$\begin{aligned} T(z,1,1) &= \frac{1}{1 - \frac{z}{1-z} \cdot \frac{1}{1-\frac{z^2}{1-z}}} \cdot \frac{z}{1-z} = z\frac{1-z-z^2}{(1-z)(1-2z-z^2)} \\ &= z + 2z^2 + 4z^3 + 9z^4 + 21z^5 + 50z^6 + 120z^7 + 289z^8 + \cdots. \end{aligned}$$

(Strangely enough, this is already listed as sequence **A024537** in Sloane's *Encyclopedia of Integer Sequences* [24].)

Finally, we make use of markers. These have size 0 (hence they do not affect the total size measured by the main variable z) and they can be replaced by variables that record useful additional information. The total number of blocks is translated into the variable u, which corresponds to the translation

$$\beta_{LR} \mapsto u, \qquad \beta_{RL} = u.$$

The variable v keeps track of the total length of LR and RL blocks where the marker μ had been purposely introduced so as to record all the relevant exceptional edges; thus the substitution

$$\mu \mapsto v$$

is also effected. In this way, we get the following trivariate GF for templates:

$$\begin{aligned} T(z,u,v) &= \left(1 - \frac{u\left(z + \frac{z^2 v}{1-vz}\right)}{1 - \frac{uz^2 v}{1-vz}}\right)^{-1} \cdot u\left(z + \frac{vz^2}{1-vz}\right) \\ &= \frac{uz(1-vz-uz^2v)}{1-(2v+u)z+v^2z^2+uv^2z^3}. \end{aligned} \tag{3}$$

C. The inclusion-exclusion transform. By fixing the way blocks of a template are chained together, one obtains a permutation with a distinguished set of exceptions to the rule defining avoiding permutations. Counting the number of ways to do so yields the relation

$$P_n^{\langle j\rangle} = \sum_k T_{n,k,j}\,\gamma(k) \tag{4}$$

where $\gamma(k)$ is the modified factorial:

$$\gamma(1) = 1, \qquad \gamma(k) = (k-2)! \quad \text{for } k \geq 2. \tag{5}$$

The reason for the factorial is that any such chaining is determined by an arbitrary permutation of the $k-2$ intermediate blocks when $k \geq 2$.

We have obtained above an explicit rational expression (3) for the trivariate GF $T(z,u,v)$ of the $T_{n,k,j}$. In terms of this GF, one can express the OGF $Q(z)$ of the Q_n as an integral transform of $T(z,u,v)$. The starting point is the simple combination of (1) and (4) into

$$Q_n = \sum_{k,j} (-1)^j \gamma(k) T_{n,k,j}, \tag{6}$$

with $\gamma(k)$ as defined in (5). The usual Eulerian integral,

$$\int_0^\infty e^{-u} u^k du = k!\,,$$

provides a way to transform a monomial u^k into a factorial $k!$ by integrating against the exponential kernel e^{-u}. It then suffices to introduce the operator $\mathcal{L}$:

$$\mathcal{L}\left(h(u)\right) = \int_0^\infty e^{-u}\left(\mathrm{h}(u) - (u-u^2)\left(\frac{\partial}{\partial u}\mathrm{h}(u)\right)_{u=0}\right)\frac{du}{u^2}. \tag{7}$$

It is easily recognized that this is a linear transformation (akin to the Laplace transform) whose effect is precisely to transform a series in u into a number according to the rule

$$u^k \mapsto \gamma(k).$$

Finally, the sign alternation in (6) is taken care of by the substitution $v \mapsto -1$. Thus, the OGF $\mathrm{Q}(z) = \sum Q_n z^n$ satisfies the main equation

$$Q(z) = \mathcal{L}(\mathrm{T}(z, u, -1)). \tag{8}$$

D. Final evaluations. Application of the $\mathcal{L}$-transformation (that counts the number of ways to connect the blocks) requires a mildly amended form of T (where terms of degrees 1 and 2 only are adjusted).From (3) used in conjunction with (7) and (8), there derives an integral representation of the ordinary generating function of avoiding permutations,

$$Q(z) = z \int_0^\infty \frac{(uz^2 + (2-u)z + 1)}{(1+z)(uz^2 + (1-u)z + 1)} \cdot e^{-u}\, du,$$

that calls for evaluation.

