The construction of the Cauchy real numbers

in the univalent foundations

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The traditional definition – the idea

Usually, the Cauchy reals are constructed as follows:

- One takes the type $\mathcal{C}$ of Cauchy sequences of rational numbers.
- Then one defines a suitable equivalence relation $\approx$ over them.
- Then $\mathbb{R}_c :\equiv \mathcal{C}/\approx$.

The traditional definition – the complications

Then we get into trouble when we want to show that $\mathbb{R}_c$ is Cauchy complete.

- Given a Cauchy sequence $x : \mathbb{N} \to \mathbb{R}_c$
- We need to lift it to a sequence of sequences $\bar{x} : \mathbb{N} \to \mathcal{C}$.
- To construct the limit of x using $\bar{x}$.

However, this lifting uses the axiom of countable choice or LEM.

We don't assume the axiom of countable choice:

- There are higher toposes where it doesn't hold.
- It doesn't hold for Prop in Coq, likewise not for our (-1)-types.
- Functions on abstract data types should not depend on representations.

The traditional definition – avoiding the trouble

Several common ways to avoid this problem are:

- Pretend that the reals are a setoid.
- Accept the axiom of countable choice.
- Use Dedekind reals instead, but this uses impredicativity.

Andrej Bauer and others at the IAS showed that there is a fourth solution, using higher inductive types!

We get a *type* of real numbers.

Propositions as mere propositions

In the univalent foundations, often we adopt the propositions as types paradigm.

This is the constructive logic in which witnesses remember the case for which an existential quantification or disjunction are proved.

As we shall see in our treatment of the reals, this is not always desired.

Luckily, several modal logics are available in the univalent foundations. Among them, we have propositions as (-1)-types.

Recall that $\mathsf{isProp}(A) :\equiv \prod_{(x,y:A)} x =_A y$. Types satisfying isProp are called mere propositions.

Squash – the basic constructors

For any type A we define the type $\|A\|$ as a higher inductive type with basic constructors

$$| - | : A \to \|A\|$$
$$\text{squashIsProp} : \prod_{(x,y:\|A\|)} x =_{\|A\|} y$$

Squash – the induction principle

For any dependent type $P : \|A\| \to \mathcal{U}$, if there are

$F : \prod_{(a:A)} P(|a|)$
$G : \prod_{(x,y:\|A\|)} \prod_{(u:P(x))} \prod_{(v:P(y))} (\mathsf{squashIsProp}(x,y))_*(u) =_{P(y)} v$

then there is a function $f : \prod_{(x:\|A\|)} P(x)$ with the property that

$$f(|a|) :\equiv F(a)$$
$$f(\mathsf{squashIsProp}(x,y)) = G(x,y,f(x),f(y))$$

Squash – the universal property

The induction principle for squash types implies the following universal property:

Theorem

For an $P : \|A\| \to \mathrm{Prop}$, *the pre-composition function*

$$\lambda f.\, f \circ |-| : \left(\prod_{(x:\|A\|)} P(x)\right) \to \left(\prod_{(a:A)} P(|a|)\right)$$

is an equivalence.

This property determines the mere proposition $\|A\|$ up to equivalence.
Hence we can use the universal property instead of the induction principle.

Propositions as (-1)-types

The squash $\|A\|$ of a type A tells whether A is inhabited without explicitly constructing a term of A

Using squashes we can implement propositions as (-1)-types.

This is useful in the implementation of mathematical concepts such as

- the law of excluded middle:

$$\prod_{(A:\mathrm{Prop})} A \vee \neg A$$

- the axiom of choice: For every type A and every family $P : A \to \mathcal{U}$

$$\left(\prod_{(a:A)} \|P(a)\|\right) \to \left\|\prod_{(a:A)} P(a)\right\|$$

Propositions as (-1)-types

$$
\begin{aligned}
\top &:\equiv \mathbf{1} \\
\bot &:\equiv \mathbf{0} \\
P \wedge Q &:\equiv P \times Q \\
P \Rightarrow Q &:\equiv P \to Q \\
P \Leftrightarrow Q &:\equiv P = Q \\
\neg P &:\equiv P \to \mathbf{0} \\
P \vee Q &:\equiv \|P + Q\| \\
\forall (x : A).\, P(x) &:\equiv \prod_{x:A} P(x) \\
\exists (x : A).\, P(x) &:\equiv \Big\| \sum_{x:A} P(x) \Big\|
\end{aligned}
$$

