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Radiocolorings in Periodic Planar Graphs: PSPACE-Completeness and Efficient Approximations for the Optimal Range of Frequencies

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Abstract. The Frequency Assignment Problem (FAP) in radio networks is the problem of assigning frequencies to transmitters exploiting frequency reuse while keeping signal interference to acceptable levels. The FAP is usually modelled by variations of the graph coloring problem. The Radiocoloring (RC) of a graph $G(V,E)$ is an assignment function $\Phi : V \to \mathbb{N}$ such that $|\Phi(u) - \Phi(v)| \geq 2$, when u, v are neighbors in G, and $|\Phi(u) - \Phi(v)| \geq 1$ when the distance of u, v in G is two. The discrete number of frequencies used is called *order* and the range of frequencies used, *span*. The optimization versions of the Radiocoloring Problem (RCP) are to minimize the span (*min span RCP*) or the order (*min order RCP*).

In this paper, we deal with an interesting, yet not examined until now, variation of the radiocoloring problem: that of satisfying frequency assignment requests which exhibit some *periodic* behavior. In this case, the interference graph (modelling interference between transmitters) is some (infinite) periodic graph. Infinite periodic graphs usually model finite networks that accept periodic (in time, e.g. daily) requests for frequency assignment. Alternatively, they can model very large networks produced by the repetition of a small graph.

A *periodic graph* G is defined by an infinite two-way sequence of repetitions of the same finite graph $G_i(V_i, E_i)$. The edge set of G is derived by connecting the vertices of each iteration G_i to some of the vertices of the next iteration G_{i+1}, the same for all G_i. The model of periodic graphs considered here is similar to the model of periodic graphs in Orlin [20], Marathe et al [17]. We focus on planar periodic graphs, because in many cases real networks are planar and also because of their independent mathematical interest.

We give two basic results:

- We prove that the *min span RCP is $\mathcal{PSPACE}$-complete for periodic planar graphs.*
- We provide an $O(n(\Delta(G_i) + \sigma))$ time algorithm, (where $|V_i| = n$, $\Delta(G_i)$ is the maximum degree of the graph G_i and σ is the number of edges connecting each G_i to G_{i+1}), which obtains a radiocoloring of a periodic planar graph G that *approximates the minimum span within a ratio which tends to 2 as $\Delta(G_i) + \sigma$ tends to infinity.*

We remark that, any approximation algorithm for the min span RCP of a finite planar graph G, that achieves a span of at most $\alpha\Delta(G) + constant$, for any α and where $\Delta(G)$ is the maximum degree of G, can be used as a subroutine in our algorithm to produce an approximation for min span RCP of asymptotic ratio α for periodic planar graphs.

1 Introduction, Previous Work and our Results

1.1 The Radiocoloring Problem

The Frequency Assignment Problem (FAP) in radio networks is a well-studied, interesting and well motivated problem, aiming at assigning frequencies to transmitters exploiting frequency reuse while keeping signal interference to acceptable levels. The FAP is usually modeled by variations of the graph coloring problem. The interference between transmitters is usually modelled by the interference graph $G(V,E)$, where the set V corresponds to the set of transmitters and E represents distance constraints. The set of colors represents the available frequencies. In addition, the color of each vertex in a particular assignment gets an integer value which has to satisfy certain inequalities compared to the values of colors of nearby nodes in the interference graph G (frequency-distance constraints).

We here study an important variation of FAP, called the Radiocoloring Problem (RCP):

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Definition 1. Radiocoloring Problem (RCP): *Given a graph $G(V, E)$ consider a function $\Phi : V \to N$ such that $|\Phi(u) - \Phi(v)| \geq 2$ if $d(u, v) = 1$ and $|\Phi(u) - \Phi(v)| \geq 1$ if $d(u, v) = 2$, where $d(u, v)$ is the distance between u and v in G. The function Φ is a radiocoloring (RC) of the graph G. We call the problem of finding such an assignment the Radiocoloring Problem (RCP).*

Two important parameters of a radiocoloring are the following:

Definition 2. order: *The number of distinct colors used in a radiocoloring assignment Φ of G is called the order of the assignment Φ.*

Definition 3. span: *The number $\nu = max_{v \in V} \Phi(v) - min_{u \in V} \Phi(u) + 1$ used in a radiocoloring assignment Φ is called the span of the assignment Φ.*

The optimization versions of the RCP corresponding to these parameters are the following:

Definition 4. min span RCP: *Given a graph G, find a radiocoloring assignment for the graph G of minimum span, denoted by $X_{span}(G)$.*

Definition 5. min order RCP: *Given a graph G, find a radiocoloring assignment for the graph G of minimum order, denoted by $X_{order}(G)$.*

Note that both versions of the RCP have been proved to be $\mathcal{NP}$-hard even for planar and other restricted families of ordinary graphs ([9, 22, 2]).

1.2 The min span RCP in periodic graphs

In this work we investigate the min span RCP for an interesting family of infinite planar graphs, called *periodic planar graphs*. A periodic graph G is defined by an infinite sequence of repetitions of the same finite graph $G_i(V_i, E_i)$. The edge set of G is derived by connecting some of the vertices of each iteration G_i to some of the vertices of the next iteration G_{i+1}, the same for all iterations.

Infinite periodic graphs usually represent finite networks that accept periodic (in time, e.g. daily) requests for frequency assignment. We note that periodic interference graphs usually represent networks of great practical interest, since in many networks the requests for frequency assignment exhibit some periodic behavior. That is, the network accepts periodic (e.g. daily) requests for frequency assignment. Each request has a starting and ending time and a node where it is applied. Two requests interfere if they apply for nearby nodes and their time intervals overlap. The assignment should be such that there is no time overlap between any two nearby requests of the same or the preceding and following periods of requests.

Alternatively, infinite periodic graphs can model very large networks produced by the repetition of a small graph. Note in this context that many real networks consist of the repetition of the same component.

We focus here on *planar* periodic graphs, because in many cases real networks are planar and because of the independent mathematical interest of this family of graphs.

Definition 6. Linear Periodic Planar Graph G: *A linear periodic planar graph is defined as follows:*

Let $\widetilde{G}$ be an arbitrary finite connected planar graph. Let V the vertex set of $\widetilde{G}$. Let also E_0 be the edge set of $\widetilde{G}$. Let E_+ be a specific set of ordered pairs (u, v) of the nodes of $\widetilde{G}$. Note that E_+ must be a set of ordered pairs of vertices whose connection according to the rule **(c2)** *below leads to planarity preservation.*

Consider the two-way infinite sequence of graphs $\ldots, G_i, G_{i+1}, \ldots$, where each G_i is isomorphic to $\widetilde{G}$. The infinite graph G is obtained from this sequence as follows:

- **(a)** *We assume a line (in fact, any 1-dimensional infinite simple curve) on which we select discrete points $\ldots, i, i+1, i+2, \ldots$, such that:*
  - **(a1)** *Each point in the line is replaced by $\widetilde{G}$.*
  - **(a2)** *Each edge $(i, i+1)$ in the line is replaced by E_+.*
  - **(a3)** *For any finite subset of consecutive points in the line, replacing the points of the line by graphs $\widetilde{G}$ end the edges between them by E_+, the resulting graph is planar.*
- **(b)** *The vertex set of G is the union of the vertex sets of the sequence $\ldots, G_i, G_{i+1}, \ldots$.*
- **(c)** *The edges of G are* **(c1)** *The union of edge sets of the sequence of G_is (i.e., the edge set E_0 of $\widetilde{G}$)*
  (c2) *For each pair of adjacent copies of $\widetilde{G}$, call them G_i, G_{i+1}, we use the E_+ specification of G to connect the nodes of G_i corresponding to the first elements of the pairs in E_+ to the nodes of G_{i+1}, corresponding to the second elements of the pairs in E_+.*

We denote a linear periodic planar graph by $G = (\widetilde{G}(V, E_0), E_+)$.

For an iteration i of G, $G_i(V_i, E_i)$, let E_{i-} and E_{i+} the set of edges connecting G_i with the previous and next iterations of G, respectively. Note that both E_{i-} and E_{i+} are the same as E_+ and that G_i is isomorphic to $\widetilde{G}$.

Note that this model of infinite periodic graphs is similar to the model of Orlin [20] for *l-dimensional periodic graphs* with $l = 1$. Our model is in fact equivalent to 1-level-restricted periodic graphs in Marathe et al [17], which is a special case of [20]. We consider the planar case of such graphs.

Definition 7. $(\widetilde{\boldsymbol{G}}, \boldsymbol{E_+})$*: The graph* $\widetilde{G}$ *along with* E_+ *is called the finite specification of* G*.*

Observe that there can be infinite periodic graphs which are not linear. For example, the periodic graph whose graph $\widetilde{G}$ in its finite specification, is a cycle connecting vertices a, b, c, d, and $E_+ = \{(a, a), (b, b), (c, c), (d, d)\}$, is not a linear periodic planar graph, since there is no line of discrete points such that each point in the line can be replaced by $\widetilde{G}$ and each edge of it can be replaced by E_+ leading to planarity. See the graph of Figure 1.

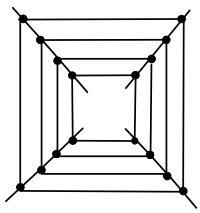

Fig. 1. This graph is not a linear periodic planar graph.

Note 1. All our results refer to linear periodic planar graphs, which we call periodic planar graphs in the sequel.