In such a situation, we can always perform a partial fraction expansion with respect to the variable u (here this is trivial as the denominator has a u-degree of 1). This reduces the integral to a canonical form that now involves the exponential integral [1, Ch. 5],

$$\mathrm{E}_1(x) := \int_x^\infty \frac{e^{-t}}{t}\, dt.$$

The following closed form is then easily obtained:

$$Q(z) = z \left(\frac{1}{z+1} + \frac{1}{z^2-1} e^{\frac{z+1}{z(z-1)}} \mathrm{E}_1\left(\frac{z+1}{z(z-1)} \right) \right).$$

Since one deals with ordinary generating functions, the last expression is to be taken as a formal (asymptotic) series as $z \to 0$. Indeed, we have from the classical expansion of the exponential integral at infinity

$$e^{1/y}\mathrm{E}_1\left(\frac{1}{y}\right) \sim \left(y - 1!y^2 + 2!y^3 - 3!y^4 + 4!y^5 - \cdots\right) \qquad (y \to 0).$$

Thus, everything can be re-expressed in terms of the hypergeometric function $\mathbf{F}$, *i.e.*, the OGF of factorial numbers (set $y = z(z-1)/(z+1)$). One gets the expression for $Q(z)$ as stated. Finally this form of $Q(z)$ is expanded using the binomial theorem, and double combinatorial sums result for the coefficients. □

Though they have no immediate bearing on the graph problem at hand, we mention two interesting consequences of this theorem.

Corollary 1 *The quantities Q_n satisfy the recurrence*

$$(n+1)Q_n + Q_{n+1} - 2nQ_{n+2} + 4Q_{n+3} + (n+3)Q_{n+4} - Q_{n+5} = 0, \tag{9}$$

where $Q_0 = 0$, $Q_1 = 1$, $Q_2 = Q_3 = Q_4 = Q_5 = 0$. Asymptotically, one has

$$\frac{Q_n}{(n-2)!} = e^{-2}\left(1 + O\left(\frac{1}{n}\right)\right). \tag{10}$$

Proof. First, the generating function $Q(z)$ is obtained from classical special functions (the exponential integral or the hypergeometric functions) by rational operations and substitutions. Many such functions fall into what Zeilberger [26] has named the "holonomic class": a function (or a power series) is *holonomic* if it satisfies a linear differential equation with coefficients that are rational (equivalently polynomial) functions. Holonomic functions enjoy a rich set of closure properties, including closure under sums and products, integration and differentiation, as well as algebraic substitutions. The MAPLE package Gfun due to Salvy and Zimmermann [23] actually implements these closure properties.

Here, since the exponential integral (also, its hypergeometric cognate) is clearly holonomic, we may take advantage of the Gfun package and build up automatically a differential equation satisfied by $Q(z)$:

$$(z^4+z^5+4z^3-1-z+4z^2)Q(z)+(-2z^4+z^2+z^6)\frac{\partial Q(z)}{\partial z}-2z^4-4z^3-z^5+z=0.$$

From this the recurrence follows by elementary properties of generating functions: multiplication by z corresponds to a shift of coefficient indices, while differentiation essentially multiplies coefficients by n. In this way, the recurrence (9) is established (it is also conveniently obtained in an automatic fashion by the Gfun package).

Regarding asymptotics, we may take advantage of the expression involving the divergent series $\mathbf{F}$. The following general principle proves especially useful: *One has*

$$[z^n]\mathbf{F}\left(z + dz^2 + O\left(z^3\right)\right) = n!\, e^d(1 + o(1))$$

provided that the argument of the hypergeometric $\mathbf{F}$ *is analytic at the origin, so that its coefficients grow at worst exponentially.* (Elementary coefficient manipulations in the style of [5, Sec. 5] establish this.) But, given this principle, the expression already obtained for $Q(z)$, and the fact that

$$\frac{z(1-z)}{1+z} = z - 2z^2 + 2z^3 - 2z^4 + 2z^5 + \mathrm{O}(z^6),$$

the main asymptotic estimate of (10) immediately results. □

The recurrence above implies the non-obvious fact that each number of avoiding permutations Q_n is computable in a constant number of arithmetic operations—a contrast with the quadratic cost of the double combinatorial sum. The GF found in this way starts as

$$z+2\,z^6+10\,z^7+68\,z^8+500\,z^9+4174\,z^{10}+38774\,z^{11}+397584\,z^{12}+4462848\,z^{13}+\cdots.$$

