Math. Log. Quart. **46** (2001) 3, 211 – 222

Mathematical Logic
Quarterly

Located Operators

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Abstract. We study operators with located graph in Bishop-style constructive mathematics. It is shown that a bounded operator has an adjoint if and only if its graph is located. Locatedness of the graph is a necessary and sufficient condition for an unbounded normal operator to have a spectral decomposition. These results suggest that located operators are the right generalization of bounded operators with an adjoint.

Mathematics Subject Classification: 3F60, 26E40, 46S30, 47S30.

Keywords: Unbounded operator, Spectral theorem, Constructive Mathematics, Hilbert space, locatedness.

1 Introduction

Unbounded operators are a natural and important extension of bounded operators. They occur in most places where bounded operators are used. As unbounded operators are discontinuous, one might be tempted to think that they can not be handled in constructive mathematics [10][9][3][11]. If this were the case, it would be a serious problem for the application of constructive mathematics in quantum mechanics, as the unbounded operators $f(x) \mapsto f'(x)$ and $f(x) \mapsto xf(x)$ on $L^2(\mathbf{R})$ play a fundamental role in quantum mechanics.

In constructive mathematics discontinuous functions are usually handled as partial functions. Since unbounded operators *are* partial functions, it is not a priori impossible to treat unbounded operators in constructive mathematics, but the challenge to give a good theory of unbounded operators still stands. Ye [16] developed a theory for unbounded self-adjoint operators, giving constructive proofs for the spectral theorem and Stone's theorem. For Ye a densely defined operator T is self-adjoint if T is symmetric ($T \subset T^*$) and $\mathrm{Ran}(T \pm iI) = H$. He did not seem to know if this definition was really stronger than the usual one: $T = T^*$. We found that it is. In fact, when $T = T^*$, the hypothesis $\mathrm{Ran}(T \pm iI) = H$ is equivalent to the locatedness of the graph of T (section 3). It turns out that locatedness of the graph is a very useful property for general unbounded operators.

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The two main results in this article are that a bounded operator has an adjoint if and only if its graph is located (section 3), and that locatedness of the graph is a necessary and sufficient condition for an unbounded normal operator to have a spectral decomposition (section 4). These results suggest that operators with located graphs are the right generalization of bounded operators with adjoints.

2 Preliminaries

In this section we introduce some notations and quote several theorems from the literature. The main references for the constructive theory are Bishop and Bridges [1] and Bridges and Ishihara [4]; we used Pedersen [12] and Rudin [14] for the classical theory.

All our arguments are constructive, and we do not use axioms that are classically false. This means that our results are acceptable in Bishop-style mathematics and intuitionistic mathematics, and also have a straightforward interpretation in recursive mathematics.

In this article H will denote a separable Hilbert space and e_n an orthonormal basis of H. If it is not known whether the Hilbert space is finite or infinite dimensional, we must allow 0 as a basis vector. In the examples H will always be infinite dimensional.

A subset $M \subset H$ is *located* if for all $x \in H$, the distance

$$\rho(x, M) := \inf\{\|x - m\| : m \in M\}$$

can be computed. A subspace is located if and only if it is the range of a projection. Locatedness is a fundamental concept in constructive mathematics.

An *operator* is a partial linear map from a Hilbert space to itself. An operator T is *bounded* if it is total and there is $M \in \mathbf{R}$ such that, for all $x \in H$, $\|Tx\| \leq M\|x\|$. *Unbounded* operators are operators which we do not require to be totally defined. In this terminology a bounded operator is unbounded. An operator T is *located* if its graph, $\mathbf{G}(T)$, is located.

Write 0 for the subspace $\{0\}$. Let T be an operator such that $(\mathrm{Dom}\, T)^{\perp} = 0$. The adjoint T^* of the operator T is the operator defined as follows. $\mathrm{Dom}\, T^*$ consists of those y such that there is y^* with $\langle Tx, y\rangle = \langle x, y^*\rangle$. Because $(\mathrm{Dom}\, T)^{\perp} = 0$, there can be only one such y^* and for $y \in \mathrm{Dom}(T^*)$, we define $T^*y := y^*$. Classically '$(\mathrm{Dom}\, T)^{\perp} = 0$' is equivalent to '$T$ is densely defined', but constructively the latter is stronger, as we shall see in example 3.5.

If T is bounded by M, then for all x in H, $\|T^*x\| \leq M\|x\|$. We can not prove in general that T^* is densely defined, but if it is, T^* is totally defined and hence a bounded operator.

Let $A, A_1, A_2, \ldots$ be a sequence of bounded operators. We say that the sequence A_n *converges to A in norm*, if there is a sequence of real numbers $M_n \downarrow 0$, such that for all $x \in H$, $\|A_n x - Ax\| \leq M_n\|x\|$. Let $B, B_1, B_2, \ldots$ be a sequence of unbounded operators. We say that B_n *converges strongly to B*, $B_n \to^s B$, if $\mathrm{Dom}\, B = \{x : \lim B_n x \text{ exists}\}$ and $Bx = \lim B_n x$ on $\mathrm{Dom}\, B$.

Let A be bounded; define the *resolvent*

$$R(A) := \{\lambda \in \mathbf{C} : A - \lambda \text{ has a total bounded inverse}\}$$

and the *spectrum* $\sigma(A) := \{\lambda \in \mathbf{C} : \lambda \neq \nu, \text{ for all } \nu \in R(A)\}$. Here and in the rest of the article $\neq$ denotes apartness, that is if (X, ρ) is a metric space and $a, b \in X$, then $a \neq b$ if and only if $\rho(a, b) > 0$.

For $n \in \mathbf{N}$, let π_n be the map $(x_k) \mapsto x_n$ of $\Pi_1^\infty[-1, 1]$ to $[-1, 1]$. A *polynomial* is a function generated from the constant function 1 and finitely many functions $\pi_{n_1}, \ldots, \pi_{n_k}$, by addition and multiplication. Let A_n be a sequence of commuting Hermitian operators. Define the *polynomial map* Ψ as the unique algebra homomorphism from the polynomials to the Hermitian operators such that $\Psi(1) = I$ and $\Psi(\pi_n) = A_n$.

Let μ be a measure. Then $L^\infty_{\mathbf{R}}(\mu)$ and $L^\infty_{\mathbf{C}}(\mu)$ denote the spaces of real and complex bounded μ-measurable functions, respectively. When it is clear which measure is meant we just write $L^\infty_{\mathbf{R}}$ and $L^\infty_{\mathbf{C}}$. Two measurable functions are *equal* if they are equal almost everywhere. All maps on the measurable functions should respect this relation, see [1, p68, p226].

We have the following constructive spectral theorem [1, p378].

