Cubical sets as a classifying topos

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Homotopy Type Theory

The homotopical interpretation of type theory:

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- types as spaces upto homotopy

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(homotopy type) theory = homotopy (type theory)

Homotopy Type Theory/Univalent foundations

Some applications:

- Synthetic Homotopy theory
- New foundation for mathematics
  inherent treatment of equivalences.
- Internal language for higher toposes
- Semantics for Type theory
  Programming languages and proof assistants

The identity type of the universe

The univalence axiom describes the identity type of the universe U.
There is a canonical function

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The univalence axiom ...

- formalizes the informal practice of substituting a structure for an isomorphic one.
- implies function extensionality ($\Pi_x f(x) = g(x) \to f = g$)
- used to reason about higher inductive types
- equivalent to (Rezk, Rijke/S):
  - universe is an object classifier
  - decent theorem

Simplicial sets

Univalence modeled in Kan fibrations of simplicial sets. (VV)
Simplicial sets are a standard example of a classifying topos.
Joyal/Johnstone: geometric realization as a geometric morphism.

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Challenge:
computational interpretation of univalence and
higher inductive types.
Solution (Coquand et al): Cubical sets
Can we extend classifying topos methods?

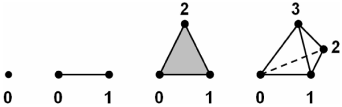
Cubical type checker

Type checker by Cohen, Coquand, Huber, Mörtberg.
`cubical`
New cubical type theory.
E.g. functional extensionality from the interval.

```
funExt (A : U) (B : A -> U) (f g : (x : A) -> B x)
       (p : (x : A) -> Id (B x) (f x) (g x)) :
                      : Id ((y : A) -> B y) f g =
  <i> \(a : A) -> (p a) @ i
```

Simplicial sets

Simplex category Δ:
finite ordinals and monotone maps
Simplicial sets $\hat{\Delta}$.
Geometric realization/Singular complex: $|-| : \widehat{\Delta} \to Top : S$



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E.g. $|-|$ is left exact (pres fin lims).
However, Top is not a topos.
Johnstone: use topological topos instead.

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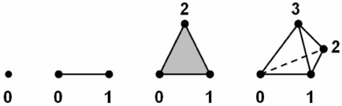

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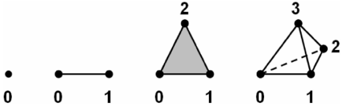
Roughly: points, equalities, equalities between equalities, ...

Geometric realization of simplicial sets

Simplices are constructed from the linear order on $\mathbb{R}$ in Set.



Can be done in any topos with a linear order.
Geometric realization becomes a geometric morphism by moving from spaces to the topological toposes.
Equivalence of cats:

$$Orders(\mathcal{E}) \to Hom(\mathcal{E}, \hat{\Delta})$$

assigns to an order I in $\mathcal{E}$, the geometric realization defined by I.
Simplicial sets classify the *geometric* theory of strict linear orders.

Cubical sets

$\mathbb{2}$: poset with two elements
$\square$: full subcategory of Cat with obj powers of $\mathbb{2}$.

$$\begin{array}{ccc} 00 & \xrightarrow{\leqslant} & 01 \\ \downarrow{\scriptstyle\leqslant} & & \downarrow{\scriptstyle\leqslant} \\ 10 & \xrightarrow{\leqslant} & 11 \end{array}$$

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Duality: finite posets and distibutive lattices.
$\mathbb{2}$ is the ambimorphic object here:
poset maps into $\mathbb{2}$ pick out ‘opens’
DL-maps select the ‘points’.
Stone duality between powers of $\mathbb{2}$ and
free finitely generate distributive lattices (copowers of DL1)

Aside: Nerve

Nerve construction:
Embedding of $\square$ into Cat.
Hence, $Cat \to \widehat{\square}$, defined by $C \mapsto \mathrm{hom}(-, C)$.
This is fully faithful (Awodey).
$\square$ is a dense subcategory of Cat.

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Alternative nerve construction, using Lawvere theories.

Lawvere theory

Classifying categories for Cartesian categories.
Alternative to monads in CS (Plotkin-Power)
For algebraic theory T, the *Lawvere theory* Θ_T^{op} is the opposite of the category of free finitely generated models.
models of T in any finite product category category E correspond to product-preserving functors $m : \Theta_T^{op} \to E$.
$m(n)$ consists of the n-tuples in the model m.
A map $1 \to T(2)$, gives a map $m^2 \to m^1$, as both are T-algebras.
E.g. $* \mapsto (x \wedge y)$, defines $(x, y) \mapsto (x \wedge y)$.