The asymptotic estimate extends properties known for permutations with excluded patterns (*e.g.*, derangements have asymptotic density e^{-1}; see [3, Sec. 4.3] for a more general result). Consequently, a nonzero proportion (about 13.53%) of all cyclic permutations that start with 1 and end with n are avoiding. Similar techniques can be employed to analyse more general avoidance rules (*e.g.*, excluding any fixed finite set of jumps); see [8, 25]. The net result is that the corresponding divergent OGF's are compositions of the $\mathbf{F}$ function with algebraic functions themselves determined by finite-state models and their associated rational functions.

2 The random graph model

We now turn to the analysis of robustness in the random graph model $\mathcal{G}_{n,p}$. A crucial step consists in enumerating what we call "avoiding paths" (Subsection 2.1) where we build upon the methods already developed for avoiding permutations. The transfer to the random graph model $\mathcal{G}_{n,p}$ is then easy (Subsection 2.2).

2.1 Avoiding paths

Define an avoiding path of type (n, j) by the fact that it satisfies the basic constraints of avoiding permutations regarding the base line $(1, 2, \ldots, n)$, but contains j "outer nodes" taken to be indistinguishable and anonymously represented by the symbol '$\star$'. Precisely, an avoiding path of type (n, j) is a sequence $[\tau_1, \ldots, \tau_n]$ such that each τ_i is in $\{1, \ldots, n\} \cup \{\star\}$ satisfying the conditions: $\tau_1 = 1$ and $\tau_n = n$; no numeric value amongst the τ_i's is repeated; $\tau_{i+1} - \tau_i \neq \pm 1$ if τ_{i+1} and τ_i are both numeric; the number of τ_i's that equal $\star$ is exactly j. For instance, for types $(n, j) = (3, 1), (4, 1), (4, 2)$, the listings are respectively

$$\{[1, \star, 3]\} \qquad \{[1, 3, \star, 4], [1, \star, 2, 4]\} \qquad \{[1, \star, \star, 4]\}\,.$$

We consider now the problem of counting the number $Q_{n,j}$ of avoiding paths of type (n, j), where n is the size (the number of nodes) and j is the number of "outer nodes".

Proposition 1 *The number of avoiding paths of type (n, j) with $j \geq 1$ is expressible as*

$$\begin{aligned} Q_{n+2,j} \quad = \quad & \sum_{k_2=0}^{n-j} \sum_{k_1=0}^{n-j-k_2} (-1)^{k_1+k_2} (n - k_1 - k_2)! \\ & \binom{n-j-k_1-k_2}{k_1} \binom{n-j+1-k_1}{k_2} \binom{n-k_1-k_2}{j}^2 . \end{aligned}$$

Note that the combinatorial sum on the right hand side extends the one for avoiding permutations in the sense that $Q_n = (-1)^{n-1} + Q_{n,0}$.

Proof. It appears convenient to relax the constraints a bit and not to impose *a priori* the number of outernodes. In so doing, we enumerate ordered pairs of paths $\alpha = (\pi_1, \pi_2)$, called "relaxed pairs", where π_1, π_2 may or may not be of the same length. The first path will be called the "ground path" and its nodes are assumed to be labelled in the canonical order $1, 2, \ldots, |\pi_1|$. The second path (*i.e.*, the "avoiding path") is *not* allowed to have any edge of type $(i, i+1)$ or $(i, i-1)$ (nor to contain any repeated label, evidently); in addition, it may contain outside nodes written as $\star$ that represent nodes *not* in the ground path. We let $\overline{Q}_{\nu,m_1,m_2}$ be the number of relaxed pairs that comprise a total of ν nodes and are such that the nodes of $\pi_2 \setminus \pi_1$ (with π_1, π_2 taken here as sets of nodes) are in number m_1 while there are m_2 nodes in $\pi_1 \setminus \pi_2$. This sequence extrapolates the sought sequence $Q_{n,j}$ in the sense that $Q_{n,j} = \overline{Q}_{2n,j,j}$.