1.3 Our results

In this work we provide the following results:

1. We first prove that *the min span radiocoloring problem is* $\mathcal{PSPACE}$*-complete for periodic planar graphs.* (The space is polynomial with respect to the size of the finite specification $\widetilde{G}$.)
2. We provide an $O(n(\Delta(G_i) + \sigma))$ time algorithm, (where $|V_i| = n$, $\Delta(G_i)$ is the maximum degree of the graph G_i and σ is the number of edges connecting G_i to G_{i+1}), which obtains a radiocoloring of a periodic planar graph G that *approximates the minimum span within a ratio which tends to* 2 *as* $\Delta(G_i) + \sigma$ *tends to infinity.*

   We remark that, any approximation algorithm for the min span RCP of a finite planar graph G, that achieves a span of at most $\alpha\Delta(G) + constant$, for any α and where $\Delta(G)$ is the maximum degree of G, can be used as a subroutine in our algorithm to produce an approximation for min span periodic planar RCP of asymptotic ratio α for periodic planar graphs.

Our results provide yet another natural $\mathcal{PSPACE}$-complete problem whose optimization version is shown here to admit a polynomial time constant ratio approximation. This answers partially an interesting open question in Condon et al [5].

Note that, the same algorithmic approach can be applied to obtain an approximation for the min order RCP of rat io that tends to 2 as well.

1.4 Previous Work

The FAP has been considered in e.g. [12, 8, 7, 1, 22]. The problem of min span RCP for ordinary planar graphs was proved to be $\mathcal{NP}$-complete in [9, 10]. In the same work, the authors provide an efficient approximation algorithm for the min order RCP of planar graphs achieving an approximation ratio tending to 2. Also, a fully polynomial time randomized approximation scheme for the number of radiocolorings of a planar graph, with a given number of colors, is given. Bodlaender et al in [2] independently proved the $\mathcal{NP}$-completeness of min span RCP for bipartite planar graphs. In the same work [2] the authors presented efficient approximations for some interesting families of graphs: outerplanar, bounded treewidth, permutation and split graphs.

A model for periodic graphs (called *l-dimensional periodic graphs*) was first presented by Orlin in [20]. The model of periodic graphs considered in this work is similar to that of Orlin for the 1-dimensional case, $l = 1$ (also called *1-dimensional periodically specified graphs* or simply *periodically specified graphs*). Our model is in fact equivalent to 1-level-restricted periodic graphs in Marathe et al [17], which is a special case of [20]. We consider the planar case of such graphs.

The complexity of various basic problems of periodically specified graphs was studied by Orlin [20] and Wanke [26]. In these works the authors discuss the $\mathcal{NP}$ and $\mathcal{PSPACE}$-hardness results for infinite and finite periodic graphs. In [20, 18] it is proved that the problems of Maximum Independent Set (MIS), Hamiltonial Path, Partition into Triangles and SAT for periodically specified graphs are $\mathcal{PSPACE}$-complete. Also, in [25] the 3-coloring problem for general periodic graphs was proved to be $\mathcal{PSPACE}$-complete.

The appoximability of basic problems on infinite periodic graphs was studied by several researchers ([3, 4, 13, 15, 21]). In these works the authors gave efficient algorithms for solving problems such as determining strongly connected components, testing the existence of cycles, bipartiteness, planarity and minimum cost spanning forests for periodically specified graphs.

Marathe et al [17] presented several $\mathcal{PSPACE}$-hardness results and also efficient approximation schemes for partitioning problems including MIS, min vertex cover and max-SAT for periodically specified graphs when restricted to planar instances. However, we remark that the general framework of Marathe et al [17] does not cover the $\mathcal{PSPACE}$-completeness of min span RCP of periodic planar graphs. This is so because the methodology of the periodic 3-SAT variation used in [17] does not trivially transfer to the 3-coloring of periodic planar graphs which we use in our reduction (see their Theorem 6.5). Also, their approximation technique for periodically specified graphs (illustrated for the MIS problem) can not be used for coloring problems, considered here, because it takes the union of partial solutions-subsets of the infinite graph and thus it does not consider all the vertices; something not allowed in coloring problems.

Organization of the paper. We provide here an overview of the rest of this paper. In Section 2 we study the embeddings of a periodic planar graph. The obtained results will be used next for the approximation algorithm. In section 3 we give the $\mathcal{PSPACE}$-completeness proof of min span radiocoloring for periodic planar graphs. In Section 4 we present an efficient approximation algorithm for min span RCP of periodic planar graphs. Because of the page limit, we have included full proofs of some Lemmas and Theorems in an Appendix.

2 Embedings of Periodic Planar Graphs

In this section we study the embeddings of a periodic planar graph. The results obtained will be used next for the approximation algorithm.

In this paper, we use the notion of an *embedding* of a planar periodic graph, with its usual meaning:

Definition 8. Planar Embedding (of a periodic graph G) *([19]): For each node v of G, there is an adjacency list, such that all neighbours of v appear in clockwise order with respect to an actual drawing of G.*

The following Lemma reveals important information about the structure of a linear periodic planar graph.

Lemma 1. *Any linear periodic planar graph G can be embedded in the plane by interchanging at most two different planar embeddings of the graph obtained by an iteration i of G, G_i (which is isomorphic to $\widetilde{G}$) and the set of edges connecting G_i with the previous and next iterations, sets E_{i-}, E_{i+} (each of which is equal to E_+), called* Extended $\widetilde{G}$.

Proof. Observe first that there are cases where we need to interchange two different embeddings of the graph Extended $\widetilde{G}$ in order to draw a linear periodic planar graph preserving planarity. As an example, consider the periodic graph whose graph $\widetilde{G}$ in its finite specification is a single edge connecting two vertices a, b, and $E_+ = \{(a, b), (b, a)\}$. See the graph of Figure 2.

We will show that interchanging at most two different embeddings of the graph Extended $\widetilde{G}$ is enough to draw any linear periodic planar graph G preserving planarity.

In order to check whether it is possible that three embeddings to be required, we need to consider any 3 consecutive iterations of G. Recall that different embeddings may be introduced because of the connections between nodes of consecutive iterations. Hence, only nodes of the exterior face of each iteration will be involved. So, we can view the three consecutive iterations as three cycles. Moreover, we can consider three

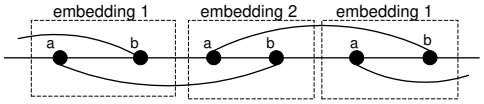

Fig. 2. For this graph we need to interchange two different embeddings of the graph Extended $\widetilde{G}$, in order to draw the periodic graph in the plane without crossings.

simple lines, since less edges are involved in the embeddings of consecutive iterations in the case of lines compared to cycles.

Finally, observe that, in order for three or more planar embeddings to be needed, we have to consider at least three nodes of G_i, assume nodes a, b, c. So, we can consider a line consisting of nodes a, b, c, having edges ab, bc (denoted by L_3).

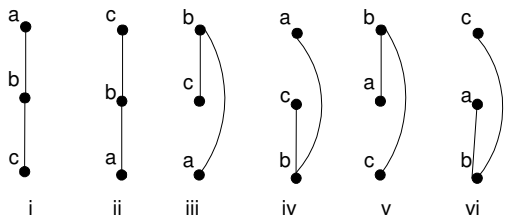

Fig. 3. The six possible 'drawings' of Line L_3.

We distinguish 6 different 'drawings' of a line L_3 in the plane. Note that some of them are equivalent embeddings with respect to definition 8 of a planar embedding. However, we consider all 6 of them since for an iteration i each such drawing combined with sets of edges E_{i-} and E_{i+} can result in a distinct embedding of the graph Extended $\widetilde{G}$. In our case the Extended $\widetilde{G}$ is the graph obtained by L_3 of iteration i and a set of edges connecting L_3 with the previous and next iterations, (sets E_{i-}, E_{i+}). In the following, we use the term 'drawing' when we refer to any of these 6 drawings and the term 'embedding' when we refer to any of them together with an instance of sets E_{i-} and E_{i+}.

To check whether three embeddings may be needed, we need to check all possible triples of these drawings. For each such triple, we check all possible sets E_+ that can lead to a linear periodic graph. For each such set, we show that interchanging at most two of these six drawings in any two consecutive iterations, is enough to draw the infinite graph in the plane. Recall that two drawings (of consecutive iterations i and $i+1$) combined with the sets of edges connecting each iteration with next and previous iterations (sets E_{i-}, E_{i+} and $E_{(i+1)-}$, $E_{(i+1)+}$) result in two different embeddings of the graph Extended $\widetilde{G}$. Henceforth, interchanging two different embeddings of the graph Extended $\widetilde{G}$ we can draw the infinite periodic planar graph in the plane without crossings.

Assume iterations G_i, G_{i+1}, having nodes a_i, b_i, c_i, and $a_{i+1}, b_{i+1}, c_{i+1}$, respectively. Observe that, in order for three embeddings to be required (a) set E_+ should contain at least 3 edges between any two drawings and (b) at least one node of each pair of nodes a_i, a_{i+1} , b_i, b_{i+1} or c_i, c_{i+1}, should have degree at least 1 in E_+.

By exhaustive check of all possible cases (see Appendix), we conclude that in all cases, interchanging at most two embeddings of the graph $\widetilde{G}$, is enough to draw the linear periodic graph in the plane, without edge crossings.

□

Lemma 2. *Assume that we construct an infinite graph as in Definition 6 except that instead of* **(a3)** *we have that for any three consecutive points of the line, replacing each point by graph $\widetilde{G}$ and the edges among consecutive points by E_+, the resulting graph is planar.*

Then, the infinite graph thus constructed is a linear periodic planar graph.

