The space of measurement outcomes as a spectrum for a non-commutative algebras

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Goal

Relate algebraic quantum mechanics to topos theory to construct new foundations for quantum logic and quantum spaces.
— A spectrum for non-commutative algebras —

Classical physics

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a *proposition* '$a \in \Delta$' by the set $a^{-1}\Delta$.
Spatial logic: logical connectives $\wedge, \vee, \neg$ are interpreted by $\cap, \cup$, complement
For a phase σ in Σ,
$\sigma \models a \in \Delta$ (in the phase σ the proposition $a \in \Delta$ holds) iff $a(\sigma) \in \Delta$

Quantum

Heunen,Landsman,S generalization to the quantum setting by

1. Identifying a quantum phase 'space' Σ.
2. Defining 'subsets' of Σ acting as propositions of quantum mechanics.
3. Describing states in terms of Σ.
4. Associating a proposition $a \in \Delta$ $(\subset \Sigma)$ to an observable a and an open subset $\Delta \subseteq \mathbb{R}$.
5. Finding a pairing map between states and 'subsets' of Σ (and hence between states and propositions of the type $a \in \Delta$).

Old-style quantum logic

von Neumann proposed:

1. A quantum phase space is a Hilbert space H.
2. Elementary propositions correspond to closed linear subspaces of H.
3. Pure states are unit vectors in H.
4. The closed linear subspace $[a \in \Delta]$ is the image $E(\Delta)H$ of the spectral projection $E(\Delta)$ defined by a and Δ.
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Von Neumann later abandoned this.
No implication, no deductive system.

Bohrification

In classical physics we have a spatial logic.
Want the same for quantum physics. So we consider two generalizations of topological spaces:

- C*-algebras (Connes' non-commutative geometry)
- toposes and locales (Grothendieck)

We connect the two generalizations by:

1. *Algebraic quantum theory*
2. *Constructive Gelfand duality*
3. *Bohr's doctrine of classical concepts*

Classical concepts

Bohr's "doctrine of classical concepts" states that we can only look at the quantum world through classical glasses, measurement merely providing a "classical snapshot of reality". The combination of all such snapshots should then provide a complete picture.

HLS proposal

Let A be a C*-algebra.
The set of as 'classical contexts', 'windows on the world':
$\mathcal{C}(A) := \{V \subseteq A \mid V \text{ commutative C*-algebra}\}$ ordered by inclusion.

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1. The quantum phase space of the system described by A is the locale $\underline{\Sigma} \equiv \underline{\Sigma}(\underline{A})$ in the topos $\mathcal{T}(A)$.
2. Propositions about A are the 'opens' in $\underline{\Sigma}$. The quantum logic of A is given by the Heyting algebra underlying $\underline{\Sigma}(\underline{A})$. Each projection defines such an open.
3. Observables $a \in A_{\mathrm{sa}}$ define locale maps $\delta(a) : \underline{\Sigma} \to \mathbb{IR}$, where $\mathbb{IR}$ is the so-called interval domain. States ρ on A yield probability measures (valuations) μ_ρ on $\underline{\Sigma}$.
4. The frame map $\mathcal{O}(\mathbb{IR}) \xrightarrow{\delta(a)^{-1}} \mathcal{O}(\underline{\Sigma})$ applied to an open interval $\Delta \subseteq \mathbb{R}$ yields the desired proposition.
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Motivation: Doering-Isham use topos theory for quantum theory.

Gelfand duality

There is a categorical equivalence (Gelfand duality):

$$\mathbf{CommC}^* \underset{C(-,\mathbb{C})}{\overset{\Sigma}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{CptHd}^{\mathrm{op}}$$

The structure space $\Sigma(A)$ is called the Gelfand spectrum of A.

C*-algebras

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Prime example:
$B(H) = \{f : H \to H \mid f \text{ bounded linear}\}$, for H Hilbert space.

is a complex vector space:	$(f+g)(x) := f(x) + g(x)$,
	$(z \cdot f)(x) := z \cdot f(x)$,
is an associative algebra:	$f \cdot g := f \circ g$,
is a Banach algebra:	$\|f\| := \sup\{\|f(x)\| \ : \ \|x\| = 1\}$,
has an involution:	$\langle fx, y \rangle = \langle x, f^* y \rangle$
satisfies:	$\|f^* \cdot f\| = \|f\|^2$,
but not necessarily:	$f \cdot g = g \cdot f$.

Slogan: C*-algebras are non-commutative topological spaces.