The counting is achieved by modifying the templates introduced in Section 1. We omit the somewhat lengthy details as they are conceptually very similar (see also [8] where detailed specifications are spelled out with ample confirmation of the formula above by exhaustive combinatorial listings). The idea is now to distinguish "inner nodes" that are in $\pi_1 \setminus \pi_2$, "outer nodes" belonging to $\pi_2 \setminus \pi_1$, and "joint" nodes from $\pi_1 \cap \pi_2$. The constraints are seen to remain of the finite-state type, corresponding to regular expressions that only involve the combinatorial constructions 'Union, Prod, Sequence'.

We can then proceed with the enumeration of modified templates. Let $\overline{T}(z, u, v, w_1, w_2)$ be the generating function in *five* variables, where z records the total number of nodes, v records the total length of LR and RL blocks (needed for inclusion-exclusion as it gives the number of exceptions), u records the number of such blocks (needed to apply the integral transform); the variables w_1, w_2 record the number of points on each one of the two paths that does not belong to its companion. The generating function $\overline{T}(z, u, v, w_1, w_2)$ then mechanically results (details omitted). For inclusion-exclusion, we must set $v = -1$, then modify T to make it comply with the form needed to apply the transform (7) and define

$$\overline{T}^{\circ}(z, u, w_1, w_2) = \overline{T}(z, u, -1, w_1, w_2) - (u - u^2)\left(\frac{\partial}{\partial u}\overline{T}(z, u, -1, w_1, w_2)\right)_{u=0}.$$

Then the integral transform technique applies via relation (8). Let $\overline{Q}(z, w_1, w_2)$ be the GF of the $\overline{Q}_{n,m_1,m_2}$ defined at the beginning of the proof as counting relaxed pairs of type (n, m_1, m_2). One obtains in this way

$$\overline{Q}(z, w_1, w_2) = \int_0^{\infty} \overline{T}^{\circ}(z, u, w_1, w_2) e^{-u}\, du, \tag{11}$$

where

$$\begin{aligned} \overline{T}^{\circ} &= \frac{z^2}{1+z^2}\left(1 + \frac{z^2}{D}\right) \\ D &= 1 - z(w_1 + w_2) + z^2(1 - u + w_1 w_2) - z^3(w_1 + w_2) + z^4(u + w_1 w_2). \end{aligned}$$

(It is comforting to note that the expression is symmetric in w_1, w_2!)

An exponential integral form is obtained which is eventually reduced to the final hypergeometric form that involves the GF of factorials:

$$\begin{aligned}\overline{Q}(z, w_1, w_2) = \frac{z^2}{1+z^2} + \\ \frac{z^4}{(1+z^2)^2(1-zw_1)(1-zw_2)} \mathbf{F}\left(\frac{z^2(1-z^2)}{(1+z^2)(1-zw_1)(1-zw_2)}\right).\end{aligned} \tag{12}$$

This is our main formula and it reduces to $Q(z^2)$, as it should, upon setting $w_1 = w_2 = 0$. From there, the expansion in terms of binomials is straightforward and $Q_{n,j}$ is determined as the coefficient $[z^{2n} w_1^j w_2^j]\overline{Q}(z, w_1, w_2)$. □

2.2 Average-case analysis of the random graph model

We discuss now how to estimate the robustness to link failures in a random graph that obeys the $\mathcal{G}_{n,p}$ model. An *avoiding pair* of length ℓ in a graph is an *unordered* pair of paths, each of length ℓ, with a common source and a common destination, that may share some nodes, but are totally edge-disjoint. We have an exact characterization of the non-asymptotic regime:

Theorem 2 *The mean number of avoiding pairs of length ℓ between a random source and a random destination in a random graph obeying the $\mathcal{G}_{n,p}$ model is*

$$N_\ell(n, p) := \frac{p^{2\ell}}{2n(n-1)} \sum_{j=0}^{\ell} Q_{\ell+1,j} \binom{n}{l+1+j} (l+1+j)!$$

where the coefficients $Q_{n,j}$ are given by Proposition 1.