Proof. The above requirement implies **(a3)** of Definition 6 because of Lemma 1. To see why, observe that there are two possible ways to draw a linear periodic planar graph: using the infinite sequence $\ldots, A, B, A, \ldots$ or $\ldots, B, A, B, \ldots$ (where A, B are the two planar embeddings of $\widetilde{G}$ needed). Our assumption guarantees

that the coexistence of AB and BA graphs (together with E_+) has a planar embedding. Thus, by induction, any longer string will also have a planar embedding.

□

3 The $\mathcal{PSPACE}$-Completeness of min span RCP for Periodic Planar Graphs

We prove have that min span radiocoloring for periodic planar graphs is $\mathcal{PSPACE}$-complete. In order to show this, we need to prove that a number of problems are $\mathcal{PSPACE}$-complete. Our $\mathcal{PSPACE}$-completeness proofs utilize known constructions for the $\mathcal{NP}$-completeness of corresponding problems for ordinary graphs. However, note that applying those constructions on the infinite periodic graph G we need infinite time to get the transformed graph G' with the desired properties. We manage to apply the transformation only to a part of the infinite graph (an iteration) and from the obtained graph to get the transformation of the whole infinite periodic graph, thus the new graph G', in time polynomial to the size of the finite specification of G. Moreover, the new graph fulfills the desired properties. These are achieved by exploiting some 'locality characteristics' that the constructions utilized here exhibit, i.e. the construction applies locally on a part (vertex or edge) of this part involving only information of the neighborhood of the part and it affects only to this neighborhood. This, combined with the repetitive structure of an infinite periodic graph enables us to get constructions of polynomial time in the size of the finite specification of the graph.

A 3-coloring of a periodic graph $G = (\widetilde{G}(V, E_0), E_+)$ is a function c that assigns a number from the set $\{1, 2, 3\}$ to each vertex of the graph G so that not two adjacent vertices get the same number. A 4-edge coloring of a periodic graph G is a function e that assigns a number from the set $\{1, 2, 3, 4\}$ to each edge of G so that no two adjacent edges get same numbers.

Let any iteration $G_i(V_i, E_i)$ of G and let E_{i+} the set of edges connecting any iteration G_i to the next iteration G_{i+1}. A *constant period 4-edge coloring* of a periodic graph G is a 4-edge coloring of G that assigns to each edge uv of E_i of any iteration G_i the same color as the color assigned to the corresponding edge $u'v'$ of any other iteration G_j of G. Also, such a 4-edge coloring assigns to each edge of E_{i+} the same color as the color of the corresponding edge of E_{j+} of any other iteration G_j. As a result, such a 4-edge coloring can be described using finite space (using the 4-edge coloring of G_i and E_{i+}).

Lemma 3. *The problem of deciding whether a periodic planar graph $G = (\widetilde{G}(V, E_0), E_+)$ is 3-colorable (also called periodic planar 3-coloring) is $\mathcal{PSPACE}$-complete.*

Proof. (a) *Membership in $\mathcal{PSPACE}$:*

The proof that this problem is in $\mathcal{PSPACE}$ is similar to that for the PERIODIC SAT [25]: The 3-coloring of two adjacent iterations must repeat after at most 3^{2n} copies, and thus the whole 3-coloring can be assumed to be periodic with exponential period. A polynomial-space machine can then guess and check such a 3-coloring.

More analytically, suppose that the given periodic graph is 3-colorable, and consider a valid 3-coloring of it. This assignment consists of a two-way infinite $\ldots, T_i, T_{i+1}, T_{i+2}, \ldots$ of valid 3-coloring assignments to the various blocks of nodes (one for each iteration). Each T_i is an element of $\{1, 2, 3\}^n$, where $n = |V|$ is the number of vertices of the graph G_i.

The ith *chunk*, where i is any integer, is the pair (T_i, T_{i+1}), of two consecutive valid 3-coloring assignments. Since there are 3^{2n} possible different chunks, there must be two chunks, not further than 3^{2n} from each other, that are identical. That is $(T_i, T_{i+1}) = (T_j, T_{j+1})$ for some i and j between $i + 2$ and $i + 3^{2n}$. But this means that there is a 3-coloring assignment consisting of a two-way infinite repetition of $(T_i, T_{i+1}, \ldots, T_{j-1})$. We conclude that *if a periodic graph G is 3-colorable, then it has a periodic 3-coloring assignment with period at most exponential in the number of nodes of one iteration.*

This crucial observation allows us to solve the problem in polynomial space: Using non-determinism we can guess valid assignments $T_1, T_2, \ldots$, always remembering the last two. After we guess T_i we check that all nodes in iteration $i - 1$ are still properly colored. Once we have successfully guessed $T_{3^{2n}+2}$ we accept: We know that there is a periodic 3-coloring assignment on G.

(b) *The $\mathcal{PSPACE}$-completeness proof:*

In order to show the completeness, we reduce from 3-coloring of periodic general graphs, which is known to be $\mathcal{PSPACE}$-complete ([25]). We adapt the construction used by [24] to prove the $\mathcal{NP}$-completeness of PLANAR-3-COLORING from 3-COLORING.

Let any periodic graph G that we are asked whether it is 3-colorable. Take the embedding (that may have crossings) of the subgraph of G obtained by an iteration G_i, and the sets of edges connecting G_i with the previous and next iterations, sets E_{i-}, E_{i+}. We consider a drawing of the periodic graph G which uses this embedding in the corresponding subgraphs of all other iterations of G.

From this graph we construct a new graph G', which is planar by eliminating the edge-crossing by replacing any edge-crossing of G with the gadget of Figure 6, where H is a subgraph presented in the same Figure. The vertices of the subgraph are named as shown in Figure 6. Note that the two vertices u, v in Figure 6, might not belong to the same iteration; that is, it might be that $u \in V_i$ and $v \in V_{i+1}$, where V_i, V_{i+1} are the vertex sets of iterations G_i and G_{i+1} of G, respectively.

We manage to construct G', which is also periodic, in time polynomial to the size of one iteration of G applying the following procedure: Consider any iteration $G_i(V_i, E_i)$ of the periodic graph G, the previous and next to it iterations, G_{i-1}, G_{i+1} and the edges between them. Call the vertices of G_i (set V_i), *original vertices of* G_i. Call the edges of G_i (set E_i) as *inside edges of* G_i. Call the edges of set E_{i+} (connecting vertices of G_i to vertices of G_{i+1}, which is isomorphic to E_+) as *next outside edges of* G_i.

We construct a new graph G'_i and a set of pairs of vertices, called E'_{i+}, as follows:

(1) *For each next outside edge* uv *of* G_i: If there are no crossings on it, the edge remains the same in G'_i. Such edges belong to a set called $OutNoCrossEdge_{G_i}$. Otherwise, that is, when there are some crossings on the edge, then we replace it by a sequence of crossovers (one for each crossing), as shown in Figure 6. Call such gadgets, *outside sequences of crossovers of* G_i, denoted by $Next_Out_SC_{G_i}$.

Moreover, for each next outside edge uv of G_i, consider each inside edge of G_{i+1}, e, (if any) crossed by the outside edge uv. Replace each such inside edge by a sequence of crossovers as shown in Figure 6. Let $Next_Edges_{uv\ e}$ the set of edges connecting the common crossover of the two sequences (of uv and e) to the sequence of crossovers of e, see Figure 4.(a). Finally, let $Next_Edges_{G_i}$ the set of all such edges, obtained by all next outside edges of G_i and the corresponding inside edges of G_{i+1} crossed by them.

(2) *For any inside edge* uv *of* G_i: We distinguish two cases as it concerns the edges crossed by any such edge. (a) to be crossed by inside and/or next outside edges of G_i only (b) to be crossed by at least one next outside edge of G_{i-1} (equivalently, a previous outside edge of G_i).

2.(a) In this case, if there are no crossings on the edge, the edge remains the same in G'_i. Such edges belong to a set called $InNoCrossEdge_{G_i}$. Otherwise, that is, when there are some crossings on the edge (corresponding to inside and/or next outside edges of G_i), then we replace it by a sequence of crossovers (one for each crossing), as shown in Figure 6. Call the set of gadgets replacing such inside edges of G_i as $In_SC_{G_i}$.

2.(b) In this case, if there are no crossings on edge uv, it remains the same in G'_i. Such edges belong also to the set $InNoCrossEdge_{G_i}$. Otherwise, that is, when the edge is crossed by at least one next outside edge of G_{i-1}, then we replace it by a sequence of crossovers (one for each crossing), as shown in Figure 6, denoted by $Previous_{uv}$. Take the subgraph obtained by graph $Previous_{uv}$ 'minus' the crossovers that belong to outside edges of G_{i-1}, see Figure 4.(b). Call this subgraph as $In_SC_Previous_{uv\ G_i}$. Call all gadgets obtained by such inside edges of G_i, (crossed by next outside edges of G_{i-1}) as $In_SC_Previous_{G_i}$.

Finally, for any sequence of crossovers SC, defined above, let $Orig_{SC}$ the original vertices of iteration G_{i+1} contained in this sequence of crossovers SC.

The graph $G'_i(V'_i, E'_i)$ is defined as follows:

$V'_i = V_i \cup (V(Next_Out_SC_{G_i}) - Orig_{Out_SC_{G_i}}) \cup V(In_SC_{G_i}) \cup V(In_SC_Previous_{G_i})$.