Theorem 2.1. *Let A_n be a sequence of commuting Hermitian operators, with common bound 1. There is a measure μ on $\Pi_1^\infty[-1, 1]$, concentrated on $\Pi_{n=1}^\infty \sigma(A_n)$, and a bound preserving homomorphism Ψ from $L^\infty_{\mathbf{R}}$ to an algebra of commuting Hermitian operators, extending the polynomial map. Moreover, if ϕ_n is a uniformly bounded sequence in $L^\infty_{\mathbf{R}}$ and $\phi_n \to \phi \in L^\infty_{\mathbf{R}}$ in measure, then $\Psi(\phi_n) \to \Psi(\phi)$ strongly.*

The fact that μ is concentrated on $\Pi_{n=1}^\infty \sigma(A_n)$ follows from [4, Prop 2.7] applied to $\mu_n(f) := \mu(f \circ \pi_n)$.

Ye [16] proved the following spectral theorem for (unbounded) self-adjoint operators.

Theorem 2.2. *Let A be a self-adjoint operator; then there is a set $\Lambda \subset \mathbf{R}$ and a family of projections $\{E_\lambda : \lambda \in \Lambda\}$ such that*

$$A = \int_{-\infty}^{\infty} \lambda dE_\lambda,$$

and any bounded operator B commutes with every E_λ if and only if B commutes with A, i.e. $BA \subset AB$.

The characteristic function, or indicator, of a set A is denoted by χ_A. Let μ be a measure and let f be μ-integrable. When, for a real number $y > 0$, the sets $[f > y] := \{x : f(x) > y\}$ and $[f \geq y]$ are integrable, then y is called *admissible.* There is a countable set $A \subset \mathbf{R}^+$ such that, if $y \neq t$ for all $t \in A$, then y is admissible for μ. The upshot is that there are enough integrable sets.

Let T be a bounded operator. Define the spectral measure for T by

$$\mu(f) := \sum_{n=1}^{\infty} 2^{-n} \langle f(T)e_n, e_n \rangle,$$

for all $f \in C(\mathbf{R})$. This measure depends on the basis e_n. If ν is the spectral measure associated with another basis, then μ and ν are equivalent, see [4]. We say that t is *admissible for T* if it is admissible for the identity function id on $\mathbf{R}$ and some spectral measure of T.

Theorem 2.3. [2, thm 4.6]*Let T be a bounded Hermitian operator and μ_T its spectral measure. Then* $\operatorname{Ran} T$ *is located if and only if* $\{0\}$ *is* μ_T*-measurable. In particular, if* $\operatorname{Ker} T = 0$ *and* $\{0\}$ *is* μ_T*-measurable, then* $\operatorname{Ran} T$ *is dense.*

3 Locatedness of the graph

In this section some basic theory of located operators is developed.

The following theorem gives necessary and sufficient conditions for the classical theorem stated as condition 1.

Recall that an operator is called *closed* if its graph is closed. Define $V : H^2 \to H^2$ by $V(x, y) = (-y, x)$.

Theorem 3.1. *Let T be a closed operator such that* $(\operatorname{Dom} T)^\perp = 0$*. The following are equivalent:*

(1) $H \times H = V\mathbf{G}(T) \oplus \mathbf{G}(T^*)$

(2) *For all a, b in H, the system*

$$\begin{cases} -Tx + y &= a \\ x + T^*y &= b \end{cases}$$

has a unique solution, with x in $\operatorname{Dom} T$ *and y in* $\operatorname{Dom} T^*$.

(3) *The operator T is located, that is* $\mathbf{G}(T)$ *is located.*

(4) $\operatorname{Ran}(T^*T + I) = H$ *and* $\operatorname{Ran}(TT^* + I) = H$.

Proof. The equivalence of 1 and 2 follows directly from the definitions.

$1 \Leftrightarrow 3$ A closed linear set $M \subset H^2$ is located if and only if $M \oplus M^\perp = H^2$. The sets $V\mathbf{G}(T)$ and $\mathbf{G}(T^*)$ are orthogonal. So the statement follows from the observation that $\mathbf{G}(T)$ is located if and only if $V\mathbf{G}(T)$ is located.

$2 \Rightarrow 4$ The solution for $(0, z)$ shows that $z \in \operatorname{Ran}(T^*T + I)$, the solution for $(z, 0)$ shows that $z \in \operatorname{Ran}(TT^* + I)$.

$4 \Rightarrow 2$ Observe that the operators $T^*T + I$ and $TT^* + I$ are injective. Let B be the right-inverse of $T^*T + I$ and let B' the right-inverse of $TT^* + I$. Now $(-T^*B'a + Bb, B'a + TBb)$ is a solution of the system. □

Suppose that $\operatorname{Ran}(T^*T + I)$ is dense. We claim that $\operatorname{Ran}(T^*T + I) = H$. Observe that $T^*T + I$ is injective. The partial inverse $B : \operatorname{Ran}(T^*T + I) \to H$ is bounded by 1, because

$$\begin{aligned} \|x\|^2 &= \|(T^*T + I)Bx\|^2 \\ &= \langle (T^*T + I)Bx, (T^*T + I)Bx \rangle \\ &= \langle T^*TBx, T^*TBx \rangle + \langle T^*TBx, Bx \rangle \\ &\quad + \langle Bx, T^*TBx \rangle + \langle Bx, Bx \rangle \\ &= \|T^*TBx\|^2 + 2\|TBx\|^2 + \|Bx\|^2 \\ &\geq \|Bx\|^2. \end{aligned}$$

Hence B can be uniquely extended to H. Observe that $(Bx, x) \in \mathbf{G}(T^*T + I)$, for all $x \in \operatorname{Ran} T^*T + I$. In fact it holds for all $x \in H$, because B is bounded, $\mathbf{G}(T^*T + I)$ is closed and $\operatorname{Ran} T^*T + I$ is dense. We conclude that if $\operatorname{Ran}(T^*T + I)$ is dense, then $\operatorname{Ran}(T^*T + I) = H$. A similar statement holds for $TT^* + I$.

One can prove classically that if T is closed and densely defined, then T^* is densely defined and $T^{**} = T$. The constructive content of this theorem is as follows. If T is closed, located and $(\operatorname{Dom} T)^\perp = 0$, then $(\operatorname{Dom} T^*)^\perp = 0$, T^* is located and $(T^*)^* = T$. Example 3.5 shows that T^* is in general not densely defined even if T is.

Let $T = T^*$; then $T^*T + I = T^2 + I = (T+i)(T-i) = (T-i)(T+i)$. So $\operatorname{Ran}(T \pm iI) = H$ if and only if $\operatorname{Ran}(T^2 + I) = H$.