A characterization of sets

Recall that a type A is a set if there is an element of type:

$$\mathsf{isSet}(A) :\equiv \prod_{x,y:A} \mathsf{isProp}(x =_A y)$$

A convenient way to test whether a type is a set is given by:

Theorem

Suppose $R : A \to A \to \mathsf{Prop}$ *is a reflexive mere relation with the property that*

$$R(x,y) \to (x =_A y)$$

for any $x, y : A$. *Then* A *is a set and* $(x =_A y) \simeq R(x,y)$ *for any* $x, y : A$.

Application: Hedberg's theorem

Corollary

A type is a set if $\neg\neg(x =_A y) \to (x =_a y)$ *for all* $x, y : A$.

Since we have $(X + \neg X) \to (\neg\neg X \to X)$ for all types X we get

Corollary (Hedberg)

A type is a set whenever it has decidable equality.

Types with proximity

Definition

A *proximity predicate* on a type A is a term

$$\sim : \mathbb{Q}_+ \to A \to A \to \mathrm{Prop}.$$

Example

$\mathbb{Q}$ has the proximity predicate $\sim$ defined by

$$(q \sim_\epsilon r) :\equiv (|q - r| < \epsilon)$$

It is even decidable.

Cauchy approximations

Definition

The type of Cauchy approximations in a type A with proximity predicate $\sim$ is defined to be

$$\mathcal{C}(A, \sim) :\equiv \sum_{(x:\mathbb{Q}_+ \to A)} \prod_{(\epsilon,\delta:\mathbb{Q}_+)} x_\delta \sim_{\epsilon+\delta} x_\epsilon.$$

Definition

Suppose A is a type with proximity predicate $\sim$. We say that A is Cauchy complete if there is an element of type

$$\sum_{(\lim:\mathcal{C}(A,\sim)\to A)} \prod_{(x:\mathcal{C}(A,\sim))} \prod_{(\epsilon,\theta:\mathbb{Q}_+)} \lim(x) \sim_{\epsilon+\theta} x_\epsilon.$$

Dependent proximity

Suppose A is a type with proximity predicate $\sim : \mathbb{Q}_+ \to A \to A \to \mathrm{Prop}$.

Definition

A dependent type over A with a proximity predicate on it consists of:

$$B : A \to \mathcal{U}$$
$$\frown : \prod_{(\epsilon:\mathbb{Q}_+)} \prod_{(x,y:A)} B(x) \to B(y) \to (x \sim_\epsilon y) \to \mathcal{U}$$

Instead of writing $\frown (\epsilon, x, y, u, v, p)$ we shall write $u \frown_\epsilon v$.

Dependent Cauchy approximations

Definition

Suppose $(B, \frown)$ is dependent over $(A, \sim)$. The type of dependent Cauchy approximations above a Cauchy approximation $x : \mathcal{C}(A, \sim)$ is defined to be

$$\mathcal{C}_{\mathsf{dep}}(B, \frown, x) :\equiv \sum_{(y:\prod_{(\epsilon:\mathbb{Q}_+)} A(x_\epsilon))} \prod_{(\epsilon,\delta:\mathbb{Q}_+)} y_\epsilon \frown_{\epsilon+\delta} y_\delta.$$

We need those for the induction principle of $\mathbb{R}_c$.

The idea of the definition of the reals

- As in the usual construction, the Cauchy reals form a quotient.
- We define $\mathbb{R}_c$ simultaneously with a proximity predicate $\sim$ over it.
- We add $\mathbb{Q}$ to $\mathbb{R}_c$.
- And for all Cauchy approximations of $\mathbb{R}_c$ we add a limit point.
- We identify reals x, y which satisfy $x \sim_\epsilon y$ for all $\epsilon : \mathbb{Q}_+$.