$E'_i = InNoCrossEdge_{G_i} \cup E(Next_Out_SC_{G_i} - Orig_{Out_SC_{G_i}})) \cup E(In_SC_{G_i})$
$\cup E(In_SC_Previous_{G_i})$.

We also define a set of pairs of nodes, called E'_{i+}, as follows:

$(v,u) \in E'_{i+}$ *if* $vu \in E(Next_Out_SC_{G_i})$ *and* $u \in V_{i+1}$,

$(v,u) \in E'_{i+}$ *if* $vu \in E(Next_Edges_{G_i})$,

$(v,u) \in E'_{i+}$ *if* $vu \in OutNoCrossEdge_{G_i}$ *and* $u \in V_{i+1}$

Now we modify set E'_{i+} so that to contains only vertices of set V'_i: (i) Replace vertex u of any edge uv of set E'_{i+} where $u \in V_{i+1}$ with the corresponding to it vertex of set V_i. Recall that $V_i \subseteq V'_i$, henceforth, now $u \in V'_i$. (ii) Replace vertex u of any pair (u, v) of subgraph $Next_Edges_{G_i}$ (contained in set E'_{i+}), where $u \in Previous_{xy\ G_{i+1}}$, with the corresponding to it vertex in subgraph $Previous_{x'y'\ G_i}$ of G_i. Recall that subgraphs $Previous_{xy\ G_{i+1}}$ and $Previous_{x'y'\ G_i}$ are isomorphic.

Applying the Procedure described above, for the graph obtained by any other iteration G_j of G and the edges connecting it to previous and next iterations, (sets E_{j+}, E_{j-}), we get a graph $G'_j(V'_j, E'_j)$ and a set of pairs of nodes E'_{j+}. In order to compute the resulting graph of iteration j, observe that G_j and G_i are isomorphic graphs and sets E_{j+}, E_{j-} are equal to E_{j+}, E_{j-}, respectively (and each of them is equal to E_+). Moreover, note that the graph obtained by G_j and the edges connecting it to previous and next

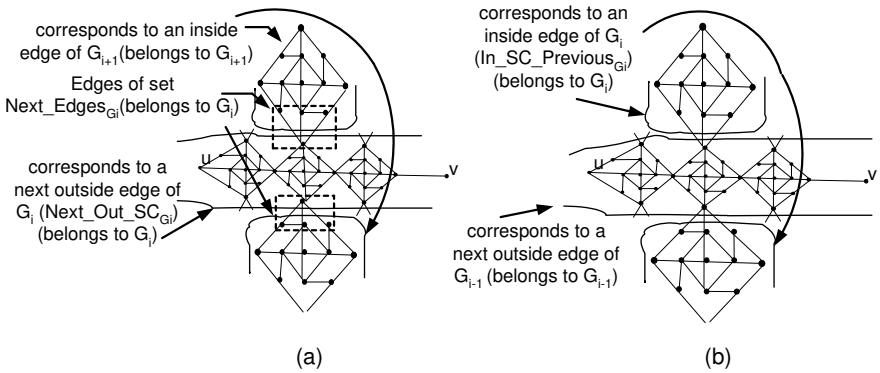

Fig. 4. The sets $Next_Edges_{G_i}$ (a) and $In_SC_Previous_{G_i}$ (b) of iteration i.

iterations, E_{j-} and E_{j+}, has the same embedding as the graph obtained by G_i and the edges connecting it to previous and next iterations (by our choice on the embedding of the initial periodic graph G). By above observations, we conclude that the graph G'_j and the set E'_{j+} obtained by iteration j are the same as G'_i and E'_{i+}, respectively. Thus, the graph G' obtained by the initial periodic graph is also a periodic graph defined using the finite specification $(\widetilde{G'}, E'_+)$, where $\widetilde{G'} = G'_i$ and $E_+ = E'_{i+}$. Moreover, obviously G' is a planar graph. Finally, in order to get G', we only need to compute the graph G'_i and the set E_{i+} only once and from these to get the new periodic planar graph G'. By this way we manage to *construct from any periodic graph G, a new periodic planar graph G' in time polynomial to the size of the finite specification of the initial graph G.*

Now, we need to prove that if the initial periodic graph G is 3-colorable, then the periodic planar G' is also 3-colorable. This property holds because the new periodic planar graph G' computed is the same as the graph obtained by [24] when the transformation of [24] applies straight in the actual infinite periodic graph G. Observe that this is due to the local characteristics of the construction and the repetitive structure of an infinite periodic graph. This completes the $\mathcal{PSPACE}$-completeness proof. □

Lemma 4. *The problem of deciding whether a given periodic planar graph $G = (\widetilde{G}(V, E_0), E_+)$ of maximal degree four is 3-colorable is $\mathcal{PSPACE}$-complete.*

Proof. (outline) We prove that the problem is in $\mathcal{PSPACE}$ using arguments similar to those used above to prove the membership of periodic planar 3-coloring in $\mathcal{PSPACE}$.

We reduce planar 3-coloring of a periodic graph with a maximum degree 4 from periodic planar 3-coloring which was proved to be $\mathcal{PSPACE}$-complete in Lemma 3. We adapt the same technique used to prove that for an ordinary planar graph G with a maximum degree four it is $\mathcal{NP}$-complete to decide whether it can be colored with 3 colors [24], which reduces the problem from the planar-3-coloring. We manage to get a construction of time polynomial in the size of the finite specification of the infinite graph, by exploiting the locality characteristics of the $\mathcal{NP}$-completeness construction and the repetitive structure of a periodic graph. *For the full proof, see the Appendix of this paper.*

□

Lemma 5. *The problem of 3-Coloring a periodic planar graph with a given constant period 4-edge coloring is $\mathcal{PSPACE}$-complete.*

Proof. (outline) The problem is proved to be in $\mathcal{PSPACE}$ using arguments similar to those used to prove the membership of 3-coloring of periodic graphs in $\mathcal{PSPACE}$ of Lemma 3.

We use a transformation from the 3-coloring for planar periodic graphs with maximum degree four which was proved to be $\mathcal{PSPACE}$-complete in Lemma 4. This reduction adapts the reduction used in [24] to show $\mathcal{NP}$-hardness of 3-coloring of planar graphs with maximum degree four. Again, the construction is of polynomial time in the size of the specification of the graph by exploiting the local characteristics of the construction and the repetitive structure of an infinite periodic graph. *For the full proof, see the Appendix.*

□

Next, we prove the main Theorem using the above results. We will show that it is $\mathcal{PSPACE}$-complete to decide whether $X_{span}(G) \leq 9$ for a given periodic planar graph $G = (\widetilde{G}(V, E_0), E_+)$.

Theorem 1. *The problem of deciding whether a periodic planar graph $G = (\widetilde{G}(V, E_0), E_+)$, of maximum degree seven, whose graph $\widetilde{G}$ in its finite specification $(\widetilde{G}, E_+)$ is a planar bipartite graph, can be radiocolored using a span of size at most 9, is $\mathcal{PSPACE}$-complete.*

Proof. We prove that the min span radiocoloring for periodic planar graphs is $\mathcal{PSPACE}$-complete, by reducing it from 3-coloring of periodic planar graphs with a given constant period 4-edge coloring which was proved to be $\mathcal{PSPACE}$-complete in Lemma 5. Our reduction adapts the construction by Bodlaender et al in [2] to show $\mathcal{NP}$-hardness of min span RCP of ordinary planar graphs reducing it from the 3-coloring of planar graphs with a given 4-edge coloring. *For the full proof of the Theorem see the Appendix.* □

As in [2], it is possible to generalize the result as follows:

Theorem 2. *Let $r \geq 8$ be an even integer. The problem of deciding whether a periodic planar graph $G = (\widetilde{G}(V, E_0), E_+)$ of maximal degree $r-2$ can be radiocolored using a span of size at most r is $\mathcal{PSPACE}$-complete.*

4 An Efficient, Constant Ratio Approximation Algorithm for min span RCP for Periodic Planar Graphs

We present an efficient time, constant ratio approximation algorithm that approximates the min span radiocoloring problem for periodic planar graphs with the same ratio as the ratio obtained by the best known approximation algorithm for planar graphs (which we use as a subroutine for the finite specification), for the same problem.

4.1 A partition of the periodic graph

We consider the following partition for each iteration of the graph G, G_i and the edges connecting it to the next iteration G_{i+1} (set E_{i+}):

The vertices of the graph can be partitioned into three disjoint sets of vertices:

1. A vertex $v \in S_3(G_i)$ iff the vertex has a neighbour vertex in graph G_{i+1}, i.e., $S_3(G_i) = \{\ u\ :\ u \in V(G_i)\ such\ that\ uv \in E_{i+}\ and\ v \in V_{i+1}\}$
2. A vertex $v \in S_2(G_i)$ iff the vertex is at distance two and not less from a vertex of graph G_{i+1}, that is, the vertex has a neighbour that belongs to set $S_3(G_i)$, i.e., $S_2(G_i) = \{\ u\ :\ u \in V(G_i),\ u \notin S_3(G_i)\ and\ \exists v :\ such\ that\ uv \in E(G_i), v \in S_3(G_i)\}$
3. A vertex $v \in S_1(G_i)$ iff all neighbour vertices of v belong either to $S_1(G_i)$ or $S_2(G_i)$, that is the vertex has no neighbour in set $S_3(G_i)$, i.e, $S_1(G_i) = \{\ u\ :\ u \in V(G_i),\ u \notin S_3(G_i),\ and\ u \notin S_2(G_i)\}$

It can be easily seen that $S_k(G_i) \bigcap S_l(G_i) = 0,\ for\ k,l = 1,2,3$. Also, $S_3(G_i) \bigcup S_2(G_i) \bigcup S_1(G_i) = V(G_i)$. Figure 11 in the Appendix shows an example of a periodic graph and its partitioning.