Example 3.2. We give an example of a densely defined, closed operator T such that $T = T^*$ and T is NOT located. That is there is no constructive proof that the operator is located. Let P be a decidable property of the natural numbers such that $P(k)$ holds for at most one natural number k, but we do not know if such k exists. Define

$$Te_n = \begin{cases} 2^{n-1}e_{n-1} + 2^n e_{n+1}, & \text{if for all } k \leq n, \text{ not } P(k); \\ 2^{k-1}e_{n-1} + 2^k e_{n+1}, & \text{if } P(k) \text{ and } n = k; \\ 2^k e_{n-1} + 2^k e_{n+1} & \text{if } P(k) \text{ and } n > k. \end{cases}$$

or in matrix form

$$\begin{pmatrix} 0 & 2 & 0 & & & \\ 2 & 0 & 4 & 0 & & \\ 0 & 4 & 0 & 8 & 0 & \\ & 0 & 8 & 0 & 16 & 0 \\ & & & \ddots & & \end{pmatrix}.$$

By Lemma 6.2 this matrix uniquely defines a self-adjoint operator. If for all k, $P(k)$ does not hold, then $T(\frac{1}{2}e_2 - \frac{1}{4}e_4 + \frac{1}{8}e_6 - \cdots) = e_1$ and $\|\frac{1}{2}e_2 - \frac{1}{4}e_4 + \frac{1}{8}e_6 - \cdots\| < \frac{1}{\sqrt{2}}$, so $\rho((0,e_1), \mathbf{G}(T)) < \frac{1}{\sqrt{2}}$. If $P(k)$, then $\|Tx - e_1\| \geq 1$, for all $x \in H$, with $\|x\| \leq 1$, so $\rho((0,e_1), \mathbf{G}(T)) \geq 1$. So, we are unable to compute the distance from $(0, e_1)$ to $\mathbf{G}(T)$. We conclude that $\mathbf{G}(T)$, and hence T, is NOT located.

3.1 Adjoints

The following lemma follows easily from the definitions.

Lemma 3.3. *Let T and S be operators such that* $0 = (\operatorname{Dom} T)^\perp = (\operatorname{Dom} S)^\perp = \operatorname{Dom}(ST)^\perp$*; then $T^*S^* \subset (ST)^*$. If S is bounded and S^* is total, then $T^*S^* = (ST)^*$.*

Remark that, even when $S = I$, we can not conclude that $(ST)^*$ is densely defined. (See 3.5.)

We now define a class of operators that is useful to prove locatedness of other operators.

Let T be a closed operator with $(\operatorname{Dom} T)^\perp = 0$. Following Pedersen [12, p195] we define $S_\lambda(T) := (I + \lambda T^*T)^{-1}$ whenever $I + \lambda T^*T$ is invertible, and similarly $\tilde{S}_\lambda(T) := (I + \lambda TT^*)^{-1}$. We write $S(T) := S_1(T)$ and $\tilde{S}(T) := \tilde{S}_1(T)$. We may drop the T when no confusion is possible.

If T is located, then $S(T)$ is bounded (Theorem 3.1) and

(1) $$I + \lambda T^*T = (I - (1-\lambda)T^*TS(T))(I + T^*T).$$

Recall that for all $z \in \mathbf{C}$ with $|z| < 1$, $\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k$. Because $T^*TS(T)$ is a bounded operator with bound 1, we see that for all λ with $|1 - \lambda| < 1$,

$$(I - (1-\lambda)T^*TS(T))^{-1} = \sum_{k=0}^{\infty} [(1-\lambda)T^*TS(T)]^k.$$

Hence, it follows from Formula 1 that

$$S_\lambda(T) = S(T) \sum_{k=0}^{\infty} [(1-\lambda)T^*TS(T)]^k.$$

So $S_\lambda(T)$ is a (total) bounded operator. In a similar way we see that $\tilde{S}_\lambda(T)$ is well-defined and bounded. Remark that $S(\lambda T) = S_{|\lambda|^2}(T)$. It follows from Theorem 3.1 that λT is located for all λ with $|1 - |\lambda|^2| < 1$.

The following theorem is important in the construction of the absolute value of an operator and in the spectral theorem. As classically, we can prove that $(\operatorname{Dom} T^*T)^\perp = 0$, when T satisfies the conditions of the theorem, but as we observed earlier, this is not enough constructively.

Theorem 3.4. *If T is densely defined, located and closed, then T^*T is densely defined.*

Proof. Write $Q := T^*T + I$; then $S(T) = Q^{-1}$ is Hermitian. Let Ψ be the spectral map for $S := S(T)$, and let $\alpha, \beta \in (0, 1]$ be admissible for S and $\alpha < \beta$. Write $P = P_{\alpha\beta} = \Psi_{\chi_{[\alpha,\beta]}} = \Psi(\chi_{[\alpha,\beta]})$. Let $f(t) := t^{-1}\chi_{[\alpha,\beta]}(t)$. Observe that for all t, $f(t) - \beta^{-1}\chi_{[\alpha,\beta]}(t) \geq 0$ and $S\Psi(f) = P$, so $\Psi(f) = QP$. Hence for all x in H,

$$\langle (Q - \beta^{-1}I)Px, Px \rangle = \langle P(Q - \beta^{-1}I)Px, x \rangle \geq 0.$$

Observe that $ST \subset TS$, so that $PT \subset TP$. It follows that for x in $\operatorname{Dom} T \subset \operatorname{Dom} TP$, $\|Tx\| \geq \|PTx\| = \|TPx\|$, and hence

$$\|Tx\|^2 + \|x\|^2 \geq \|TPx\|^2 + \|Px\|^2 = \langle QPx, Px \rangle \geq \beta^{-1}\|Px\|^2.$$

This shows that $\|Px\|^2 \leq \beta(\|Tx\|^2 + \|x\|^2)$. This estimate is independent of α, so if we take admissible $\beta_n \downarrow 0$, then for all $m < n$,

$$0 \leq \|\Psi_{\chi_{[\beta_n,1]}}x - \Psi_{\chi_{[\beta_m,1]}}x\|^2 \leq \|\Psi_{\chi_{[\beta_n,\beta_m]}}x\|^2 \leq \beta_m(\|Tx\|^2 + \|x\|^2).$$

Therefore $y = \lim_n \Psi_{\chi_{[\beta_n,1]}}x$ exists. Because $\|S(I - \Psi_{\chi_{[\beta_n,1]}})\| \leq \beta_n$, we see that $S(x - y) = 0$; it follows that $x = y$. As $\operatorname{Ran} \Psi_{\chi_{[\beta_n,1]}} \subset \operatorname{Dom} T^*T$, we see that $\operatorname{Dom} T^*T$ is dense in $\operatorname{Dom} T$, and hence in H. $\square$

Theorem 3.10 implies that T^*T is also located.

Example 3.5. There exists a densely defined, located and closed operator T such that T^* (and therefore TT^*) is NOT densely defined.

Suppose that $P(n)$ is decidable and holds for at most one n, but we do not know if such n exists; define, by Theorem 6.1, the matrix operator

$$Te_n = \begin{cases} 0 & \text{if } \neg P(n); \\ ne_1 & \text{if } P(n). \end{cases}$$

We claim that T is located, closed and densely defined.

To see that T is located; observe that if $P(n)$, then

$$(I+T^*T)^{-1}e_j = \begin{cases} 1/(1+n^2)e_1 & \text{if } j=1; \\ e_j & \text{if } j \neq 1. \end{cases}$$

and

$$(I+TT^*)^{-1}e_j = \begin{cases} 1/(1+n^2)e_n & \text{if } j=n; \\ e_j & \text{if } j \neq n. \end{cases}$$

This tells us how to define $(I+T^*T)^{-1}$ in general. By theorem 3.1 T is located.