Toposes

Let A be a C*-algebra. Put

$$\mathcal{C}(A) := \{V \subseteq A \mid V \text{ commutative C*-algebra}\}.$$

It is a order under inclusion. Elements V can be viewed as 'classical contexts', 'windows on the world'

The associated topos is the functor topos:

$$\mathcal{T}(A) := \mathbf{Set}^{\mathcal{C}(A)}$$

Sets varying over the classical contexts.

Internal C*-algebra

Internal C*-algebras in $\mathbf{Set}^{\mathbf{C}}$ are functors of the form $\mathbf{C} \to \mathbf{CStar}$.
'Bundle of C*-algebras'.

We define the Bohrification of A as the internal C*-algebra

$$\underline{A} : \mathcal{C}(A) \to \mathbf{Set}, \\ V \mapsto V.$$

in the topos $\mathcal{T}(A) = \mathbf{Set}^{\mathcal{C}(A)}$, where
$\mathcal{C}(A) := \{V \subseteq A \mid V \text{ commutative C*-algebra}\}$.

The internal C*-algebra $\underline{A}$ is commutative!
This reflects our Bohrian perspective.

Kochen-Specker

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Isham-Döring: a certain *global* section does not exist.
We can still have neo-realistic interpretation by considering also non-global sections.
These global sections turn out to be global points of the internal Gelfand spectrum of the Bohrification $\underline{A}$.

Pointfree Topology

We want to consider the phase space of the Bohrification.
Use internal constructive Gelfand duality.
The classical proof of Gelfand duality uses the axiom of choice (only) to construct the points of the spectrum.
Solution: use topological spaces without points (locales)!

Pointfree Topology

Choice is used to construct ideal points (e.g. max. ideals).
Avoiding points one can avoid choice and non-constructive reasoning (Joyal, Mulvey, Coquand).
Slogan: using the axiom of choice is a choice!
(Tychonoff, Krein-Millman, Alaoglu, Hahn-Banach, Gelfand, Zariski, ...)
Point free approaches to topology:

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These formal objects model basic observations:

- Formal opens are used in computer science (domains) to model observations.
- Formal continuous functions, self adjoint operators, are observables in quantum theory.

Phase object in a topos

Phase space = constructive Gelfand dual Σ (spectrum) of the Bohrification. (motivated by Döring-Isham).

Kochen-Specker = Σ has no (global) point.
However, Σ is a well-defined interesting compact regular locale.
Pointless topological space of hidden variables.

States in a topos

An integral is a pos lin functional I on a commutative C*-algebra, with $I(1) = 1$.
A state is a pos lin functional ρ on a C*-algebra, with $\rho(1) = 1$.

In the foundations of QM one uses quasi-states (linear only on commutative parts)
Theorem(Gleason): Quasi-states = states (dim $H > 2$)

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Segal-Kunze developed integration theory using states, with intended interpretation:
an expectation defined on an algebra of observables.
We will present a variation on this.

Constructive integration

Integral on commutative C*-algebras of functions (Daniell,Segal/Kunze)
An integral is a positive linear functional on a space of continuous functions on a topological space

Prime example: Lebesgue integral $\int$
Linear: $\int af + bg = a \int f + b \int g$
Positive: If $f(x) \geq 0$ for all x, then $\int f \geq 0$

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Other example: Dirac measure $\delta_t(f) := f(t)$.

Riesz representation theorem

Riesz representation: Integral = Regular measure = Valuation

A valuation is a map $\mu : O(X) \to \mathbb{R}$, which is lower semicontinuous and satisfies the modular laws.

Theorem (Coquand/Spitters)

The locales of integrals and valuations are homeomorphic.

Proof The integrals form a compact regular locale, presented by a *geometric* theory. Only $(\wedge, \bigvee)$.
Similarly for the theory of valuations.
By the classical RRT the models(=points) are in bijective correspondence.
Hence by the completeness theorem for geometric logic
(Truth in all models $\Rightarrow$ provability)
we obtain a bi-interpretation/a homeomorphism.

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Once we have first-order formulation (no DC), we obtain a transparent constructive proof by 'cut-elimination'.

Giry monad in domain theory in logical form (cf Jung/Moshier)

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Thus an open '$\delta(a) \in \Delta$' can be assigned a probability. In general, this probability is only partially defined, it is in the interval domain.