4.2 The modified graph

Furthermore, consider the following partition of the periodic graph G: group together every four consecutive iterations of the graph, call the $j-th$ such group $G_{group\ j}$.

More specifically,

$$G_{group\ j} = \{G(i) \bigcup G(i+1) \bigcup G(i+2) \bigcup G(i+3)\},\ \forall\ i = \ldots, -7, -3, 1, 5, 9, \ldots$$

where $j = 1, 2, 3, 4, \ldots$, (respectively, i.e. $j = \lceil i/4 \rceil$). See Figure 11 in the Appendix for an example.

Consider the graph $G_{group\ j}$ of G. Denote the first graph of the group as $G_{(j)1}$ or G_1, the second as $G_{(j)2}$ or G_2 and so on until the fourth. The algorithm produces a new graph, called $G'_{group\ j}$ defined as follows: The graph has the same vertex and edge set as the graph $G_{group\ j}$ except from the following modifications on the first and the fourth graphs of group $G_{group\ j}$:

Consider an edge $uv \in E_{4+}$ of graph G_4. Recall that $u \in V_4$ and the vertex v belongs to the next iteration of G, that is $v \in V_{(j*4)+1}$. For each such edge uv of the graph G_4 we do the following:

- Delete edge uv from G_4.
- Add a new edge uv' connecting the vertex $u \in G_{43}$ to vertex v', where $v' \in V_1$ is the corresponding to v ($v \in V_{(j*4)+1}$) vertex in graph G_1. Recall that the graphs $G_1, V_{(j*4)+1}$ are isomorphic.

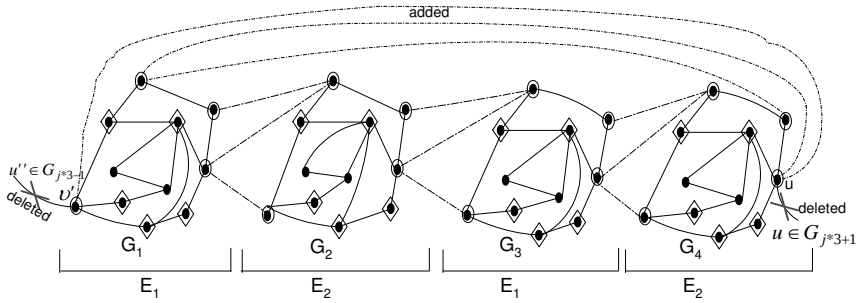

Fig. 5. The graph $G'_{group\ j}$ produced by the $Group\ j$ of the periodic graph G

– Delete edge $u''v'$ of graph G_1, where $v' \in G_1$ and u'' is the corresponding to u vertex ($u \in V_4$) in iteration G_{j*4-1} of G.

The produced graph for the example of a periodic graph of Figure 11 is illustrated in Figure 5.

The graph $G'_{group\ j}$ has two critical properties compared to the initial periodic planar graph G: (i) it has the same maximum degree as the initial graph G, i.e. $\Delta(G'_{group\ j}) = \Delta(G)$ and (ii) as the next Lemma proves, it is a planar graph.

Lemma 6. *The modified graph $G'_{group\ j}$ is a planar graph.*

Proof. Observe that, by Lemma 1 any four consecutive iterations can be drawn in a plane using embeddings E_1, E_2, E_1, E_2, where it might be that $E_1 = E_2$. The modified graph $G'_{group\ j}$ is the same as graph $G_{group\ j}$ (which is planar), with the only difference that G_4 is connected to G_1 instead of G_{j*4+1}. Assuming that embedding E_1 is used be G_1, graph G_4 uses embedding E_2. In the periodic graph G, G_4 is connected to G_{j*4+1} which uses embedding E_1. In the modified graph $G'_{group\ j}$, G_4 is connected to G_1, which also uses embedding E_1. Thus, there is no crossing in edges connecting G_4 to G_1. We conclude that $G'_{group\ j}$ is a planar graph. □

The proposed radiocoloring algorithm for a periodic graph G, called Periodic Radiocoloring Partitioning Algorithm (PRPA), can now be described.

4.3 The Periodic Radiocoloring Partitioning Algorithm (PRPA)

The following definition is needed by the Algorithm. The definition uses the observation that the optimal span, S^*, of a radiocoloring of a graph G with maximum degree $\Delta(G)$ is clearly $S^* \geq \Delta(G)$.

Definition 9. RC Algorithm: *Let an RC Algorithm be any known min span radiocoloring polynomial time approximation algorithm for finite planar graphs with performance ratio R (when $\Delta(G)$ is used as a lower bound), i.e. if S^* is the optimal span and S_{RC} is the span obtained by the algorithm then there are constants $R > 1$ and b such that*

$$\Delta(G) \leq S^* \leq S_{RC} \leq R \cdot \Delta(G) + b$$

For example the algorithm of [14] is an RC algorithm with $R = 2$ and $b = 35$.

Our algorithm is then described as follows:

Algorithm PRPA*:*

1. Run an RC algorithm, on graph $G'_{group\ j}$.
   Let S_{RC} be the span obtained by RC on $G'_{group\ j}$.

2. For all $j = 1, 2, \ldots$ color the four graphs $G_{(j-1)*4+1}$, $G_{(j-1)*4+2}$, $G_{(j-1)*4+3}$, , $G_{(j-1)*4+4}$ of the group $G_{group\ j}$ as follows: Set the color of each vertex of graph $G_{(j-1)*4+k}$, $k = 1, 2, 3, 4$ to the color of its corresponding vertex, in V_k of $V(G'_{group\ j})$.
   Step 2 produces a radiocoloring of the whole periodic graph G with span S_{RC}.

4.4 Analysis of Algorithm PRPA: Correctness and Performance

Theorem 3. (**Correctness**) *The algorithm PRPA produces a radiocoloring of a periodic graph G.*

Proof. (outline) It can be proved that there is no conflict either between the colors of vertices inside a group $G_{group\ j}$ or between the colors of vertices in neighbour groups. *For the full proof, see the Appendix.* □

Theorem 4. *The Algorithm PRPA runs in time $O(T(RC))$ and approximates the span within an asymptotic ratio of R, where $T(RC)$ is the time needed for the RC Algorithm to run on $G'_{group\ j}$.*

Proof. Let $\Delta(G)$ the maximum degree of the periodic graph G. Recall that the graph $G'_{group\ j}$ has maximum degree $\Delta(G'_{group\ j}) = \Delta(G)$. Denote by S the span obtained by our algorithm and by $S^*(G'_{group\ j})$ the optimal span of the graph $G'_{group\ j}$.

Note that, (a) $S^* \leq S$.

(b) Also that, $S \leq R \cdot \Delta(G'_{group\ j}) + b$, by the definition of the RC Algorithm.

(c) Since, $\Delta(G'_{group\ j}) = \Delta(G)$, we get $S^* \leq S \leq R \cdot \Delta(G) + b$

Also, since $S^* \geq \Delta(G)$, we get that, $1 \leq \frac{S}{S^*} \leq R + \frac{b}{\Delta(G)}$

Finally, since $\Delta(G) \geq \Delta(\widetilde{G})$, we have $1 \leq \frac{S}{S^*} \leq R + \frac{b}{\Delta(\widetilde{G})}$.

Also, the algorithm, clearly, needs $O(T(RC))$ time, where $T(RC)$ is the time needed for algorithm RC to run on $G'_{group\ j}$. □

Remark. If the RC Algorithm is the algorithm in [14], then algorithm $PRPA$ has $R = 2$ and $b = 35$ and runs in $O(n(\Delta(G_i) + \sigma))$ time, where $n = |V_i|$, and σ is the number of edges with which each G_i is connected to G_{i+1} and $\Delta(G_i)$ is the maximum degree of G_i.

Consider now the min order RCP of a periodic planar graph G. We conjecture that the problem is $\mathcal{PSPACE}$-complete. Note that the following modification of $PRPA$ works also for min order RCP: Modify the algorithm $PRPA$ as follows: In Step 1, apply a known min order radiocoloring algorithm (instead of a min span RC algorithm), e.g. [9], for the radiocoloring of $G'_{group\ j}$. The same analysis gives that the modified $PRPA$ algorithm approximates the min order RCP of G within an asymptotic approximation ratio of 2. Furthermore, the modified $PRPA$, for periodic graphs of large maximum degree (≥ 749), approximates the min order RCP within an asymptotic approximation ratio of 1.8 (using an algorithm implied by [1] in Step 1).

Lemma 7. *Any approximation algorithm for the min span RCP of a finite planar graph G, that achieves a span of at most $\alpha\Delta(G) + constant$, for any α, can be used as a subroutine in algorithm PRPA to produce an approximation algorithm for min span periodic planar RCP of asymptotic ratio α.*

Proof. *Using the same arguments as in Theorem 4.* □

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APPENDIX

A Lemma 1. (The rest of the proof)

As one case of all possible cases (the other cases are similar), we consider the triple of drawings (i), (ii), (iii). For this triple, we check all possible sets of E_+ that lead to planarity. For each such set E_+, we show that interchanging at most two different drawings is enough to draw the infinite graph in the plane. Recall that two drawings (of consecutive iterations i and $i+1$) combined with the sets of edges connecting each iteration with next and previous iteration (sets E_{i-}, E_{i+} and $E_{(i+1)-}$, $E_{(i+1)+}$) result to two different embeddings of the graph Extended $\widetilde{G}$. Henceforth, interchanging two different embeddings of the graph Extended $\widetilde{G}$ we can draw the infinite periodic planar graph in the plane without crossings. For the rest possible triples of drawings we get similar results. See [11] for an exhaustive check of all possible cases.