The idea of the definition of the reals

Thus $\mathbb{R}_c$ consists of

- $\mathbb{Q}$...
- Limits of Cauchy approximations in $\mathbb{Q}$...
- Limits of Cauchy approximations of limits of Cauchy approximations in $\mathbb{Q}$...
- Limits of Cauchy approximations of limits of Cauchy approximations of limits of Cauchy approximations in $\mathbb{Q}$...
- ...and so on, continuing like that transfinitely many steps.

We establish the simultaneous construction of $\mathbb{R}_c$ and $\sim$ with higher induction-induction.

The point and path constructors of $\mathbb{R}_c$

We simultaneously define $\mathbb{R}_c$ and $\sim : \mathbb{Q}_+ \to \mathbb{R}_c \to \mathbb{R}_c \to \mathrm{Prop}$.
The constructors of $\mathbb{R}_c$ are:

- *including the rationals:* $\mathrm{rat} : \mathbb{Q} \to \mathbb{R}_c$
- *including limit points:* $\lim : \mathcal{C}(\mathbb{R}_c, \sim) \to \mathbb{R}_c$
- *taking the quotient:*

$$\mathrm{eq}_{\mathbb{R}_c} : \prod_{x,y:\mathbb{R}_c} \left(\prod_{(\epsilon:\mathbb{Q}_+)} x \sim_\epsilon y\right) \to x =_{\mathbb{R}_c} y$$

The proximity relation on $\mathbb{R}_c$

The constructors of $\sim : \mathbb{Q}_+ \to \mathbb{R}_c \to \mathbb{R}_c \to \mathsf{Prop}$ are:

- The rationals in $\mathbb{R}_c$ inherit the proximity predicate from $\mathbb{Q}$:

$$\prod_{q,r:\mathbb{Q}} (-\epsilon < q - r) \to (q - r < \epsilon) \to (\mathsf{rat}(q) \sim_\epsilon \mathsf{rat}(r)).$$

- Proximity of a rational q to a limit $\lim(y)$ of a Cauchy approximation y is tested by the proximity of q to the components y_δ:

$$\prod_{(q:\mathbb{Q})} \prod_{(y:\mathcal{C}(\mathbb{R}_c,\sim))} \prod_{(\epsilon,\delta:\mathbb{Q}_+)} (\mathsf{rat}(q) \sim_{\epsilon-\delta} y_\delta) \to (\mathsf{rat}(q) \sim_\epsilon \lim(y)).$$

The proximity relation on $\mathbb{R}_c$

- ...and vice versa:

$$\prod_{(r:\mathbb{Q})} \prod_{(x:\mathcal{C}(\mathbb{R}_c,\sim))} \prod_{(\epsilon,\delta:\mathbb{Q}_+)} (x_\delta \sim_{\epsilon-\delta} \mathrm{rat}(r)) \to (\lim(x) \sim_\epsilon \mathrm{rat}(r)).$$

- To see whether $\lim(x)$ is proximate to $\lim(y)$, we need to know whether x_δ is sufficiently proximate to y_η:

$$\prod_{(x,y:\mathcal{C}(\mathbb{R}_c,\sim))} \prod_{(\epsilon,\delta,\eta:\mathbb{Q}_+)} (x_\delta \sim_{\epsilon-\delta-\eta} y_\eta) \to (\lim(x) \sim_\epsilon \lim(y))$$

The induction principle

Since we defined $\mathbb{R}_c$ and $\sim$ simultaneously, the induction principle says how we can prove a property of $\mathbb{R}_c$ simultaneously with a property of $\sim$.

