Internal Universes in Models of Homotopy Type Theory

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HoTT

Voevodsky: Simplicial model of univalent type theory
Coquand: Cubical model of univalent type theory

Cubical type theory

Axiomatics: What makes this model tick?
New models: How can we generalize this?

E.g. zoo of cubical models
Adding features: guarded types

Cubical model

Cube category $\Box$: Lawvere theory of De Morgan algebras
$=$ opposite of the category of finitely gen DM algebras
Topos of cubical sets: $\hat{\Box}$
The internal (extensional) type theory has interval type $\mathbb{I}$
$\wp A = \mathbb{I} \to A$
Coquand: internal statement of uniformity condition, fibrations, ...
Cf: Two level type theory HTS, internal models of sets

Full model and axiomatic treatment: OP, GCTT
Universe of fibrant types is axiomatized.
Can we construct it from the Hofmann-Streicher universe?

Cofibrations

The predicate $\cdot = 1 : \mathbb{I} \to \Omega$ defines a collection of propositions $\mathrm{cof} \subset \Omega$, the *face* lattice.
Maps $A \to \mathrm{cof}$ are called *cofibrations*
OP: axiomatization of cof.
extension relation, 'x extends t':

$$\_ \nearrow \_ : \{\varphi : \mathrm{Set}\}\{A : \mathrm{Set}\}(t : \varphi \to A)(x : A) \to \mathrm{Set}$$
$$t \nearrow x = (u : \varphi) \to t\, u \equiv x$$

Composition structure

The type isFib A of *fibration structures* for a family of types $A : \Gamma \to \mathrm{Set}$ over context $\Gamma : \mathrm{Set}$ consists of functions taking any path $p : \wp\,\Gamma$ in the base type to a *composition structure* in $\mathrm{C}(A \circ p)$:

$$\begin{aligned} \mathrm{isFib} &: (\Gamma : \mathrm{Set})(A : \Gamma \to \mathrm{Set}) \to \mathrm{Set} \\ \mathrm{isFib}\,\Gamma\,A &= (p : \wp\,\Gamma) \to \mathrm{C}(A \circ p) \end{aligned}$$

$$\begin{aligned} \mathrm{CCHM}\,P = \;& (\varphi : \mathrm{Set})(\_ : \mathrm{cof}\,\varphi)(p : (i : \mathbb{I}) \to \varphi \to P\,i) \to \\ & (\Sigma\, a_0 : P\,\mathtt{0}\,,\, p\,\mathtt{0} \nearrow a_0) \to (\Sigma\, a_1 : P\,\mathtt{I}\,,\, p\,\mathtt{I} \nearrow a_1) \end{aligned}$$

Nogo theorem

Internal model:
Consider the subCwF of fibrant families

Thm: There is no internally defined universe of fibrant types

Proof.
It would be stable under weakening.
This leads to a contradiction (agda) □

Modal type theory

Universe needs to be defined in the empty context
Idea: Modal type theory (Pfenning/...).

Simplicial sets is a cohesive topos ($\int \dashv \flat \dashv \sharp$)
Very general setting for topology
$\int$ monad: shape (connected components)
$\flat$ comonad: discrete topology
$\sharp$ monad: codiscrete topology
Lawvere: sythetic differential geometry
Schreiber/Shulman: cohesive type theory (for physics)

Proposition: Cubical sets is cohesive too

spatial type theory

Shulman: synthetic homotopy theory
HoTT has two circles:
Homotopical (1-type) and topological (0-type)
Use cohesive type theory to connect them
Spatial type theory: the $\flat, \sharp$ fragment
Conjecture(Shulman): can be interpreted in *local* higher toposes
Here: by UIP restrict to (1-)toposes
Cf. Awodey/Birkedal

crisp modal type theory

Dual context modal type theory:
$\Delta \mid \Gamma \vdash a : A$
Γ the usual local elements
Δ new global elements
This can be interpreted in connected toposes:
Here: the comonad $\flat : \widehat{\Box} \to \widehat{\Box}$ that sends a presheaf A to the constant presheaf on the set of global sections of A; thus
$\flat A(X) \cong A(1)$

Families over $\Sigma_{\flat\Delta}\Gamma$

crisp modal type theory

The crisp variable and (admissible) substitution rules:

$$\frac{}{\Delta, x :: A, \Delta' \mid \Gamma \vdash x : A} \qquad \frac{\Delta \mid \diamond \vdash a : A \qquad \Delta, x :: A, \Delta \mid \Gamma \vdash b : B}{\Delta, \Delta[a/x] \mid \Gamma[a/x] \vdash b[a/x] : B[a/x]}$$

Global elements can be used locally.

Agda $\flat$

Parametricity is proved using the model of reflexive relations
Reflexive relations are truncated simplicial/cubical sets
Cohesive type theory
Vezzosi: a lot of parametricity results can be obtained this way
agda-$\flat$.
(Nuyts, Vezzosi, Devriese)

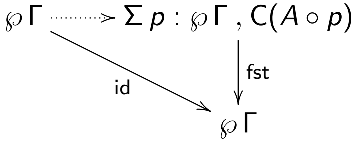
Amazing right adjoint

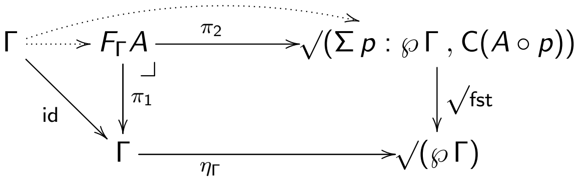
Final ingredient:
In $\hat{\square}$, $\mathbb{I}$ is *tiny* (Lawvere):
$^{\mathbb{I}}$ has a (global) *right* adjoint $\surd$

$$
\begin{aligned}
(\mathbb{I} \to F)\, X &\cong \widehat{\square}(\mathrm{y}X, \mathbb{I} \to F) \\
&\cong \widehat{\square}(\mathrm{y}X \times \mathrm{y}\mathrm{I}, F) \cong \widehat{\square}(\mathrm{y}(X \times \mathrm{I}), F) = ((\_ \times \mathrm{I})^* F)\, X
\end{aligned}
$$

Dependent right adjoint

Fibrations structures are sections:





This suggests taking $\Gamma = \mathsf{Set}$ and $A = \mathsf{id} : \mathsf{Set} \to \mathsf{Set}$ to get a universe $\mathsf{U} = F_{\mathsf{Set}}\mathsf{id}$ and family $\pi_1 : F_{\mathsf{Set}}\mathsf{id} \to \mathsf{Set}$ to classify fibrations.

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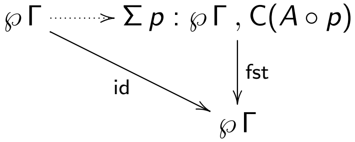

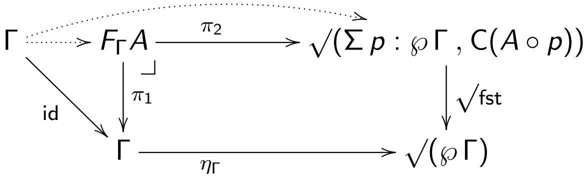

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Challenges

Coherence $\to$ Voevodsky coherence construction
local/global $\to$ crisp type theory

Theorem

Internal universe construction

Applications

Our constructions are modular,
and we expect them to be useful in related models too,
E.g. cartesian cubes, directed type theory

Conclusion

Internal construction of the cubical model using:

- Crisp type theory
- Amazing dependent right adjoint