Observe that there must not be an edge connecting c_i to any node of G_{i+1}, since this would eliminate the possibility of a third drawing. Also, there must be an edge from some node of G_i to c_{i+1}, since otherwise, two drawings would be sufficient. Thus, there are three possible cases for the set of edges between any two consecutive iterations that preserve planarity.

A. There exists an edge (a_i, c_{i+1}). In this case, by planarity (the resulting graph should be planar), it is not possible to have both edges (a_i, a_{i+1}) and (b_i, b_{i+1}). Hence, the only cases remaining are to have edges $\{(a_i, b_{i+1}), (b_i, b_{i+1}), (b_i, a_{i+1})\}$ or edges $\{(a_i, b_{i+1}), (a_i, a_{i+1}), (b_i, a_i + 1)\}$. Both of them lead to a linear periodic planar graph by interchanging drawings (i) and (ii). Obviously, for any other subset of those cases at most two embeddings are sufficient to preserve the planarity of the periodic planar graph.

B. There exists an edge (b_i, c_{i+1}). In this case, by planarity (the resulting graph should be planar), it is not possible to have both edges (a_i, b_{i+1}) and (b_i, a_{i+1}). Hence, the only cases remaining are to have edges $\{(a_i, a_{i+1}), (a_i, b_{i+1}), (b_i, b_{i+1})\}$ or edges $\{(a_i, a_{i+1}), (b_i, a_{i+1}), (b_i, b_{i+1})\}$. Both of them lead to a linear periodic planar graph by using drawing (ii). Obviously, for any other subset of those cases interchanging at most two drawings (which combined with sets E_{i-} and E_{i+} result to two embeddings of the graph Extended $\widetilde{G}$) is sufficient to preserve the planarity of the periodic planar graph.

C. There exist both edges (a_i, c_{i+1}) and (b_i, c_{i+1}). In this case, by planarity (the resulting graph should be planar), the only possible extra edges are the following sets:
(-) $\{(a_i, b_{i+1}), (a_i, a_{i+1})\}$. Then we can use drawing (i).
(-) $\{(b_i, b_{i+1}), (b_i, a_{i+1})\}$. Then we can interchange drawings (i) and (ii).
(-) $\{(a_i, a_{i+1}), (b_i, b_{i+1})\}$. Then we can use drawing (iii).
By planarity, any other supersets of those cases are not possible. For any other subset of those cases interchanging at most two drawings (which combined with sets E_{i-} and E_{i+} result to an interchange of two embeddings of the graph Extended $\widetilde{G}$) is sufficient to draw the infinite periodic graph in the plane.

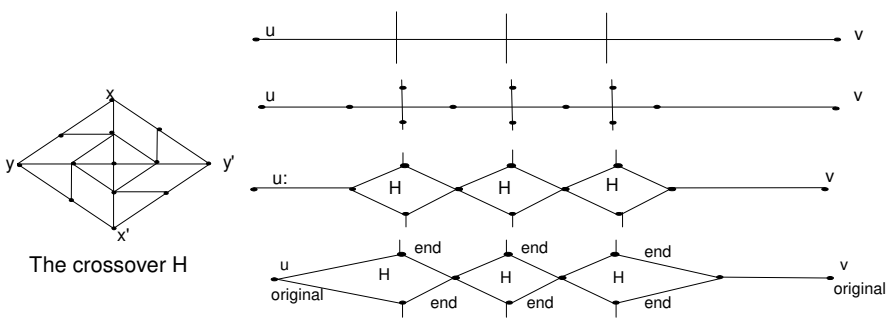

Fig. 6. Figure used for Lemma 3.
The Crossover H and the construction of a planar G' from a given periodic graph G, using the crossover H.

B Lemma 4

Lemma 4. *The problem of deciding whether a given periodic planar graph $G = (\widetilde{G}(V, E_0), E_+)$ of maximal degree four is 3-colorable is $\mathcal{PSPACE}$-complete.*

Proof. We prove that the problem is in $\mathcal{PSPACE}$ using arguments similar to those used above to prove the membership of periodic planar 3-coloring in $\mathcal{PSPACE}$.

We reduce planar 3-coloring of a periodic graph with a maximum degree 4 from periodic planar 3-coloring which we proved to be $\mathcal{PSPACE}$-complete in Lemma 3. We adapt the construction used to prove that for a planar G with a maximum degree four it is NP-complete to decide whether it can be colored with 3 colors [24] which reduces the problem from the planar-3-coloring.

Let any periodic planar graph G of arbitrary maximum degree that we are asked whether it is 3-colorable. We construct a new planar graph G' of maximum degree is 4 as in the reduction of [24].

The construction uses a vertex substitute, called H_3, shown in Figure 7. The graph H_3 has three "outlets" labeled 1, 2, 3. It is designed so that all of its outlets should take the same color. Moreover, each outlet has degree 2. A vertex of degree k in G is replaced by a subgraph H_k, which is defined by $k-2$ repetitions of the subgraph H_3, as shown in Figure 7, for the case where $k = 5$. In the subgraph H_k all of its outlets should be colored with the same color and each of them is of degree 2.

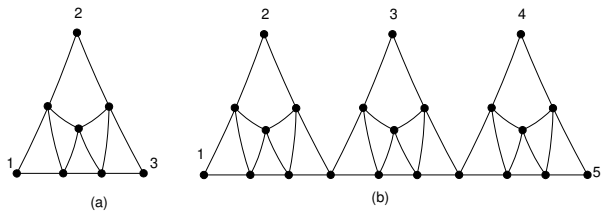

Fig. 7. The subgraphs H_3 (a) and H_5 (b)

We construct from G a *periodic* planar graph G' as follows. (Observe that the construction of G' can be performed in time polynomial to the size of one iteration of G.)

Consider any iteration $G_i(V_i, E_i)$ of the periodic graph G. Let E_{i+} the set of edges connecting vertices of G_i to vertices of G_{i+1}.

The transformation is described by the following Procedure: (1) Replace any vertex v of G_i of degree k by the subgraph H_{k_v}. (2) Replace each edge (v, u_j), with $v, u_j \in V_i$, by an edge joining u_j to the j-th outlet of H_{k_v}. (3) Replace each edge (v, u_j), with $v \in V_i$ but $u_j \in V_{i+1}$, by an edge joining u_j to the j-th outlet of H_{k_v}.

Define a graph $G'_i(V'_i, E'_i)$ as follows:

$\forall u \in V_i : V(H_{k_u}) \subseteq V'_i$
$\forall u \in V_i : E(H_{k_u}) \subseteq E'_i$
$\forall (u, v_j) \in E_i \; and \; u, v_j \in V_i : (v_j, j) \in E_{i'}$

where j is the j-th outlet of H_{k_u} and k is vertex's u degree.

Also, define a set of pairs of nodes, called E'_{i+}, of as follows:
$(u, v_j) \in E_i \; and \; u \in V_i \; but \; v_j \in V_{i+1} : (v_j, j) \in E'_{i+}$

Applying the same Procedure, described above, for any other iteration G_j of G and the edges connecting G_j to G_{j+1}, we get the same graph $G'_i(V'_i, E'_i)$ and set of pairs of nodes E'_{i+}. These two parameters conclude that the new graph G' obtained, is *a periodic graph* $G' = (G'_i(V'_i, E'_i), E'_{i+})$. This also concludes *that the graph G' can be constructed in polynomial time to the size of the finite specification of G.*

We need to prove that if the initial periodic graph G is 3-colorable, then the periodic planar G' of maximum degree 4, is also 3-colorable. Recall that the reduction of [24] guarantees that assuming any 3-colorable graph G, the new graph G' is 3-colorable. In our case, we are given a periodic graph G. Consider any vertex of it. The vertex is replaced by a subgraph H_k, (k is vertex's degree). The subgraph has the same properties as when applied for ordinary graphs. Moreover, it is local; that is uses and changes only the structure and the properties of the vertex applied. The rest of the graph "sees" exactly the same constraints, considering this vertex as before the vertex substitution. Hence, if the initial periodic graph G is 3-colorable the new graph G' is also 3-colorable.

Now, we only need to prove that if G' is 3-colorable then G is also 3-colorable. This is proved using the same arguments used in [24] and above observations on the locality of the transformation.

This completes the $\mathcal{PSPACE}$-completeness proof.

□

C Lemma 5

Lemma 5 *The problem of 3-Coloring a periodic planar graph with a given constant period 4-edge coloring is $\mathcal{PSPACE}$-complete.*

Proof. The problem is proved to be in $\mathcal{PSPACE}$ using arguments similar to those used to prove the membership of 3-coloring of periodic graphs in $\mathcal{PSPACE}$ of Lemma 3.

We reduce from the 3-coloring for planar periodic graphs with maximum degree four which was proved to be $\mathcal{PSPACE}$-complete in Lemma 4. The reduction adapts the reduction used in [24] to show $\mathcal{NP}$-hardness of 3-coloring of planar graphs with maximum degree four. Suppose we have a periodic planar graph G with maximum degree four.

