Modal DTT and the cubical model

Bas Spitters

Modal Dependent Type Theory and Dependent Right Adjoints

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Outline

More examples of modalities: nominal type theory, guarded and clocked type theory, and spatial and cohesive type theory.

Modal dependent type theory with an operator satisfying the K-axiom of modal logic.

$$\Box(A \to B) \to \Box A \to \Box B$$

We provide semantics and syntax for a DTT with universes.

Since every finite limit category with an adjunction of endofunctors gives rise to model, we call this *dependent right adjoint*

Dependent K

$$\Box(\Pi y : A.\, B) \to \Pi x : \Box A.\, \Box B[\mathsf{open}\, x/y]$$

Dependent right adjoint

A *dependent right adjoint* then extends the definition of CwF with a functor on contexts L and an operation on families R, intuitively understood to be left and right adjoints:

Definition (**category with a dependent right adjoint**)

A *CwDRA* is a CwF C equipped with the following extra structure:

1. An endofunctor $\mathsf{L} : \mathsf{C} \to \mathsf{C}$ on the underlying category.
2. For each object $\Gamma \in \mathsf{C}$ and family $A \in \mathsf{C}(\mathsf{L}\Gamma)$, a family $\mathsf{R}_\Gamma A \in \mathsf{C}(\Gamma)$, stable under re-indexing in the sense that for all $\gamma \in \mathsf{C}(\Delta, \Gamma)$ we have $(\mathsf{R}_\Gamma A)[\gamma] = \mathsf{R}_\Delta(A[\mathsf{L}\gamma]) \in \mathsf{C}(\Delta)$
3. For each object $\Gamma \in \mathsf{C}$ and family $A \in \mathsf{C}(\mathsf{L}\Gamma)$ a bijection

$$\mathsf{C}(\mathsf{L}\Gamma \vdash A) \cong \mathsf{C}(\Gamma \vdash \mathsf{R}_\Gamma A) \tag{1}$$

We write the effect of this bijection on $a \in \mathsf{C}(\mathsf{L}\Gamma \vdash A)$ as $\overline{a} \in \mathsf{C}(\Gamma \vdash \mathsf{R}_\Gamma A)$ and write the effect of its inverse on $b \in \mathsf{C}(\Gamma \vdash \mathsf{R}_\Gamma A)$ also as $\overline{b} \in \mathsf{C}(\mathsf{L}\Gamma \vdash A)$. Thus

$$\overline{\overline{a}} = a \qquad (a \in \mathsf{C}(\mathsf{L}\Gamma \vdash A)) \tag{2}$$

$$\overline{\overline{b}} = b \qquad (b \in \mathsf{C}(\Gamma \vdash \mathsf{R}_\Gamma A)) \tag{3}$$

The bijection is required to be stable under re-indexing.

Mode theory

Conjecture Licata:
Our works fits within the mode theory framework
Checked for STT

Dradjoint as a guide for the general case ?

Syntax

Context formation rules:

$$\frac{}{\diamond \vdash} \qquad \frac{\Gamma \vdash \quad \Gamma \vdash A}{\Gamma, x : A \vdash}\ x \notin \Gamma \qquad \frac{\Gamma \vdash}{\Gamma, 🔓 \vdash} \qquad \frac{\Gamma, x : A, y : B, \Gamma' \vdash}{\Gamma, y : B, x : A, \Gamma' \vdash}\ x \text{ NOT FREE IN } B$$

Type formation rules:

$$\frac{\Gamma \vdash A \quad \Gamma, x : A \vdash B}{\Gamma \vdash \Pi x : A.\, B} \qquad \frac{\Gamma, 🔓 \vdash A}{\Gamma \vdash \Box A}$$

$$\frac{\Gamma \vdash t : A \quad \Gamma \vdash A = B}{\Gamma \vdash t : B} \qquad \frac{\Gamma, x : A, \Gamma' \vdash}{\Gamma, x : A, \Gamma' \vdash x : A}\ 🔓 \notin \Gamma' \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : \Pi x : A.\, B}$$

$$\frac{\Gamma \vdash t : \Pi x : A.\, B \quad \Gamma \vdash u : A}{\Gamma \vdash t\, u : B[u/x]}$$

$$\frac{\Gamma, 🔓 \vdash t : A}{\Gamma \vdash \mathsf{shut}\ t : \Box A} \qquad \frac{\Gamma \vdash t : \Box A \quad \Gamma, 🔓, \Gamma' \vdash}{\Gamma, 🔓, \Gamma' \vdash \mathsf{open}\ t : A}\ 🔓 \notin \Gamma'$$

Term equality rules

$$\frac{\Gamma \vdash (\lambda x.t)u : A}{\Gamma \vdash (\lambda x.t)u = t[u/x] : A} \qquad \frac{\Gamma \vdash \mathsf{open}\ \mathsf{shut}\ t : A}{\Gamma \vdash \mathsf{open}\ \mathsf{shut}\ t = t : A} \qquad \frac{\Gamma \vdash t : \Pi x : A.\, B}{\Gamma \vdash t = \lambda x.t\, x : \Pi x : A.\, B}\ x \notin \Gamma$$

$$\frac{\Gamma \vdash t : \Box A}{\Gamma \vdash t = \mathsf{shut}\ \mathsf{open}\ t : \Box A}$$

Sytax and semantics

Sound interpretation
Term model (conjecture: initiality)
Normalization and canonicity for simply typed case.
Conjecture: extends to dependent case.