To see that T is closed suppose that $x_n \to x \in \mathrm{Dom}T$ and $Tx_n \to y$. If $y \neq Tx$, then $y \neq 0$ or $Tx \neq 0$. In either case we can compute n with $P(n)$ and show that $y \neq Tx$ is impossible.

Now suppose that T^* is densely defined. In particular, there is $x \in \mathrm{Dom}T^*$ close to e_1, in fact we can make sure that $\langle x, e_1\rangle = 1$. Suppose that $P(n)$; then $\|T^*x\| \geq \langle e_n, T^*x\rangle = \langle Te_n, x\rangle = \langle \mathrm{T}e_n, e_1\rangle = n$, so we would have an upper bound for n.

Theorem 3.6. *Let T be a (total) bounded operator. Then T is located if and only if* $\mathrm{Dom}\, T^* = H$.

Proof. Suppose that T is located; then $\tilde{S}_\lambda(T) \to I$ in norm as $\lambda \downarrow 0$. It follows that $\mathrm{Ran}\, \tilde{S}_\lambda = \mathrm{Dom}\, TT^*$ is dense and hence T^* is total.

Conversely, if $\mathrm{Dom}\, T^* = H$, then T^*T and TT^* are Hermitian. So it follows from the spectral theorem for bounded Hermitian operators that $(I+T^*T)^{-1}$ and $(I+TT^*)^{-1}$ exist. Hence T is located. □

Example 3.7. Richman [13] showed that, for a bounded operator T, $\mathrm{Dom}\, T^* = H$ if and only if the image of the unit ball is located. Theorem 3.6 implies that in this case T is also located. In the unbounded case, it is possible that the image of the ball is located and the adjoint is densely defined, but T is NOT located. Indeed, define the matrix operator T such that for all $n \in \mathbf{N}$, $Te_n = ne_n$. Let $v = e_1 + \frac{1}{2}e_2 + \cdots$. Let P be some unsolved problem. Let T_v be the extension of T defined as follows. If P holds define $Tv = e_1$. So $\mathrm{Dom}\, T_v = \mathrm{Span}(\mathrm{Dom}\, T \cup \{v : P\})$.

To see that T_v is closed, suppose that there are $x_n \in \mathrm{Dom}\, T_v$ and $x, y \in H$ with $x_n \to x$ and $T_v x_n \to y$. The elements x_n are of the form $a_{n,v}v + \sum a_{n,m}e_m$. Observe that for $m \geq 2$, $a_{n,v} + a_{n,m} \to \langle x, e_m\rangle$ and $a_{n,m} \to \frac{1}{m}\langle y, e_m\rangle := a_m$. So, $a_{n,v} \to \langle x, e_2\rangle - \frac{1}{2}a_2 := a_v$. Finally $a_{n,1} \to \langle x, e_1\rangle - a_v := a_1$. It follows that $x = a_v v + \sum a_m e_m$ and $y = a_v e_1 + \sum m a_m e_m$. So $x \in \mathrm{Dom}\, T_v$ and $y = T_v x$.

Remark that for all $n \in \mathbf{N}$, $e_n \in \mathrm{Dom}\, T^*$, so T^* is densely defined.

Now $T_v(B_1 \cap \mathrm{Dom}\, T_v)$ is located, but $\rho((v, e_1), \mathbf{G}(T_v))$ can not be computed.

We prove some basic properties of self-adjoint operators.

Definition 3.8. An operator T is *self-adjoint* if T is densely defined, located and $T = T^*$. A set A is a *core* for an operator T if the closure of the graph of $T|_A$ is equal to the graph of T.

It follows from Theorem 3.1 and the discussion following it that our definition of self-adjointness is equivalent with Ye's definition.

Theorem 3.9. *If T is self-adjoint, injective and has dense range, then T^{-1} has the same properties.*

Proof. Recall that $V(x,y) := (-y, x)$ and observe that $V\mathbf{G}(-T) = \mathbf{G}(T^{-1})$. □

Classically the hypothesis that T has dense range is not necessary, as it follows from the fact that T is injective. But there is a recursive example of an injective bounded Hermitian operator that does not have a dense range (See [7]). So the extra hypothesis is necessary constructively.

Theorem 3.10. *If T is densely defined, closed and located, then T^*T is self-adjoint and* $\operatorname{Dom} T^*T$ *is a core for T.*

Proof. The operator $(I + T^*T)^{-1}$ is positive and bounded, so by the previous theorem $I + T^*T$ is self-adjoint. Hence T^*T is self-adjoint.

It is not hard to prove that $S_\lambda(T)x \to^s x$, when λ tends to 0. Moreover, for all $x \in \operatorname{Dom} T$, $TS_\lambda x = \tilde{S}_\lambda Tx \to Tx$. So $\operatorname{Dom} T^*T$ is a core for T, because $\operatorname{Ran} S_\lambda = \operatorname{Dom} T^*T$. See [12, Thm 5.1.9] for a more detailed proof. □

3.2 The absolute value

We show that a densely defined closed located operator has an absolute value.

An operator T is *positive* if it is self-adjoint and for all $x \in \operatorname{Dom} T$, $\langle Tx, x\rangle \geq 0$.

Lemma 3.11. *Let A be a positive operator and let $\{E_\lambda : \lambda \in \Lambda\}$ be a spectral family for A, i.e. $A = \int_{-\infty}^{\infty} \lambda dE_\lambda$. Then $E_\kappa = 0$ for all $\kappa < 0$.*

Proof. Let $\lambda < 0$ and $x \in H$; then

$$0 \leq \langle AE_\lambda x, E_\lambda x\rangle = \int_{-\infty}^{\lambda} \kappa d \langle E_\kappa x, E_\kappa x\rangle \leq 0$$

and hence $\|E_\kappa x\|^2 = 0$ for all $x \in H$ and all $\kappa < \lambda$. □

Theorem 3.12. *Let T be densely defined, closed and located. There is a unique positive operator, denoted by $|T|$, such that $|T|^2 = T^*T$ and* $\operatorname{Dom}|T| = \operatorname{Dom} T$. *Moreover $\|Tx\| = \||T|x\|$ for all $x \in \operatorname{Dom} T$.*

Proof. The operator T^*T is positive, so $|T| = (T^*T)^{1/2}$ is well-defined, by the spectral theorem 2.2 and Lemma 3.11. Because $T^*T = |T|^*|T|$, it follows from Theorem 3.10 that the set $\operatorname{Dom} T^*T$ is a core for both T and $|T|$. Because for all $x \in \operatorname{Dom} T^*T$,

$$\|Tx\|^2 = \langle T^*Tx, x\rangle = \||T|x\|^2.$$

It follows that $\operatorname{Dom} T = \operatorname{Dom}|T|$ and $\|Tx\| = \||T|x\|$, for all $x \in \operatorname{Dom} T$.