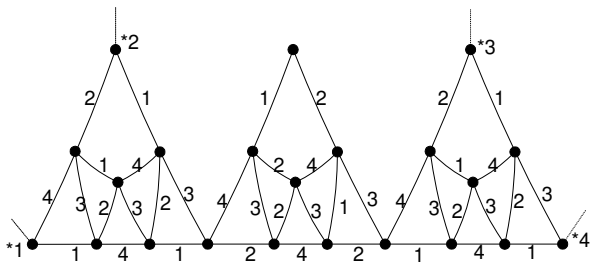

Fig. 8. The subgraph S

We construct from G a *periodic* planar graph G' as follows.

Consider any iteration $G_i(V_i, E_i)$ of the periodic graph G. Let E_{i+} the set of edges connecting vertices of G_i to vertices of G_{i+1}.

The transformation is described by the following Procedure:
(1) Replace every vertex u of G_i by a copy of the subgraph S, shown in Figure 8.
(2) Consider any vertex u in V_i which is replaced in G' by a subgraph S. We replace each edge (u, v), with $v \in V_i$, incident to u with an edge joining v with one of the four marked vertices (with an $*i$, for $i = 1 \ldots 4$) in the subgraph S such that the graph stays planar.
(3) Similarly, we replace each edge (u, v), with $u \in V_i$ but $v \in V_{i+1}$, incident to u with an edge joining v with one of the four marked vertices in the subgraph S (replacing u in G') so that the graph stays planar.

The above Procedure can be used to construct G' in time polynomial to the size of the finite specification of G. Moreover, the constructed G' is a periodic graph as we explain below.

Define a graph $G'_i(V'_i, E'_i)$ as follows:

$\forall u \in V_i : V(S_u) \subset V'_i$
$\forall\ v\ \in V_i\ :\ E(S_u) \subset E'_i$ and
$\forall (u, v_j) \in E_i\ and\ u, v_j \in V_i : (v_j, j) \in E_{i'}$

where j is the j-vertex marked as $*j$ in S_u.

Also define a set of pairs of nodes, called E'_{i+}, of as follows:
$\forall (u, v_j) \in E_i\ and\ u \in V_i\ \ but\ v_j\ \in V_{i+1} : (v_j, j) \in E'_{i+}$

Applying the same Procedure, described above, for any other iteration G_j of G and the edges connecting G_j to G_{j+1}, we get the same graph $G'_i(V'_i, E'_i)$ and set of edges E'_{i+}. These two parameters conclude that the new graph G' obtained, is *a periodic graph* $G' = (G'_i(V'_i, E'_i), E'_{i+})$. This also concludes *that the graph G' can be constructed in polynomial time to the size of the finite specification of G.*

Let G' be the resulting periodic graph. An 4-edge coloring of G' can be constructed by coloring the edges in the replacement subgraph S as in Figure 8, and coloring the original edges of G (the dotted lines in Figure 8) by the following procedure: if e is an edge between two subgraphs, since both endpoints have degree three and one edge is already colored with 1, there is at least one color from { 2, 3, 4 } left for the edge.

We need to prove that the 4-edge-coloring of the periodic graph G' can be constructed in time polynomial to the size of one iteration of G'. That is, to prove that it is a constant period 4-edge coloring. To see why, consider any iteration $G'_i(V'_i, E'_i)$ of G'. The edge coloring procedure described above may apply to produce a 4-edge coloring of edges of set E'_i in time polynomial to $n = |V'_i|$. Now use the same edge-coloring assignment of E'_i to edge-color the edge sets E'_j of each other iteration G'_j of the periodic graph G'. Observe, that this assignment results to no conflicts between successive iterations because none of the edges of E'_j is adjacent to an edge of $E'_{(j+1)}$.

Consider next, the edges of set E_{i+} connecting some vertices of G_i to some vertices of G_{i+1}. The procedure described above may apply to produce a 4-edge coloring of edges of set E_{i+} in time polynomial to $n = |V'_i|$. Now as before, use the same edge-coloring assignment of E'_{i+} to edge-color the edges of sets E'_{j+} of the each other iteration G'_j of the periodic graph G'. Observe again that, this assignment results to no conflicts between successive iterations because none of the edges of E'_{j+} is adjacent to an edge of $E'_{(j+1)+}$. *Hence this is a constant period 4-edge-coloring G' and thus it can be computed in time polynomial to the size of one iteration of G'.*

Now, we need to prove that if the initial periodic planar graph G, with a given 4 edge coloring is 3-colorable, then the periodic planar G', is also 3-colorable and has 4 edge coloring. Recall that the reduction of [24] guarantees that assuming any 3-colorable graph G, the new graph G' is 3-colorable. In our case, we are given a periodic planar graph G. Consider any vertex of it. The vertex is replaced by a subgraph S. The subgraph S has the same properties as when applied for ordinary graphs. Moreover, it is local; that is uses and changes only the structure and the properties of the vertex applied. The rest of the graph "sees" exactly the same constraints, considering this vertex as before the vertex replacement. Hence, if the initial periodic graph G with a given 4 edge coloring, is 3-colorable the new graph G' is also 3-colorable and has a 4 edge coloring.

Now, we only need to prove that if G' is 3-colorable then G is also 3-colorable. This is proved using the same arguments used in [24] and above observations on the locality of the transformation.

This completes the $\mathcal{PSPACE}$-completeness proof.

□

D Theorem 1

Theorem 1 *The problem of deciding whether a periodic planar graph $G = (\widetilde{G}(V, E_0), E_+)$, of maximum degree seven, whose iteration is a bipartite graph, can be radiocolored using a span of size at most 9, ($X_{span}(G) \leq 9$) is in $\mathcal{PSPACE}$.*

Proof. (a) Membership in $\mathcal{PSPACE}$:

The proof that it is in $\mathcal{PSPACE}$ is the same as for PERIODIC SAT: The radiocoloring of two adjacent iterations must repeat after at most k^n copies, and thus the whole radiocoloring can be assumed to be periodic with exponential period. A polynomial-space machine can then guess and check such a radiocoloring.

More analytically, suppose that the given periodic graph is k-radiocolorable, and consider its valid radio-coloring assignment. This assignment consists of a two-way infinite $\ldots, T_i, T_{i+1}, T_{i+2}, , \ldots.$ of valid radio-coloring assignments to the various blocks of nodes (one for each iteration). Each T_i is an element of $\{1, 2, \ldots, k\}^n$, where $n = |V|$ is the number of vertices of the graph G_i.

The ith iteration, where i is any integer, is the pair (T_i, T_{i+1}), of two consecutive valid radiocoloring assignments. Since there are k^{2n} possible different chunks, there must be two chunks, not further than k^{2n} from each other, that are identical. That is $(T_i, T_{i+1}) = (T_j, T_{j+1})$ for some i and some j between $i+2$ and $i + k^{2n}$. But this means that there is a a valid radiocoloring assignment consisting of a two-way infinite repetition of $(T_i, T_{i+1}, \ldots, T_{j-1})$. We conclude that *if a periodic graph G is k-radiocolorable, then it has a periodic radiocoloring assignment with period at most exponential in the number of nodes of one iteration.*

The crucial observation allows us to solve the problem in polynomial space: Using non-determinism we can guess valid assignments $T_1, T_2, \ldots$, always remembering the last two. After we guess T_i we check that all nodes in the $i-1$ iteration are still valid radiocolored. Once we have successfully guesses $T_{k^{2n+2}}$ we accept: We know that there is a periodic radiocoloring assignment that uses a span of size no more than k.

(b) *The $\mathcal{PSPACE}$-completeness proof:*

To show $\mathcal{PSPACE}$-completeness, we reduce from 3-coloring of periodic planar graphs with a given 4-edge coloring which was proved to be $\mathcal{PSPACE}$-complete in Lemma 5. we adapt the construction of [2] to show $\mathcal{NP}$-hardness of min span RCP of ordinary planar graphs reducing it from the 3-coloring of planar

graphs with a given 4-edge coloring. Suppose we are given a periodic planar graph $G = (\widetilde{G}(V, E_0), E_+)$ with a 4-edge coloring f. We transform G to get a new graph G' using a sequence of rules as described below:

Consider any iteration $G_i(V_i, E_i)$ of the periodic graph G. Let E_{i+} the set of edges connecting vertices of G_i to vertices of G_{i+1}. Apply the following rules on G_i and the edges of set E_{i+}.

(1) For each vertex in V_i of G, we add a new vertex in G', called *original* vertex.

Next we subdivide every edge kl of E_i or E_{i+} to get vertex u_{kl}. The resulting vertices are named as follows:

(2) Every vertex resulting from the subdivision of an edge colored 1 is called *0-vertex.*

(3) Every vertex resulting from the subdivision of an edge colored 2 is called *1-vertex.*

(4) Every vertex resulting from the subdivision of an edge colored 3 is called *7-vertex.*

(5) Every vertex resulting from the subdivision of an edge colored 4 is called *8-vertex.*

(6) Now, if an original vertex has degree less than four, it receives new neighbors: if original vertex v has no neighbor that is a 0-vertex, we add one new vertex and connect it only to v. Call this vertex an *extra 0-vertex.* Similarly, original vertices that have no neighbor that is a 1-vertex, 7-vertex, or 8-vertex get a new neighbor of degree one that is an *extra 1-vertex, extra 7-vertex*, or *extra 8-vertex.*

(7) Now we add, to every 0-vertex and 8-vertex, five new neighbors of degree one each, as in Figure 9.

(8) To every extra 0-vertex and extra 8-vertex, we add six new neighbors of degree one each, as in Figure 9.