Since the $\mathcal{G}_{n,p}$ model implies isotropy, the quantity $N_\ell(n, p)$ is also the mean number of avoiding pairs between *any fixed* source and destination s, t.

Proof. The coefficient $1/2$ corresponds to the fact that one takes unordered pairs of paths; the coefficient $1/(n(n-1))$ averages over all possible sources and destinations; the factor $p^{2\ell}$ provides the edge weighting corresponding to $\mathcal{G}_{n,p}$; the arrangement numbers $\binom{n}{l+1+j}(l+1+j)!$ account for the number of ways to embed an avoiding path into a graph by choosing certain nodes and assigning them in some order to an avoiding path; the coefficients $Q_{\ell+1,j}$ provide the basic counting of avoiding paths that build up avoiding pairs. □

Robustness. A short table of initial values of $N_\ell(n, p)$ follows:

$$\begin{aligned}
&N_2 = \tfrac{1}{2}(n-2)(n-3)p^4, \qquad N_3 = \tfrac{1}{2}(n-2)(n-3)^2(n-4)p^6,\\
&N_4 = \tfrac{1}{2}(n-1)(n-2)(n-3)(n-4)(n-5)^2 p^8,\\
&N_5 = \tfrac{1}{2}(n-2)(n-3)(n-4)(n-5)^2(n^3 - 11n^2 + 25n + 32)p^{10}.
\end{aligned}$$

From developments in the previous section, the formulæ are computable in low polynomial time (as a function of ℓ) and they describe exactly the *non-asymptotic regime*. This makes it possible to determine the mean number of avoiding pairs in graphs of a given size for all reasonable values of n, p, ℓ. Take for instance a graph with $n = 10^5$ nodes and an edge probability $p = 5 \cdot 10^{-5}$. This corresponds to a mean node degree that is extremely close to 5, so that, on average, each node has 5 neighbours. Then the mean values are

$$N_2 = 3.1 \cdot 10^{-8},\ N_3 = 7.8 \cdot 10^{-7},\ N_4 = 1.9 \cdot 10^{-5},\ N_5 = 4.8 \cdot 10^{-4},\ N_6 = 1.2 \cdot 10^{-2},$$
$$N_7 = 0.30,\ N_8 = 7.6,\ N_9 = 190,\ N_{10} = 4763,\ N_{11} = 119052,\ N_{12} = 2.9 \cdot 10^6.$$

Thus, in this example, one expects to have short and multiple connections between source and destination provided paths of length 8 are allowed. This numerical example also shows that there are rather sharp transitions. The formula of Theorem 2, that entails the following rough approximation

$$N_\ell(n,p) \approx \frac{1}{2}\, n^{2\ell-2} p^{2\ell} \tag{13}$$

precisely accounts for such a sharpness phenomenon.

In the introduction, we have defined ℓ–robustness as multiple connectivity by edge-disjoint paths of length *at most* ℓ. In fact, Equation (12) gives access to explicit expressions for relaxed pairs of type (ℓ_1, ℓ_2) that are made of two paths, of lengths ℓ_1, ℓ_2. It can then be seen that the bottleneck for existence of pairs (ℓ_1, ℓ_2) with ℓ_1, ℓ_2 at most ℓ is in fact the case (ℓ, ℓ). Thus, since $N_\ell(n,p) \to 0$ when $\frac{p}{P_M(n,\ell)} \to 0$, the function

$$P_M(n,\ell) = 2^{\frac{1}{2\ell}} n^{-1+\frac{1}{\ell}}$$

is a "cut-off" point for ℓ–robustness (in a mean value sense) and an $(\leq \ell, \leq \ell)$-avoiding pair is expected or not depending on whether p/P_M tends to 0 or to ∞.