To see that $|T|$ is unique, suppose that A is positive, $A^2 = T^*T = |T|^2$ and $\operatorname{Dom} A = \operatorname{Dom} T$. Theorem 2.2 supplies spectral projections E_λ such that

$$\int_0^\infty \lambda dE_\lambda = A.$$

So $T^*T = \int \lambda^2 dE_\lambda$. Fix $\lambda \in \mathbf{R}^+$ and let $P_\lambda := E_\lambda - E_{-\lambda}$; then T^*T commutes with P_λ, and hence so does $|T|$. So $(|T|P_\lambda)^2 = T^*TP_\lambda = (AP_\lambda)^2$. Because the absolute value is unique for bounded operators, $|T|P_\lambda$ is equal to AP_λ, for all $\lambda \in \mathbf{R}^+$. So for all $x \in \operatorname{Dom} A$, $|T|x = \lim |T|P_\lambda x = \lim AP_\lambda x = Ax$. □

4 The spectral theorem

In this section we prove the spectral theorem for unbounded normal operators. A bounded operator N is called *normal* if its adjoint is a total and $N^*N = NN^*$.

We can extend the homomorphism Ψ in the spectral theorem to complex measurable functions in the following way.

T h e o r e m 4.1. *Let $N_1, N_2, N_3, \ldots$ be a commuting sequence of normal operators with a common bound 1. There is a measure μ on $\Pi_1^\infty[-1,1]^2$, concentrated on $\Pi_{i=1}^\infty \sigma(N_i)$ and a homomorphism Ψ from $L^\infty_{\mathbf{C}}(\mu)$ to an algebra of commuting normal operators, extending the polynomial map. Moreover, if ϕ_n is a uniformly bounded sequence in $L^\infty_{\mathbf{C}}$ and $\phi_n \to \phi \in L^\infty_{\mathbf{C}}$ in measure, then $\Psi(\phi_n) \to \Psi(\phi)$ strongly.*

P r o o f. By the Fuglede-Putnam-Rosenblum theorem [14, 12.16] we see that for all n, m, $N_n N_m^* = N_m^* N_n$. Write $N_n = N_n^r + iN_n^i$, where $N_n^r := \frac{1}{2}(N_n^* + N_n)$ and $N_n^i := \frac{1}{2}i(N_n^* - N_n)$ are commuting Hermitian operators. Apply the spectral theorem for Hermitian operators to the sequence $N_1^r, N_1^i, N_2^r \ldots$ We identify $[-1,1]^2$ with a subset of $\mathbf{C}$. For $f \in L^\infty_{\mathbf{C}}$ define $\Psi_{\mathbf{C}}(f) := \Psi_{\mathbf{R}}(\operatorname{Re} f) + i\Psi_{\mathbf{R}}(\operatorname{Im} f)$. Observe that $\Psi_{\mathbf{C}}(\bar{z}) = \Psi_{\mathbf{C}}(z)^*$. The fact that μ is concentrated on $\Pi_{n=1}^\infty \sigma(N_n)$ follows from carefully substituting squares for intervals in [4, Lemma 2.3 to 2.7]. □

Recall that a bounded operator T is *positive* if for all $x \in H$, $\langle Tx, x\rangle \geq 0$.

R e m a r k 4.2. Let N be bounded and normal. If N is self-adjoint, then $\sigma(N) \subset \mathbf{R}$ and if N is positive, then $\sigma(N) \subset \mathbf{R}^+$ (see [4, Cor. 3.3]). The converse is also true: if $\sigma(N) \subset \mathbf{R}^+$, then the functions $x \mapsto x$ and $x \mapsto |x|$ are equal almost everywhere. So $N = |N|$ is positive.

D e f i n i t i o n 4.3. Two self-adjoint operators *commute* if all their spectral projections commute.

T h e o r e m 4.4. *For a densely defined, located, closed operator T, the following conditions are equivalent:*

(1) $\operatorname{Dom}(T) = \operatorname{Dom}(T^*)$, *and* $\|Tx\| = \|T^*x\|$ *for every* x *in* $\operatorname{Dom}(T)$.

(2) $T^*T = TT^*$.

(3) *There are commuting, self-adjoint operators A and B such that $T = A + iB$, $T^* = A - iB$, and $\|Tx\|^2 = \|Ax\|^2 + \|Bx\|^2$ for every x in* $\operatorname{Dom} T$.

P r o o f. Pedersen's proof [12, 5.1.10] is constructive. We give a sketch.

$1 \Rightarrow 2$: Fix $x \in \operatorname{Dom} T^*T$ and $y \in \operatorname{Dom} T$. Then

$$\begin{aligned} 4\langle T^*Tx, y\rangle &= 4\sum_{k=0}^{3} i^k \|T(x + i^k y)\|^2 \\ &= 4\sum_{k=0}^{3} i^k \|T^*(x + i^k y)\|^2 = \langle T^*x, T^*y\rangle. \end{aligned}$$

So $T^*x \in \operatorname{Dom} T^{**} = \operatorname{Dom} T$ and $TT^*x = T^*Tx$. Hence $T^*T \subset TT^*$; the other inclusion is obtained by a symmetry argument.

$2 \Rightarrow 1$: For all $x \in \operatorname{Dom} T^*T$, $\|Tx\|^2 = \|T^*x\|^2$. So $\operatorname{Dom} T \subset \operatorname{Dom} T^*$, because T^* is closed and for $x \in \operatorname{Dom} T$ and $\epsilon \downarrow 0$: $S_\epsilon x \to x$, $TS_\epsilon x \to Tx$. The converse inclusion follows from a symmetry argument.

$1 \Rightarrow 3$ and $3 \Rightarrow 1$ follow from long, but not very difficult computations. □

Let T be densely defined, closed and located. If T satisfies the conditions in Theorem 4.4, we say that T is *normal.* Note that in this case T^* is densely defined.

When T is bounded, it follows from Theorem 4.4 that this definition of 'normal' is equivalent with the definition at the beginning of this section.

Remark 4.5. A normal operator is maximally normal, i.e. if N and T are normal and $T \subset N$, then $T = N$. Indeed, $T \subset N$, so $N^* \subset T^*$, hence $\operatorname{Dom} T = \operatorname{Dom} T^* \supset \operatorname{Dom} N^* = \operatorname{Dom} N \supset \operatorname{Dom} T$, i.e. $T = N$.

The following theorem extends the spectral theorem to unbounded measurable functions. We need a lemma first.

Lemma 4.6. *Let μ be a spectral measure and let f be a measurable function. Suppose that there are sequences $f_n, g_n \in L^\infty$ such that $f_n \to f$ in measure, $g_n \to f$ in measure and for all $n \in \mathbf{N}$, $|f_n| \leq |f|$ and $|g_n| \leq |f|$. Let $x \in H$. Suppose that $y := \lim_n \Psi(f_n)x$ exists; then $\lim \Psi(g_n)x$ exists and equals y.*

Proof. There is a increasing sequence A_m of measurable sets such that $\chi_{A_m} \to 1$ in measure and for all m, there is N_m such that: if $n \geq N_m$, then $|f - f_n|\chi_{A_m} < 2^{-m}$ and $|f - g_n|\chi_{A_m} < 2^{-m}$. Note that f is bounded on A_m.

Define for all m, $\chi_m := \chi_{A_m}$, $Q_m := \Psi(\chi_m)$ and $P_m := \Psi(\chi_m - \chi_{m-1})$. Note that for all $z \in H$, $z = \lim Q_m z$ and $z = \sum P_m z$.