Suppose we have the dependent types

$$A : \mathbb{R}_c \to \mathcal{U}$$

$$\frown : \prod_{(x,y:\mathbb{R}_c)} \prod_{(\epsilon:\mathbb{Q}_+)} A(x) \to A(y) \to (x \sim_\epsilon y) \to \mathrm{Prop}$$

The induction principle tells what to do to define:

$$f : \prod_{x:\mathbb{R}_c} A(x)$$

$$g : \prod_{(x,y:\mathbb{R}_c)} \prod_{(\epsilon:\mathbb{Q}_+)} \prod_{(p:x\sim_\epsilon y)} f(x) \frown_\epsilon f(y)$$

The hypotheses of the induction principle – the constructors of $\mathbb{R}_c$

- The property A should hold on the rationals:

$$F_{\mathsf{rat}} : \prod_{q:\mathbb{Q}} A(\mathsf{rat}(q))$$

- For any Cauchy approximation $x : \mathcal{C}(\mathbb{R}_c, \sim)$, the property A holds on $\lim(x)$ whenever there is a dependent Cauchy approximation $a : \mathcal{C}_{\mathsf{dep}}(A, \frown, x)$, i.e. we can find a function of type

$$F_{\lim}(a) : \mathcal{C}_{\mathsf{dep}}(A, \frown, x) \to A(\lim(x))$$

The hypotheses of the induction principle – the constructors of $\mathbb{R}_c$

- For any $x, y : \mathbb{R}_c$, any $p : \prod_{(\epsilon:\mathbb{Q}_+)} x \sim_\epsilon y$, any $u : A(x)$ and $v : A(y)$ a term

$$F_{\mathsf{eq}} : \left(\prod_{(\epsilon:\mathbb{Q}_+)} u \frown_\epsilon v\right) \to \left(\mathsf{eq}_{\mathbb{R}_c}(p)_*(u) =_{A(y)} v\right)$$

The hypotheses of the induction principle – the constructors of $\sim$

- For any $q, r : \mathbb{Q}$ and $\epsilon : \mathbb{Q}_+$ with $-\epsilon < q - r < \epsilon$ an element

$$G_{\mathsf{rat,rat}}(q, r, \epsilon) : F_{\mathsf{rat}}(q) \frown_\epsilon F_{\mathsf{rat}}(r)$$

- For any $q : \mathbb{Q}$ and $y : \mathcal{C}(\mathbb{R}_c, \sim)$, any dependent cauchy approximation $b : \mathcal{C}_{\mathsf{dep}}(A, \frown, y)$ and $\epsilon, \delta : \mathbb{Q}_+$ a function

$$G_{\mathsf{rat,lim}}(q, b, \epsilon, \delta) : (F_{\mathsf{rat}}(q) \frown_{\epsilon-\delta} b_\delta) \to (F_{\mathsf{rat}}(q) \frown_\epsilon F_{\mathsf{lim}}(b))$$

The hypotheses of the induction principle – the constructors of $\sim$

- For any $x : \mathcal{C}(\mathbb{R}_c, \sim)$ and dependent Cauchy approximation $a : \mathcal{C}_{dep}(A, \frown, x)$, any $r : \mathbb{Q}$ and any $\epsilon, \delta : \mathbb{Q}_+$ a function

$$G_{\mathrm{lim,rat}}(a, r, \epsilon, \delta) : (a_\delta \frown_{\epsilon-\delta} F_{\mathrm{rat}}(r)) \to (F_{\mathrm{lim}}(a) \frown_\epsilon F_{rcrat}(r))$$

- For any $x : \mathcal{C}(\mathbb{R}_c, \sim)$ and $a : \mathcal{C}_{dep}(A, \frown, x)$, any $y : \mathcal{C}(\mathbb{R}_c, \sim)$ and $b : \mathcal{C}_{dep}(A, \frown, y)$ and $\epsilon, \delta, \eta : \mathbb{Q}_+$ a function

$$G_{\mathrm{lim,lim}}(a, b, \epsilon, \delta, \eta) : (a_\delta \frown_{\epsilon-\delta-\eta} b_\eta) \to (F_{\mathrm{lim}}(a) \frown_\epsilon F_{\mathrm{lim}}(b))$$