Corollary 2 *Any fixed pair in a $\mathcal{G}_{n,p}$ graph is almost surely not ℓ–robust if $p/P_M(n,\ell) \to 0$.*

Proof. When $\frac{p}{P_M(n,\ell)} \to 0$, then the expected number $N_\ell(n,p)$ of the desired pairs of paths tends to 0 and so does the probability of existence of at least one such pair of paths (by Markov's inequality or by direct reasoning). Thus, with probability tending to 1, there is no pair of edge-disjoint paths between the two vertices and these two vertices are, almost certainly, not ℓ–robust. □

3 Thresholds in the random graph model

In this section, we examine properties that hold "almost surely" (a.s.), a term synonymous to "with probability tending to 1 as $n \to \infty$". We provide bounds for the probability (and thus the threshold) of existence, between pairs of vertices, of two edge-disjoint paths of length at most ℓ, by proving the following:

— We give an estimation of the "lower threshold" value $P_L \equiv P_L(n, \ell)$ such that $\mathcal{G}_{n,p}$ graphs with $p \leq P_L$ do not satisfy the desired property of the existence, between all pairs of nodes, of two edge-disjoint paths with probability tending to 1 as n goes to infinity.

— We present an "upper threshold" value $P_U \equiv P_U(n, \ell)$ such that almost every $\mathcal{G}_{n,p}$ graph with $p \geq P_U$ has almost all its (source-destination) pairs of vertices connected by at least two edge-disjoint paths of length at most ℓ.

Theorem 3 *Define*

$$P_L(n, \ell) = \left(\log \frac{n^2}{\log n}\right)^{\frac{1}{\ell}} n^{-1+\frac{1}{\ell}}.$$

Then, for $p \leq P_L(n, \ell)$, almost surely, there exists a pair of vertices in the $\mathcal{G}_{n,p}$ graph that does not have the ℓ–robustness property.

Proof. Use the threshold function for diameter ℓ (see [7], Theorem 10, p. 233), and the fact that the property of having diameter at most ℓ is a monotone increasing property for random graphs. □

Theorem 4 *Define*

$$P_U(n, \ell) = 2\left(\log\left(n^2 \log n\right)\right)^{\frac{1}{\ell}} n^{-1+\frac{1}{\ell}}.$$

Then, for $p \geq P_U(n, \ell)$, almost surely, almost all pairs of vertices of a $\mathcal{G}_{n,p}$ graph have the ℓ–robustness property.

Proof. Consider two independent distributions $\mathcal{G}_{n,p_1}$ and $\mathcal{G}_{n,p_2}$ on the same set of vertices. Let E_i $(i = 1, 2)$ be the events "$\mathcal{G}_{n,p_i}$ has diameter ℓ".

Consider the graph $\widetilde{G}$ obtained when we superimpose an instance $G' \in \mathcal{G}_{n,p_1}$ and an instance $G'' \in \mathcal{G}_{n,p_2}$ and OR them (*i.e.*, $\widetilde{G}$ has an edge joining u, v iff at least one of G', G'' has). Clearly $\widetilde{G}$ is a $\mathcal{G}_{n,p}$ object with

$$p = p_1(1 - p_2) + p_2(1 - p_1) + p_1 p_2 = p_1 + p_2 - p_1 p_2.$$

In fact, if u, v are joined in G' by a path π_1 and in G'' by a path π_2, then these two paths both exist in $\widetilde{G}$. For p equal to the threshold for constant diameter ℓ of $\mathcal{G}_{n,p}$, the number of pairs u, v of $\widetilde{G}$ for which the paths of G', G'' overlap in some edge is $o(n^2)$; thus the vast majority of pairs of vertices (there are $n^2 - o(n^2)$ of them) in $\widetilde{G}$ are connected via two edge-disjoint paths of length at most ℓ.

If suffices to take $p = p_1 + p_2 - p_1 p_2$ with $p_1 = p_2 = p_0^{(\ell)}$ and $p_0^{(\ell)}$ a threshold for diameter ℓ, so that

$$p \leq 2p_0^{(\ell)} - \left(p_0^{(\ell)}\right)^2 .$$

Precisely, we can then adopt for p the value

$$P_U = 2(2\log n - \log c)^{\frac{1}{\ell}} n^{\frac{1}{\ell}-1}$$

where c is adjusted to $1/\log n$ (see [7], Corollary 12, p. 237), so that the diameter is almost surely ℓ. □

Finally, we show how to transfer results relative to the probability of robustness of a fixed pair to an all–pairs property. This starts with an easy combinatorial lemma.