Suppose that A is a measurable set on which f is bounded. Then $|f|^2\chi_A \geq |g_n|^2\chi_A$. In fact this holds for every measurable set A. So, it follows from Remark 4.2 and Lemma 3.11 that for all $z \in H$,

$$\left\langle \Psi(|f|^2\chi_A - |g_n|^2\chi_A|)z, z \right\rangle \geq 0,$$

$$\left\langle \Psi(|f|^2\chi_A)z, z \right\rangle \geq \left\langle \Psi(|g_n|^2\chi_A|)z, z \right\rangle,$$

$$\|\Psi(f\chi_A)z\|^2 \geq \|\Psi(g_n\chi_A)z\|^2. \tag{2}$$

Fix $\varepsilon > 0$. Choose m such that $\|Q_m y - y\| < \varepsilon$. Choose N such that for all $n \geq N$, $\|\Psi(g_n\chi_m)x - \Psi(f\chi_m)x\| < \varepsilon$. Remark that

$$Q_m y = Q_m \lim \Psi(f_n)x = \lim_n \Psi(f_n\chi_m)x = \Psi(f\chi_m)x$$

and $\Psi(g_n\chi_m)x = Q_m\Psi(\chi_m)x$. So $\|Q_m\Psi(g_n)x - Q_m y\| < \varepsilon$. Note that

$$\Psi(g_n)x - \Psi(g_n)Q_m x = \sum_{k>m} P_k\Psi(g_n)x$$

and $y - Q_m y = \sum_{k>m} P_n y$. Moreover for all $k \in \mathbf{N}$, $\|P_k\Psi(g_n)x\| \leq \|P_k y\|$, by (2). So $\|\Psi(g_n)x - Q_m\Psi(g_n)x\| \leq \|y - Q_m y\| \leq \varepsilon$. It follows that for all $n \geq N$,

$$\begin{aligned} \|\Psi(g_n)x - y\| &\leq \|\Psi(g_n)x - \Psi(g_n)Q_m x\| \\ &\quad + \|\Psi(g_n)Q_m x - Q_m y\| \\ &\quad + \|Q_m y - y\| \\ &\leq 3\varepsilon. \end{aligned}$$

Hence $\lim \Psi(g_n)x = y$. $\square$

Remark 4.7. In the previous lemma the hypothesis that for all $n \in \mathbf{N}$, $|f_n| \leq |f|$ is necessary. Indeed, define the multiplication operator M_h on $L_2[0,1]$ by $M_h g = h \cdot g$. Consider the spectral map Ψ for the Hermitian operator M_{id}, where id is the identity map. Note that $f_n := n\chi_{[0,1/n]}$ converges to 0 in measure. But the operators $\Psi(f_n) = M_{f_n}$ do not converge to M_0 in the strong topology.

Theorem 4.8. *Let A_n be a sequence of commuting normal operators bounded by 1. The homomorphism in the spectral theorem can be extended to the unbounded measurable functions. This map has the following properties. If f is measurable, then $\Psi(f)$ is a normal operator. If f, g are measurable, then $\Psi(\bar{f}) = \Psi(f)^*$, $\Psi(f)+\Psi(g) \subset \Psi(f+g)$, $\Psi(f)\Psi(g) \subset \Psi(fg)$ and $\operatorname{Dom}(\Psi(f)\Psi(g)) = \operatorname{Dom}\Psi(g) \cap \operatorname{Dom}\Psi(fg)$.*

Proof. Let f be a measurable function. Choose a sequence $f_n \in L^\infty$, such that $f_n \to f$ in measure and for all $n \in \mathbf{N}$, $|f_n| \leq |f|$. Define

$$\operatorname{Dom}\Psi(f) := \{x : \lim \Psi(f_n)x \text{ exists}\}$$

and $\Psi(f)x := \lim \Psi(f_n)x$. Lemma 4.6 implies that this definition does not depend on the choice of the sequence f_n.

We claim that $\Psi(f)$ is densely defined. Indeed, choose $\alpha_n \uparrow \infty$ such that $\chi_n := \chi_{[|f| \leq \alpha_n]}$ is measurable. Note that $\chi_n \to 1$ in measure, so $Q_n := \Psi(\chi_n) \to^s I$, by [1, 8.22]. Finally observe that for all $n \in \mathbf{N}$, $\operatorname{Ran} Q_n \subset \operatorname{Dom}\Psi(f)$.

Let g be measurable. We claim that $\Psi(f)\Psi(g) \subset \Psi(fg)$ and $\operatorname{Dom}(\Psi(f)\Psi(g)) = \operatorname{Dom}\Psi(g) \cap \operatorname{Dom}\Psi(fg)$. Indeed, let $x \in \operatorname{Dom}\Psi(g) \cap \operatorname{Dom}\Psi(fg)$ and choose sequences $f_n, g_n \in L_\infty$ such that $f_n \to f$ in measure, $g_n \to g$ in measure and for all n, $|f_n| \leq |f|$ and $|g_n| \leq |g|$. Then $|f_n g_m| \leq |fg|$ and

$$\begin{aligned} |fg - f_n g_m| &\leq |fg - f_n g| + |f_n g - f_n g_m| \\ &\leq |f - f_n||g| + |f_n||g - g_m| \\ &\leq |f - f_n||g| + |f||g - g_m|, \end{aligned}$$

which converges to 0 in measure when $n, m \to \infty$. So

$$\begin{aligned} \Psi(fg)x &= \lim_{n,m} \Psi(f_n g_m)x \\ &= \lim_{n,m} \Psi(f_n)\Psi(g_m)x \\ &= \lim_{n} \Psi(f_n)\Psi(g)x. \end{aligned}$$

This implies that $\Psi(g)x \in \operatorname{Dom}\Psi(f)$, so $\operatorname{Dom}\Psi(g) \cap \operatorname{Dom}\Psi(fg) \subset \operatorname{Dom}\Psi(f)\Psi(g)$. The other inclusion is straightforward.

The reader is invited to check that $\Psi(f) + \Psi(g) \subset \Psi(f+g)$.

To show that $\Psi(f)$ is closed we show that $\Psi(\bar{f}) = \Psi(f)^*$. For all n, $\Psi(f\chi_n)$ is normal and $\Psi(\bar{f}\chi_n) = \Psi(f\chi_n)^*$, so for all $x \in H$,

$$\|\Psi(f\chi_n)x\| = \|\Psi(f\chi_n)^*x\| = \|\Psi(\bar{f}\chi_n)x\|.$$

Hence $\operatorname{Dom}\Psi(f) = \operatorname{Dom}\Psi(\bar{f})$ and $\Psi(\bar{f}) \subset \Psi(f)^*$. It follows from Lemma 3.3 that $Q_n\Psi(f)^* \subset (\Psi(f)Q_n)^* = \Psi(f\chi_n)^* = \Psi(\bar{f}\chi_n)$. For all $z \in \operatorname{Dom}\Psi(f)^*$, $Q_n\Psi(f)^*z = \Psi(\bar{f}\chi_n)z$, whence $z \in \operatorname{Dom}\Psi(\bar{f})$.