The outcome of the induction principle for $\mathbb{R}_c$

Then there are functions

$$f : \prod_{x:\mathbb{R}_c} A(x)$$

$$g : \prod_{(x,y:\mathbb{R}_c)} \prod_{(\epsilon:\mathbb{Q}_+)} f(x) \frown_\epsilon f(y)$$

such that

$$f(\mathrm{rat}(q)) :\equiv F_{\mathrm{rat}}(q)$$

$$f(\lim(x)) :\equiv F_{\lim}(\lambda\delta.\, f(x_\delta))$$

A special case

A special case for the induction principle arises when we take

$$A : \mathbb{R}_c \to \mathrm{Prop}$$
$$\frown :\equiv \lambda\_.\,\mathbf{1}.$$

In this case, to show $\prod_{(x:\mathbb{R}_c)} A(x)$, it suffices to show

$$\prod_{q:\mathbb{Q}} A(\mathrm{rat}(q))$$
$$\prod_{x:\mathcal{C}(\mathbb{R}_c,\sim)} \Big(\prod_{\delta:\mathbb{Q}_+} A(x_\delta)\Big) \to A(\lim(x))$$

Basic properties of $\sim$

Lemma

The proximity $\sim_\epsilon$ is reflexive and symmetric:

$$(u \sim_\epsilon u)$$
$$(u \sim_\epsilon v) \to (v \sim_\epsilon u).$$

Hence $\mathbb{R}_c$ is a set.

Proof of reflexivity.

- To show $\prod_{(q:\mathbb{Q})} \mathrm{rat}(q) \sim_\epsilon \mathrm{rat}(q)$, note that $|q - q| < \epsilon$.
- To show $\prod_{(x:\mathcal{C}(\mathbb{R}_c,\sim))} \lim(x) \sim_\epsilon \lim(x)$, assume $\prod_{(\delta:\mathbb{Q}_+)} x_\delta \sim_\epsilon x_\delta$. Then we have $x_{\epsilon/3} \sim_{\epsilon/3} x_{\epsilon/3}$. By the fourth constructor of $\sim$ we get $\lim(x) \sim_\epsilon \lim(x)$.

□

Lipschitz continuity

Definition

A function $f : \mathbb{Q} \to \mathbb{R}_c$ is called Lipschitz if there is an element $L : \mathbb{Q}_+$ such that

$$|q - r| < \epsilon \Rightarrow (f(q) \sim_{L\epsilon} f(r))$$

for all $\epsilon : \mathbb{Q}_+$ and $q, r : \mathbb{Q}$.
Similarly, a function $g : \mathbb{R}_c \to \mathbb{R}_c$ is called Lipschitz if there is an element $L : \mathbb{Q}_+$ such that

$$(u \sim_\epsilon v) \Rightarrow (g(u) \sim_{L\epsilon} g(v))$$

for all $\epsilon : \mathbb{Q}_+$ and $q, r : \mathbb{Q}$.

An extension theorem

Theorem

Suppose $f : \mathbb{Q} \to \mathbb{R}_c$ *is Lipschitz with constant* $L : \mathbb{Q}_+$*. Then there exists a Lipschitz map* $\bar{f} : \mathbb{R}_c \to \mathbb{R}_c$*, also with constant* L*, such that* $\bar{f}(\mathrm{rat}(q)) \equiv f(q)$ *for all* $q : \mathbb{Q}$*.*

Using the extension theorem we can define

- absolute values of reals
- operations $+$, $-$, $\cdot$, $^{-1}$, min, max and $<$.

Theorem

$(u \sim_\epsilon v) \simeq (|u - v| < \mathrm{rat}(\epsilon))$ *for all* $u, v : \mathbb{R}_c$ *and* $\epsilon : \mathbb{Q}_+$*.*

Theorem (Archimedean principle for $\mathbb{R}_c$)

For every $u, v : \mathbb{R}_c$ *such that* $u < v$ *there merely exists* $q : \mathbb{Q}$ *such that* $u < q < v$*.*

Theorem

$\mathbb{R}_c$ *is an archimedian ordered field.*

Theorem

$\mathbb{R}_c$ *is Cauchy complete.*