Local universe

Definition

Let C be a cartesian category. The **Giraud CwF of** C ($\mathbb{G}$C) is the CwF whose underlying category is C, and where a family $A \in \mathbb{G}C(\Gamma)$ is a pair of morphisms

$$\begin{array}{ccc} & & E \\ & & \downarrow{\scriptstyle v} \\ \Gamma & \xrightarrow[u]{} & U \end{array} \tag{4}$$

and an element is a map $a : \Gamma \to E$ such that $v \circ a = u$.
Reindexing

$$A[\gamma] \triangleq (u \circ \gamma, v) \in \mathbb{G}C(\Delta)$$
$$a[\gamma] \triangleq a \circ \gamma \in \mathbb{G}C(\Delta \vdash A[\gamma])$$

The comprehension $\Gamma.A \in C$ is given by the pullback of diagram (4).

CwDRA from cartesian category

Definition

A **weak CwF morphism** R between CwFs consists of a functor $\mathsf{R} : \mathsf{C} \to \mathbf{C}D$ between the underlying categories preserving the terminal object, an operation on families mapping $A \in \mathsf{C}(\Gamma)$ to a family $\mathsf{R}A \in \mathbf{C}D(\mathsf{R}\Gamma)$, an operation on elements mapping $a \in \mathsf{C}(\Gamma \vdash A)$ to an element $\mathsf{R}a \in \mathbf{C}D(\mathsf{R}\Gamma \vdash \mathsf{R}A)$, and an isomorphism $\nu_{\Gamma,A} : \mathsf{R}\Gamma.\mathsf{R}A \to \mathsf{R}(\Gamma.A)$, inverse to $(\mathsf{Rp}_A, \mathsf{Rq}_A)$. Required to commute with reindexing: $\mathsf{R}A[\mathsf{R}\gamma] = \mathsf{R}(A[\gamma])$ and $\mathsf{R}t[\mathsf{R}\gamma] = \mathsf{R}(t[\gamma])$.

CwFs as discrete comprehension categories and then using pseudo-maps of comprehension cats.

Theorem

$\mathbb{G}$ is a (fully faithful) functor from the category of cartesian categories and finite limit preserving functors, to the category of CwFs with weak morphisms.

Theorem

If C is a cartesian category and $\mathsf{L} \dashv \mathsf{R}$ are adjoint endofunctors on C, then $\mathbb{G}\mathsf{C}$ has the structure of a CwDRA.

Examples

- Nominal sets
- Guarded and Clocked Type Theory
- Cohesive toposes
- $\mathbb{I}$ is *tiny* if exponentation by it has a right-adjoint $\surd$. Dependent right-adjoint plays is important in the construction of the universe in cubical sets.

Universes

Extension to universes using Coquand's CwU:
Presheaf models interpret an *inverse* $\ulcorner\urcorner$ to El from codes to types.

Let $\tilde{U} \to U$ be a universe.
Suppose that R preserves small fibers.
Then R can be lifted to a CwDRA with universes.
The image under R of maps with U-small fibers is classified by the universe with codes RU.

Examples include essential geometric morphisms given by functors on the underlying category.

Convenient universe polymorphic category seems to be missing.

Conclusions

Dependent right-adjoint

- Syntax, semantics
- Many examples from the literature and from adjunction on lex category.

Internal Universes in Models of Homotopy Type Theory

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Cubical type theory

Axiomatics: What makes this model tick?
New models: How can we generalize this?

E.g. zoo of cubical models

Adding features: guarded types, nominal, ..., realizability, directed type theory

Cubical model

Cube category $\Box$: Lawvere theory of De Morgan algebras
$=$ opposite of the category of finitely gen DM algebras
Topos of cubical sets: $\hat{\Box}$
The internal (extensional) type theory has interval type $\mathbb{I}$
$\wp A = \mathbb{I} \to A$
Coquand: internal statement of uniformity condition, fibrations, ...
Cf: Two level type theory HTS, internal models of ZF

Full model and axiomatic treatment: OP, GCTT
Universe of fibrant types is axiomatized *externally*.
Can we construct it from the Hofmann-Streicher universe?