Lemma 1 *For every graph $G(V,E)$, if vertices u, v are each connected to a specific vertex $x \in V$ via two edge-disjoint paths each of length ℓ, then u, v are connected in G via two edge-disjoint paths, each of length at most 3ℓ.*

Proof. For simplicity, let the two (edge-disjoint) paths from u to x be coloured blue and the two (edge-disjoint) paths from v to x be coloured red. Take one of the two red paths and mark the first red-blue intersection vertex x_1 of it (there always exists such a vertex since at worst one may take $x_1 = x$). Now take the other red path and mark the first red–blue intersection vertex x_2 (again this vertex can be x). There are two cases:

Case 1. Vertices x_1, x_2 are in different blue paths. Then the lemma is easily proved by simply following the two different blue parts and then continuing with the two different red ones. Note that the two blue parts are edge-disjoint, the two red continuations are also edge-disjoint and there is no red-blue edge.

Case 2. Both x_1, x_2 are on the same blue path. Let x_1 the closest to u on this blue path. Take the first $u - v$ path to be from u (on this blue path) to x_1 and then from x_1 to v (by the same red path which defined x_1) and the second $u - v$ path be composed by the other red path from v to x_2, then the blue part from x_2 to x and then the unused other blue path returning to u. Again, there is obviously no edge intersection.

With respect to length, the worst case is clearly Case 2, where the second constructed path has pieces from three of the four initial paths, leading to length at most 3ℓ. □

Lemma 1 can be restated as follows: For every graph $G(V,E)$ if there exists a vertex $x \in V$ such that for all vertices $u, v \in V (u, v \neq x)$ each of u, v connects to x via two edge-disjoint paths of length at most ℓ, then the diameter of G is at most 3ℓ and each $u, v \in V$ is connected via two edge-disjoint paths of length at most 3ℓ. We use this in our last result:

Theorem 5 *Given $\mathcal{G}_{n,p}$, if $p(n,\ell)$ is such that the probability that two specific nodes of G are connected via two edge-disjoint paths of length at most ℓ is at least $1-\theta$ (where $\theta = o\left(\frac{1}{n}\right)$), then all pairs of nodes u, v of G are connected via two edge-disjoint paths of length at most 3ℓ with probability at least $1 - n\theta$.*

Proof. Consider a specific vertex $x \in V$ and let $Y(u,x)$ be the indicator random variable of the event "*u is connected to x via two edge-disjoint paths*". Let also $Z(x)$ be the sum of $Y(u,x)$ for all $u \neq x$. Then:

$$\begin{aligned}
\Pr\{\exists u : Z(x) < n-1\} &\leq \sum_{\forall u \neq x} \Pr\{Y(u,x) = 0\} \\
&= (n-1)\Pr\{Y(u,x) = 0\} \\
&= (n-1)(1 - \Pr\{Y(u,x) = 1\}).
\end{aligned}$$

If $\Pr\{Y(u,x) = 1\} \geq 1 - \theta$ (where $\theta = o\left(\frac{1}{n}\right)$), then

$$\Pr\{\exists u : Z(x) < n-1\} \leq (n-1)\theta$$

and all pairs of nodes u, v of G are each connected via two edge-disjoint paths of length at most 3ℓ with probability at least $1 - n\theta$. $\square$

Theorem 5 potentially provides an upper bound for the all pairs problem, by way of a bound χ such that for $p \geq \chi$, in an instance of $\mathcal{G}_{n,p}$, any fixed (or random) pair has the ℓ–robustness property with probability tending to 1 as n tends to infinity. The derivation of such a bound could conceivably be approached by a determination of the Second Moment of the ℓ–robustness distribution, a computation that seems to represent a major technical difficulty.

Conclusions. We have estimated here tightly and also asymptotically the mean number of ways to get at least two edge-disjoint paths between any two specific nodes of $\mathcal{G}_{n,p}$ graphs. We pose as an open problem the calculation of the second moment (this would provide bounds for the all-pairs problem). Another question of interest is the extension of the analysis to the existence of k simultaneously edge-disjoint paths.

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