Nerve construction and Lawvere theories

Alternative nerve construction, using Lawvere theories.
Consider DL the free distributive lattice monad on Fin.
Then Θ_{DL}^{op} is the Lawvere theory for distributive lattices.
The inclusion of distributive lattices into $\widehat{\Theta_{DL}}$ is fully faithful.
The image consists of those presheaves satisfying the Segal condition.
$\widehat{\Theta_{DL}} = \check{\Box}$ are the cocubical sets.

Classifying topos

Λ_T: finitely *presented* T-models.
$\Lambda_T \to Set$ is the classifying topos.
This topos contains a generic T-algebra.
T-algebras in any topos $\mathcal{F}$ correspond to *left exact left adjoint* functors from the classifying topos to $\mathcal{F}$.

Classifying topos

Example:
T a propositional geometric theory (=formal topology).
$Sh(T)$ is the classifying topos.

Set^{Fin} classifies the Cartesian theory with one sort.
Used for variable binding (Fiore, Plotkin,Turi, Hofmann).
Replaces Pitts' use of nominal sets for the cubical model.
Nominal sets classify decidable infinite sets.

Moerdijk: Connes' cyclic sets classify abstract circles.

Classifying topos of cubical sets

Let $\Theta = \square^{op}$ be the category of *free finitely generated* DL-algebras
Let Λ the category of *finitely presented* ones.
We have a fully faithful functor $f : \Theta \to \Lambda$.
This gives a geometric embedding $\phi : \check{\Theta} \to \check{\Lambda}$

Classifying topos of cubical sets

The subtopos $\check{\Theta}$ of the classifying topos for DL-algebras is given by a quotient theory, the theory of the model $\mathbb{I} := \phi^* M$,
the DL-algebra $\mathbb{I}(m) := m$ for each $m \in \Theta$.

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Each free finitely generated DL-algebra has the disjunction property $(a \vee b = 1 \vdash a = 1, b = 1)$
This properties is geometric and hence also holds for $\mathbb{I}$.

Geometric realization for cubical sets

Theorem (Johnstone-Wraith)

Let T be an algebraic theory, then the topos $\breve{\Theta}_{DL}$ classifies the geometric theory of flat T-models.

In particular, $\widehat{\square}$ classifies flat distributive lattices.

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Need to show that (classically) [0,1] is a flat DL-algebra.

Geometric realization as a geometric morphism

Prop: Every linear order D defines a flat distributive lattice.
Hence, we have a geometric morphism $\widehat{\Delta} \to \widehat{\square}$.
Let $\mathcal{E}$ be Johnstone's topological topos.

Theorem (Cubical geometric realization)

There is a geometric morphism $r : \mathcal{E} \to \hat{\square}$ defined using the flat distributive lattice $[0, 1]$.

We obtain the familiar formulas for both simplicial and topological realization.

Related work

Independently, Awodey showed that Cartesian cubical sets (without connections or reversions) classify strictly bipointed objects.
Much of Awodey's constructions of the cubical methods can be extended based on $\mathbb{I}$ and should give Coquand's model?

Categorical models of Id-types

Application: We have an ETT with an internal 'interval' $\mathbb{I}$.
van den Berg, Garner path object categories.
Usual path composition is only h-associative.
Moore paths can have arbitrary length.
category freely generated from paths of length one.
Moore paths: strict associativity, but non-strict involution.
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Apply vdB/G-construction. However, work internally in the topos of cubical sets using the generic DL-algebra $\mathbb{I}$.
Simplifies computation substantially.
North: Π-h-tribe.

Categorical models of Id-types

Coquand (MFPS): much of the cubical model can be carried out in the internal logic of $\hat{\Box}$.
Observation (WIP):
Need a topos with an DL with disjunction property and $\forall : \mathbb{I}^{\mathbb{I}} \to \mathbb{I}$.
E.g. sSets, $\widehat{\Box \times \omega}$

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$PathA := A^{\mathbb{I}}$. Univalent model.
No judgmental computation rule for J (Path-recursion).
No: $J_{A,D}(d, a, a, r_A(a)) = d(a) : D(a, a, r_A(A))$
only: $Path_{D(a,a,r_A(A))}(J_{A,D}(d, a, a, r_A(a)), d(a))$
Coquand/Swan: Id
Coquand: Univalence for Id

Path,Id,Moore

Path,Id,Moore are all equivalent types ...

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Path,Id,Moore are all equivalent types ...
Path-equivalent, Id-equivalent, Moore-equivalent in CTT

Two premodel structures on the fibrant objects
in the internal logic of $\hat{\square}$.
Using the cubical type theory.
Uses GG factorization for *Id* or Moore
and mapping cylinder wrt Path
$(\mathcal{C}, \mathcal{W}, \mathcal{F})$ are the same, but factorizations are not.

Conclusion

- Cubical sets as a classifying topos.
- Cubical model in the internal logic.
- Premodel structure in the internal logic
- Cubical geometric realization