Cofibrations

The predicate $\cdot = 1 : \mathbb{I} \to \Omega$ defines a collection of propositions cof $\subset \Omega$, the *face* lattice.
Maps $A \to$ cof are called *cofibrations*
OP: axiomatization of cof.
extension relation, 'x extends t':

$$\_ \nearrow \_ : \{\varphi : \mathrm{cof}\}\{A : \mathrm{Set}\}(t : \varphi \to A)(x : A) \to \mathrm{Set}$$
$$t \nearrow x = (u : \varphi) \to t\, u \equiv x$$

Composition structure

The type $\mathsf{isFib}\,A$ of *fibration structures* for a family of types $A : \Gamma \to \mathsf{Set}$ over context $\Gamma : \mathsf{Set}$ consists of functions taking any path $p : \wp\,\Gamma$ in the base type to a *composition structure* in $\mathsf{C}(A \circ p)$:

$$\begin{aligned} \mathsf{isFib} &: (\Gamma : \mathsf{Set})(A : \Gamma \to \mathsf{Set}) \to \mathsf{Set} \\ \mathsf{isFib}\,\Gamma\,A &= (p : \wp\,\Gamma) \to \mathsf{C}(A \circ p) \end{aligned}$$

$$\begin{aligned} \mathsf{CCHM}\,P = \ & (\varphi : \mathsf{Set})(\_ : \mathsf{cof}\,\varphi)(p : (i : \mathbb{I}) \to \varphi \to P\,i) \to \\ & (\Sigma\, a_0 : P\,\mathtt{0}\,,\, p\,\mathtt{0} \nearrow a_0) \to (\Sigma\, a_1 : P\,\mathtt{I}\,,\, p\,\mathtt{I} \nearrow a_1) \end{aligned}$$

Nogo theorem

Internal model:
Consider the subCwF of fibrant families

Thm: There is no internally defined universe of fibrant types

Proof.
It would be stable under reindexing.
This leads to a contradiction (agda) □

Modal type theory

Universe needs to be defined in the empty context
Idea: Modal type theory (Pfenning/...).

Simplicial sets is a cohesive topos ($\int \dashv \flat \dashv \sharp$)
Very general setting for topology
$\int$ monad: shape (connected components)
$\flat$ comonad: discrete topology
$\sharp$ monad: codiscrete topology
Lawvere: sythetic differential geometry
Schreiber/Shulman: cohesive type theory (for physics)

Proposition: Cubical sets is cohesive too

Spatial type theory

Shulman: synthetic homotopy theory
HoTT has two circles:
Homotopical (1-type) and topological (0-type)
Use cohesive type theory to connect them
Crisp type theory: the $\flat$ fragment
Conjecture(Shulman): can be interpreted in *connected* higher toposes
Here: by UIP restrict to (1-)toposes

Crisp modal type theory

Dual context modal type theory:
$\Delta \mid \Gamma \vdash a : A$
Γ the usual local elements
Δ new global elements
This can be interpreted in connected toposes:
Here: the comonad $\flat : \widehat{\square} \to \widehat{\square}$ that sends a presheaf A to the constant presheaf on the set of global sections of A; thus $\flat A(X) \cong A(1)$

Types are interpreted as families over $\Sigma_{\flat\Delta}\Gamma$

Crisp modal type theory

The crisp variable and (admissible) substitution rules:

$$\frac{}{\Delta, x :: A, \Delta' \mid \Gamma \vdash x : A} \qquad \frac{\Delta \mid \diamond \vdash a : A \qquad \Delta, x :: A, \Delta' \mid \Gamma \vdash b : B}{\Delta, \Delta'[a/x] \mid \Gamma[a/x] \vdash b[a/x] : B[a/x]}$$

Global elements can be used locally.

Agda $\flat$

Parametricity is proved using the model of reflexive relations
Reflexive relations are truncated simplicial/cubical sets
Cohesive type theory
Vezzosi: a lot of parametricity results can be obtained this way
agda-$\flat$.
(Nuyts, Vezzosi, Devriese)

Amazing right adjoint

Final ingredient:
In $\hat{\square}$, $\mathbb{I}$ is *tiny* (Lawvere):
$^{\mathbb{I}}$ has a (global) *right* adjoint $\surd$

$$\begin{aligned}(\mathbb{I} \to F)\,X &\cong \widehat{\square}(\mathrm{y}X, \mathbb{I} \to F) \\ &\cong \widehat{\square}(\mathrm{y}X \times \mathrm{yI}, F) \cong \widehat{\square}(\mathrm{y}(X \times \mathrm{I}), F) = ((\_ \times \mathrm{I})^{*} F)\,X\end{aligned}$$

Sattler: $\surd$ should be useful for constructing universes.