Finally observe that $\Psi(f)\Psi(f)^* \subset \Psi(|f|^2) \supset \Psi(f)^*\Psi(f)$, so that $S(\Psi(f)) = \Psi((|f|^2+1)^{-1})$. By Theorem 3.1 $\Psi(f)$ is located and because it satisfies (1) of Theorem 4.4 it is normal. □

Theorem 4.8 tells us how to go from bounded operators to unbounded operators. We would also like to have a spectral theorem *starting* from an unbounded operator.

Remark that if we would have a spectral theorem, then $T^*T = TT^* = \Psi(z\bar{z})$ and

$$(T^*T + I)^{-1} = \Psi((|z|^2 + 1)^{-1}) = (TT^* + I)^{-1}.$$

So we can only hope to prove the spectral theorem for (located) normal operators.

Let $\zeta(z) := z(1 - |z|)^{-1}$ for z with $|z| < 1$; then $\zeta^{-1}(z) := z(1 + |z|)^{-1}$. We will use this function as a substitute for the Cayley transform, which was used in the case where the operators are self-adjoint. The function ζ^{-1} maps $\mathbf{C}$ into its unit ball.

The proof of the following lemma can be simplified if T has a polar decomposition. Unfortunately, it can not be proved constructively that every operator has a polar decomposition [5], not even for bounded operators.

Recall that if T is self-adjoint and $T = \int \lambda dE_\lambda$, then we say that a bounded operator B *commutes* with T if $BT \subset TB$, or equivalently $E_\lambda B = BE_\lambda$, for all $\lambda \in \mathbf{R}$.

Lemma 4.9. *Let T be normal; then $\zeta^{-1}(T) := T(I + |T|)^{-1}$ is a bounded normal operator which commutes with the bounded operator $(I + |T|)^{-1}$, and $(I + |T|)^{-1}T \subset \zeta^{-1}(T)$.*

Proof. The operator T^*T is self-adjoint, so $A := (I + |T|)^{-1}$ is a well-defined bounded operator commuting with T^*T. We claim that the operator $\zeta^{-1}(T) = TA$ is bounded. Indeed, $\operatorname{Ran} A = \operatorname{Ran}(I + |T|)^{-1} = \operatorname{Dom}(|T|) = \operatorname{Dom} T$, so by Theorem 3.12, $\|TAx\| = \||T|Ax\|$. It follows that TA is a bounded operator, because $|T|A$ is. We claim that TA is also normal. Indeed, $TAT^*T \subset TT^*TA = T^*TTA$, so $TAE = ETA$, for all E in the spectral family of T^*T, whence $TAA = ATA$, as A is a bounded continuous function of T^*T. This implies that $TA = AT$ on the dense set $\operatorname{Ran} A = \operatorname{Dom} T$, similarly $T^*A = AT^*$ on $\operatorname{Dom} T^* = \operatorname{Dom} T$. Note that $(TA)^* \supset AT^* \subset T^*A$, so for all x in the dense set $\operatorname{Dom} T^*$, $\|(TA)^*x\| = \|T^*Ax\| = \|TAx\|$. Because TA is bounded and $(TA)^*$ is closed, $(TA)^*$ is bounded and $\|(TA)^*x\| = \|TAx\|$, for all $x \in H$. □

Definition 4.10. Unbounded normal operators T and A *commute* if $\zeta^{-1}(T)$ and $\zeta^{-1}(A)$ commute. Define the spectrum $\sigma(T)$ of T by $\sigma(T) := \zeta(\sigma(\zeta^{-1}(T)))$.

It will follow from the next theorem and Lemma 4.14 that these definitions are equivalent to other reasonable definitions. In particular with definition 4.3.

Theorem 4.11. *Let $T_1, T_2, \dots$ be a sequence of (unbounded) commuting normal operators. There is a measure μ on $\Pi_{i=1}^\infty \mathbf{C}$, concentrated on $\Pi_{i=1}^\infty \sigma(T_i)$ and a *homomorphism Ψ mapping $L^\infty_{\mathbf{C}}(\mu)$ isometrically to an algebra of commuting bounded normal operators, and mapping measurable functions to normal operators. The map Ψ extends the polynomial map. It can be extended to all μ-measurable functions such that the following properties hold. If $f \in L^\infty$ and f_n is a uniformly bounded sequence in L_∞ which converges to f in measure, then $\Psi(f_n) \to \Psi(f)$ in norm. If f_n, f are measurable, $|f_n| \leq |f|$ and $f_n \to f$ in measure, then $\Psi(f_n) \to^s \Psi(f)$. Moreover $\Psi(\bar{f}) = \Psi(f)^*$, $\Psi(f) + \Psi(g) \subset \Psi(f + g)$, $\Psi(f)\Psi(g) \subset \Psi(fg)$ and $\operatorname{Dom}(\Psi(f)\Psi(g)) = \operatorname{Dom} \Psi(g) \cap \operatorname{Dom} \Psi(fg)$.*

Proof. Denote $f^\zeta(x_1, x_2, \dots) := f(\zeta(x_1), \zeta(x_2), \dots)$. Let ν be the spectral measure on $\Pi_{i=1}^\infty[-1, 1]^2$ for the sequence $\zeta^{-1}(T_1), \zeta^{-1}(T_2), \dots$ Define for f on $\Pi_{i=1}^\infty \mathbf{C}$, $\mu(f) := \nu(f^\zeta)$ and $\Psi_\mu(f) = \Psi_\nu(f^\zeta)$. We have to prove that for all $i \in \mathbf{N}$, $\Psi_\mu(\pi_i) = T_i$.

Fix $i \in \mathbf{N}$. Set $T := T_i$, $A := (I + |T|)^{-1}$ and $T_0 := TA$, note that A and T_0 are bounded.

The operators $|T_0|$ and $|T|A$ are equal, because the absolute value is unique and $|T_0|^2 = T_0^* T_0 = T^* ATA = |T|^2 A^2 = (|T|A)^2$. So $I - |T_0| = (I + |T|)A - |T|A = A$. That is $A = \Psi_\phi(T_0)$, where $\phi(t) := 1 - |t|$. Let μ_0 be a spectral measure for T_0 and let $\mu_A(f) := \mu_0(f \circ \phi)$.

From Theorem 2.3 and the fact that $\operatorname{Ran} A = \operatorname{Dom}|T| = \operatorname{Dom} T$ is dense, it follows that $\{0\}$ is μ_A-measurable. Hence

$$\mu_A(0,1] = 1 - \mu_A\{0\} = 1,$$

because $\chi_{\{0\}}(A)$ is the projection on the kernel and A is injective.

Choose $t_0 := 1$ and a sequence $t_i \downarrow 0$ which is admissible for μ_A. Define $P_i := \chi_{(t_i, t_{i-1}]}(A)$; then for all $x \in H$, $\sum_i P_i x = x$. Define $g_i(t) = \frac{1}{t}\chi_{(t_i,t_{i-1}]}(t)$. Note that $P_i T = g_i(A)AT \subset g_i(A)TA = TAg_i(A) = TP_i$, by Lemma 4.9. Similarly $P_i T^* \subset T^* P_i$.