(9) To every 1-vertex and 7-vertex, we add two subtrees of a form, as shown in 10. To every extra 1-vertex and extra 7-vertex we add three of these subtrees, as also shown in Figure 10. Observe that the construction

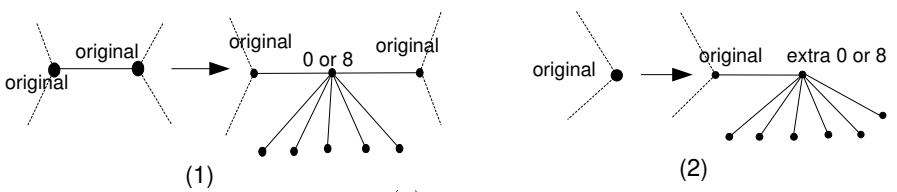

Fig. 9. The vertices added to each 0 or 8-vertex (1), and the vertices added to each extra 0 or 8-vertex (2).

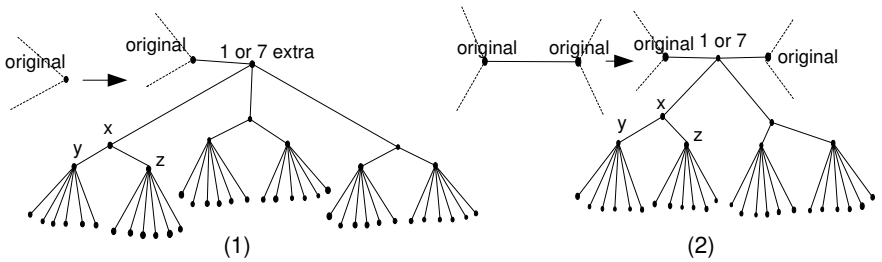

Fig. 10. The vertices added to each 1-vertex and 7-vertex (1), to each extra 1-vertex and extra 7-vertex (2).

of G' can be performed in time polynomial to the size of the finite specification of G and also that the constructed G' is a periodic graph. The reason is explained below.

For each original vertex k of G_i we add extra 0, 1, 7, 8 vertices until its degree gets equals to 4 (rule (6)). Also, for each extra vertex we add a subgraph of six vertices of the form of Figure 9 or a tree of the form of Figure 10. Call the induced subgraph defined by the original vertex k, the extra vertices added for it and their neighbors, as O_k. Also, call the induced subgraph, defined by a new vertex u_{kl} (obtained by the edge subdivision of edge kl) and the new vertices added to this vertex ((by rules (7),(9)), as subgraph S_{kl}.

Define a graph $G'_i(V'_i, E'_i)$ as follows:

(a) $\forall k \in V_i : V(O_k) \subseteq V'_i$ (by rules 1, 8, 9)
(b) $\forall (k,l) \in E_i$ *and* $k, l \in V_i : u_{kl} \in V'_i$ (by rules 2, 3, 4, 5).
(c) $\forall k \in V_i : E(O_k) \subseteq E'_i$ (by rules 8, 9)
(d) $\forall (k,l) \in E_i : V(S_{kl}) \subseteq V'_i$ (by rules 7, 9)
(e) $\forall (k,l) \in E_i : E(S_{kl}) \subseteq E'_i$ (by rules 7, 9)
(f) $\forall (k,l) \in E_i$ *and* $k, l \in V_i : (k, u_{kl}), (u_{kl}, l) \in E'_i$ (by rules 2, 3, 4, 5).
(g) $\forall (k,l) \in E_i$ *and* $k \in V_i$ *but* $l \in V_{i+1} : (k, u_{kl}) \in E'_i$ (by rules 2, 3, 4, 5).

Also, define a set of pairs of nodes, called E'_{i+}, as follows:
$(k,l) \in E_i$ *and* $k \in V_i$ *but* $l \in V_{i+1} : (u_{kl}, l) \in E'_{i+}$ (by rules 2, 3, 4, 5).

Applying the Procedure described above, for any other iteration G_j of G and the edges connecting G_j to G_{j+1} (set E_{j+}), we get a graph $G'_j(V'_j, E'_j)$ and a set of pairs of nodes E'_{i+}. Recall that G_j, E_{j+} are isomorphic to G_i and E_{i+} respectively. Hence, the resulting graph G_j is isomorphic to G'_i and the set of pairs of nodes E'_{j+} is the same with the corresponding set in E'_{i+}. By this repetitive behavior of our procedure, we conclude that the new graph G' obtained, is *a periodic graph* $G' = (G'_i(V'_i, E'_i), E'_{i+})$. We also conclude *that the graph* G' *can be constructed in polynomial time to the size of the finite specification of* G.

Observe also that, if the initial graph G is a planar graph, then the resulting graph G' is also a planar graph. To see why, observe that the extra edges added to G' form a star or a tree. Hence the planarity of the initial graph G remains.

The $\mathcal{PSPACE}$-completeness result follows from the next claim.

Claim 1. G is 3-colorable, if and only if $X_{span}(G') \leq 9$.

Proof. The claim holds because the new periodic planar graph G' computed is the same graph as the graph obtained by [2] when the transformation of [2] applies straight in the actual infinite periodic graph G. □

By the results above, we have shown the $\mathcal{PSPACE}$-completeness of min span RCP for periodic planar graphs of maximum degree seven, with a span of size 9. □

E The PRPA Algorithm

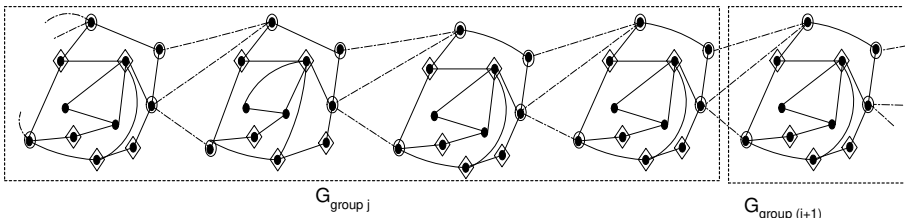

Fig. 11. The partitioning of each iteration G_i into three sets of vertices, S_1, S_2, S_3; the vertices of set S_3 are shown by cycles, of set S_2 by diamonds, and of set S_1 with no surrounding box. Also, the partitioning of graph G into groups of four consecutive iterations in each group.

F Theorem 3

Theorem 3 *The algorithm PRPA produces a radiocoloring of a periodic graph* G.

Proof. Observe first that there is no conflict between the colors of vertices inside a group $G_{group\ j}$. This is so because the neighborhood (consisting of vertices in the same group) of distance 1 and 2 of any vertex in the group is the same as the neighborhood of the corresponding vertex in $G_{group\ j'}$. The vertices take the same colors as the colors of their corresponding vertices computed by the algorithm RC, which is correct. Thus, there are no conflicts between the vertices of $G_{group\ j}$.

What about the colors of vertices in neighbour groups?

Consider again the partitioning defined before of each iteration of the periodic graph into three disjoint sets of vertices: vertices neighbour to vertices of the next iteration graph (set $S_3(G_i)$), vertices neighbour to the vertices of set $S_3(G_i)$, (set $S_2(G_i)$) and vertices that are no neighbour to vertices of either set $S_2(G_i)$ or $S_3(G_i)$ $(S_1(G_i))$.

Consider two consecutive groups $G_{group\ j}$,$G_{group\ j+1}$.

It can be easily seen that there is no conflict between the vertices of graph G_2 of $G_{group\ j}$, and vertices of graph G_2 of $G_{group\ j+1}$.

Consider now the vertices of the graph G_4 of group $G_{group\ j}$ and the vertices of graph G_1 of $G_{group\ j+1}$. For each set of vertices of the two graphs we have:

- *Vertices of set S_3 of graph G_3 of $G_{group\ j}$ (set s_3) and vertices of set S_3 of graph G_1 of $G_{group\ j+1}$* (set s'_3):
  A vertex u of set s_3 is at distance one from some vertices of set s'_3, let v one of them. However, the vertex v has the same color as the corresponding to it vertex in graph G_1 of $G_{group\ j'}$, vertex v'. Moreover, the vertex u takes the same color as the corresponding to it vertex of graph G_4 in $G_{group\ j'}$, vertex u'. Thus, the two vertices u, v take the same colors as their corresponding vertices in $G_{group\ j'}$. Since, the coloring produced by the algorithm RC for the two vertices u', v' in $G_{group\ j'}$, has no conflicts, the coloring of the two vertices u, v in G has also no conflicts.
- *Vertices of set S_3 of graph G_4 of $G_{group\ j}$ (set s_3) and vertices of set S_2 of graph G_1 of $G_{group\ j+1}$* (set s_2):
  A vertex u of set s_3 is at distance two to some vertices of set s_2, let w one of them. However, the vertex w has the same color as the corresponding to it vertex of graph G_2 in $G_{group\ j'}$, vertex w'. Moreover, the vertex u takes the same color as the corresponding to it vertex of graph G_4 in $G_{group\ j'}$, vertex u'. Thus, the two vertices u, w take the same colors as their corresponding vertices in $G_{group\ j'}$. Since, the coloring produced by the algorithm RC for the two vertices u', w' in $G_{group\ j'}$, has no conflicts, the coloring of the two vertices u, w has no conflicts.
- *Vertices of set S_2 of graph G_4 of $G_{group\ j}$ (set s'_2) and vertices of set S_3 of graph G_1 of $G_{group\ j+1}$* (set s'_3). Same reasons as above apply, by replacing s_2 to s'_2 and s_3 to s'_3. Again, we conclude that there are no conflicts between the colors of these vertices.

The above observations are enough to conclude that there are no conflicts between vertices of neighbour groups, and as a result of the whole periodic graph G.

□