Externalizing

There is an external locale Σ such that $Sh(\underline{\Sigma})$ in $\mathcal{T}(A)$ is equivalent to $Sh(\Sigma)$ in Set.
HLS proposal for intuitionistic quantum logic.
When applied to the lattice of projections of a Hilbert space we turn old style quantum logic into a Heyting algebra.

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When applied to the lattice of projections of a Hilbert space we turn old style quantum logic into a Heyting algebra.
Problem: $\Sigma(C(X))$ is not X. Here we propose a refinement.
First, a concrete computation of a basis for the Heyting algebra.

Externalization

Theorem (Moerdijk)

Let $\mathbb{C}$ be a site in $\mathcal{S}$ and $\mathbb{D}$ be a site in $\mathcal{S}[\mathbb{C}]$, the topos of sheaves over $\mathbb{C}$. Then there is a site $\mathbb{C} \ltimes \mathbb{D}$ such that

$$\mathcal{S}[\mathbb{C}][\mathbb{D}] = \mathcal{S}[\mathbb{C} \ltimes \mathbb{D}].$$

Presentation using forcing conditions

$$\mathcal{C}(A) := \{C \mid C \text{ is a commutative C*-subalgebra of } A\}.$$

Let $\mathbb{C} := \mathcal{C}(A)^{\mathrm{op}}$ and $\mathbb{D} = \Sigma$ the spectrum of the Bohrification.

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The objects (forcing conditions): (C, u),
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Covering relation $(C, u) \lhd (D_i, v_i)$: for all i, $C \subset D_i$ and $C \Vdash u \lhd V$, where V is the pre-sheaf generated by the conditions $D_i \Vdash v_i \in V$. This is a Grothendieck topology.

Theorem

The points of the locale generated by $\mathbb{C} \ltimes \mathbb{D}$ are consistent ideals of partial measurement outcomes.

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For $C(X)$, the points are points of the spectrum of a subalgebra.

Measurements

In algebraic quantum theory, a measurement is a (maximal) Boolean subalgebra of the set of projections of a von Neumann algebra. The outcome of a measurement is the consistent assignment of either 0 or 1 to each element (test, proposition) of the Boolean algebra: an element of the Stone spectrum.

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C*-algebras need not have enough projections. One replaces the Boolean algebra by a commutative C*-subalgebra and the Stone spectrum by the Gelfand spectrum.

Definition

A *measurement outcome* is a point in the spectrum of a maximal commutative subalgebra.

How to include maximality?

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Extra: allows classical logic internally (Boolean valued models).
The dense topology on a poset P is defined as $p \lhd D$ if D is dense below p: for all $q \leq p$, there exists a $d \in D$ such that $d \leq q$.
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This topos of $\neg\neg$-sheaves satisfies the axiom of choice.
The associated sheaf functor sends the presheaf topos $\hat{P}$ to the sheaves $\mathrm{Sh}(P, \neg\neg)$.
The sheafification for $V \rightarrowtail W$:

$$\neg\neg V(p) = \{x \in W(p) \mid \forall q \leq p \exists r \leq q . x \in V(r)\}.$$

Eventually

The covering relation for $(\mathcal{C}(A), \neg\neg) \ltimes \underline{\Sigma}$ is $(C, u) \lhd (D_i, v_i)$ iff $C \subset D_i$ and $C \Vdash u \lhd V_{\neg\neg}$, where $V_{\neg\neg}$ is the sheafification of the presheaf V generated by the conditions $D_i \Vdash v_i \in V$. Now, $V \rightarrowtail L$, where L is the spectral lattice of the *presheaf* $\underline{A}$.

$$V_{\neg\neg}(C) \quad = \quad \{u \in L(C) \mid \forall D \leq C \exists E \leq D. u \in V(E)\}.$$

So, $(C, u) \lhd (D_i, v_i)$ iff

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Theorem

The locale MO generated by $(\mathcal{C}(A), \neg\neg) \ltimes \underline{\Sigma}$ *classifies measurement outcomes.*

$MO(C(X)) = X$!

Theorem (Kochen-Specker)

Let H be a Hilbert space with $\dim H > 2$ and let $A = B(H)$. Then the $\neg\neg$-sheaf $\sum$ does not allow a global section.

Conclusions

Bohr's doctrine suggests a functor topos making a C*-algebra commutative

- Spatial quantum logic via topos logic
- Phase space via internal Gelfand duality
- Intuitionistic quantum logic
- Spectrum for non-commutative algebras.
- States (non-commutative integrals) become internal integrals.

Classical logic and maximal algebras