Define $\psi_n(z) := \frac{z}{\phi(z)}\chi_{(t_n,1]}(\phi(z))$. Remark that

$$\sum_{i=1}^{n} TP_i = \sum_{i=1}^{n} T_0 g_i(A) = \psi_n(T_0)$$

converges to a normal operator, with adjoint $\sum (T_0 g_i(A))^* = \sum T^* P_i$. For $x \in \operatorname{Dom} T$, $Tx = T\sum P_i x = \sum TP_i x$, so $T \subset \sum TP_i$, but both T and $\sum TP_i$ are normal and hence these operators are equal, by Remark 4.5. □

Recall that, when T is bounded, the resolvent is defined as

$$R(T) := \{\lambda \in \mathbf{C} : T - \lambda \text{ has a bounded inverse}\}.$$

Lemma 4.12. *If T is bounded by M and f is holomorphic on a region containing the closed ball $\overline{B}_M$, then $R(f(T)) \subset f(R(T))$.*

Proof. This follows easily from the fact that for all $\lambda \in \mathbf{C}$,

$$f(T) - f(\lambda) = \sum_{n=1}^{\infty} a_n(\lambda^n I - T^n) = (\lambda I - T)A_\lambda,$$

where $f(z) = \sum a_n z^n$ and A_λ is a suitable bounded operator commuting with T. □

Definition 4.13. For an unbounded normal operator T define

$$R(T) := \{\lambda \in \mathbf{C} : T - \lambda I \text{ has a bounded inverse}\}.$$

Lemma 4.14. *Let T be normal; then $R(T) = \zeta(R(\zeta^{-1}(T))$.*

Proof. Define $\zeta_n(z) := \frac{z}{1+\frac{1}{n}-|z|}$; then $\zeta_n^{-1}(z) = \frac{(1+\frac{1}{n})z}{1+|z|}$. The function ζ_n^{-1} maps the ball with radius n onto the unit-ball and the rest of the complex plane into the ball with radius $1 + \frac{1}{n}$. The maps ζ_n and ζ_n^{-1} are holomorphic on $\overline{B}_{1+\frac{1}{2n}}$ and $\mathbf{C}$, respectively. Let Ψ be the spectral map for T. Define $T_n = \Psi_{\mathrm{id}\chi_{B_n}}$ for all admissible $n > 0$. Then $\mathrm{R}(T) \cap B_n = \mathrm{R}(T_n) = \zeta_n R(\zeta_n^{-1}(T_n))$, by the previous lemma. □

Lemma 4.14 shows that our definition of the spectrum coincides with the usual definition.

5 Some other approaches to the spectral theorem.

Finally we discuss three classical approaches to the spectral theorem that do not seem to work constructively.

First consider the classical theorem: Let T be Hermitian; there are a measure μ, a unitary operator $U : H \to L_2(\mu)$ and a measurable function h, such that for all $x \in H$, $Tx = U^{-1}(h \cdot Ux)$. Even the 2-dimensional version of this theorem: 'every Hermitian matrix is unitarily equivalent to a diagonal matrix' can not be proved constructively [6, p21]. This is the constructive analogue of the classical fact that the eigenvectors do not depend continuously on the elements of the matrix. This approach was promoted by Halmos [8].

A second approach uses Gelfand theory, see for instance [12]. This approach seems to work only when all the operators are normable [1, 7.8.28].

Finally, consider the approach used by Rudin [14, Ch. 13]. Roughly, the idea is as follows. If A is Hermitian, then $A = \int \lambda dE_\lambda$. Define for bounded measurable functions f, $E_{xy}(f) := \langle \Psi(f)x, y\rangle$; then $\langle Ax, y\rangle = E_{xy}(\mathrm{id})$. Let f be an unbounded measurable function. We define $\mathrm{Dom}(\Psi(f)) := \{x : E_{xx}(|f|^2) < \infty\}$. Remark that for $x \in \mathrm{Dom}(\Psi(f))$, $y \mapsto E_{xy}(f)$ is a bounded linear functional. So by the Riesz representation theorem, there exists $z_x \in H$ such that $E_{xy}(f) = \langle z_x, y\rangle$. Finally we define $\Psi(f)x := z_x$.

To generalize this approach to the case where we start with countably many Hermitian operators. We could use the theorem:

Theorem 5.1. *If $A_1, A_2, \ldots$ is a sequence of commuting Hermitian operators, then there is an operator B and a sequence of continuous functions $j_1, j_2, \ldots$ on σ_B such that for all n, $j_n(B) = A_n$.*

Finally, define $f(A_1, A_2, \ldots)$ as $f(j_1(B), j_2(B), \ldots)$.

There are several problems when we want to make this approach constructive. First, it is difficult in constructive mathematics to define a class of measurable functions without defining a measure simultaneously. Second, in the constructive Riesz representation theorem, we need the hypothesis that the functional is normable, not just bounded. Finally, Theorem 5.1 can be proved constructively, but in general $f \circ j$ is not measurable in constructive mathematics. For instance, let $j = c$ be constant and $c \notin \mathrm{Dom}\, f$.

6 Appendix on matrix operators

In this appendix we assume that the Hilbert space H is infinite dimensional.

Recall the following theorem on matrix operators[15, p149].

Theorem 6.1. *If a_{jk} is a matrix and $\sum_j |a_{jk}|^2 < \infty$ for all k, and A is the corresponding matrix operator, then $D(A)$ is dense, A^* is a restriction of the operator associated to the adjoint matrix. If $\sum_k |a_{jk}|^2 < \infty$ for all j, then A is closed and A^* is densely defined.*

In fact it was proved in [15] that, if $\sum_k |a_{jk}|^2 < \infty$ for all j, then $A = (A_0^+)^*$. Here A^+ is the operator associated with the adjoint matrix and A_0^+ the restriction of A^+ to $\mathrm{Span}\{e_1, e_2, \ldots\}$. We see that for all $n \in N$, $e_n \in \mathrm{Dom}\, A^*$.

Lemma 6.2. *If a_{jk} is a symmetric real matrix and $\sum_j |a_{jk}|^2 < \infty$ for all k, and A is the corresponding matrix operator, then A is self-adjoint.*

Proof. It follows from the previous theorem that $A^* \subset A$. But from the definition of A we see that $\mathrm{Span}\{e_1, e_2, \ldots\}$ is a core for A. Hence $A = A^*$, because A^* is closed. $\square$

Acknowledgement. Most of this research was carried out when I visited Douglas Bridges at the University of Canterbury, New Zealand. I would like to thank him for his hospitality and his valuable suggestions. This visit was partially supported by NWO, the Marsden Fund of the Royal Society of New Zealand, and the University of Nijmegen. I would also like to thank my PhD-advisor, Wim Veldman, for equally valuable suggestions. I would like to thank Arnoud van Rooij for his comments on a draft of this paper. Finally, I would like to thank the referee for comments on the presentation of this paper.

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