Cubical sets as a classifying topos

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Homotopy Type Theory

The homotopical interpretation of type theory:

- types as spaces upto homotopy
- dependent types as fibrations (continuous families of spaces)
- identity types as path spaces

(homotopy type) theory = homotopy (type theory)

Homotopy Type Theory/Univalent foundations

Some applications:

- Synthetic Homotopy theory
- New foundation for mathematics
  inherent treatment of equivalences.
- Internal language for higher toposes
- Semantics for Type theory
  Programming languages and proof assistants

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The univalence axiom describes the identity type of the universe U. There is a canonical function

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The univalence axiom ...

- formalizes the informal practice of substituting a structure for an isomorphic one.
- implies function extensionality $(\Pi_x f(x) = g(x) \to f = g)$
- used to reason about higher inductive types
- equivalent to (Rezk, Rijke/S):
  - universe is an object classifier
  - decent theorem

Simplicial sets

Univalence modeled in Kan fibrations of sSets. (Voevodsky)
Simplicial sets are a standard example of a classifying topos.
Joyal/Johnstone: geometric realization as a geometric morphism.

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Challenge:
computational interpretation of univalence and
higher inductive types.
Solution (Coquand et al): Cubical sets
Can we extend classifying topos methods?

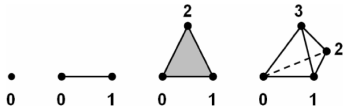
Cubical type checker

Type checker by Cohen, Coquand, Huber, Mörtberg.
`cubical`
New cubical type theory.
Primitive type I for interval.
$PathA := A^I$.
E.g. functional extensionality from the interval.

```
funExt (A : U) (B : U) ( f g : A -> B )
       (p : (x : A) -> Id B (f x) (g x)) :
                      : Id ( A -> B ) f g =
  <i> \(a : A) -> (p a) @ i
```

Simplicial sets

Simplex category Δ:
finite ordinals and monotone maps
Simplicial sets $\hat{\Delta}(=\Delta^{\mathrm{op}} \to Set)$. Free cocompletion.
Geometric realization/Singular complex: $|-| : \widehat{\Delta} \to Top : S$
$S_n(X) = Hom_{Top}(\Delta^n_{Top}, X)$



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(=continuous function between toposes).
E.g. $|-|$ is left exact (pres fin lims).
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Johnstone: use topological topos instead.

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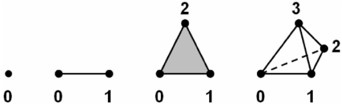

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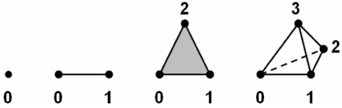
Roughly: points, equalities, equalities between equalities, ...

Geometric realization of simplicial sets

Simplices are constructed from the linear order on $\mathbb{R}$ in Set.



Can be done in any topos with a linear order.
Geometric realization becomes a geometric morphism by moving from spaces to the topological toposes.
Equivalence of cats:

$$Orders(\mathcal{E}) \to Hom(\mathcal{E}, \hat{\Delta})$$

assigns to an order I in $\mathcal{E}$, the geometric realization defined by I.
Simplicial sets classify the *geometric* theory of strict linear orders.

Cubical sets

$\mathbb{2}$: poset with two elements

$\Box$: full subcategory of Cat with obj powers of $\mathbb{2}$.

$$\begin{array}{ccc} 00 & \xrightarrow{\leqslant} & 01 \\ \downarrow{\scriptstyle\leqslant} & & \downarrow{\scriptstyle\leqslant} \\ 10 & \xrightarrow{\leqslant} & 11 \end{array}$$

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Duality: finite posets and distibutive lattices.
$\mathbb{2}$ is the ambimorphic object here:
poset maps into $\mathbb{2}$ pick out 'opens'
DL-maps select the 'points'.
Stone duality between powers of $\mathbb{2}$ and
free finitely generate distributive lattices (copowers of DL1)

Lawvere theory

Classifying categories for Cartesian categories.
Alternative to monads in CS (Plotkin-Power)

For algebraic theory T, the *Lawvere theory* Θ_T^{op} is the opposite of the category of free finitely generated models.

models of T in any finite product category E correspond to product-preserving functors $m : \Theta_T^{op} \to E$.
$m(n)$ consists of the n-tuples in the model m.

A map $1 \to T(2)$, gives a map $m^2 \to m^1$, as both are T-algebras.
E.g. $* \mapsto (x \wedge y)$, defines $(x, y) \mapsto (x \wedge y)$.

Lawvere theories

Consider DL the free distributive lattice monad on Fin.
Then Θ_{DL}^{op} is the Lawvere theory for distributive lattices.

Classifying topos

T an algebraic theory.
Λ_T: finitely *presented* T-models.
$\Lambda_T \to Set$ is the classifying topos.
This topos contains a generic T-algebra.
T-algebras in any topos $\mathcal{F}$ correspond to *left exact left adjoint* functors from the classifying topos to $\mathcal{F}$.

Classifying topos

Example:
T a propositional geometric theory (=formal topology).
$Sh(T)$ is the classifying topos.

Set^{Fin} classifies the Cartesian theory with one sort.
Used for variable binding (Fiore, Plotkin,Turi, Hofmann).
Replaces Pitts' use of nominal sets for the cubical model.
Nominal sets classify decidable infinite sets.

Geometric realization for cubical sets

Theorem (Johnstone-Wraith)

Let T be an algebraic theory, then the topos $\widetilde{\Theta_{DL}}$ classifies the geometric theory of flat T-models.

In particular, $\widehat{\square}$ classifies flat distributive lattices.

Flat distributive lattices (like flat modules).
For every $(d_1, \ldots, d_n)$ and lattice 'polynomials' ϕ, ψ,
if $\phi(\vec{d}) = \psi(\vec{d})$,
then there are χ and $\vec{d}'$ such that $\phi\chi = \psi\chi$ and $\vec{d} = \chi(\vec{d}')$

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Need to show that (classically) [0,1] is a flat DL-algebra.

Geometric realization as a geometric morphism

Prop: Every linear order D defines a flat distributive lattice.
Hence, we have a geometric morphism $\widehat{\Delta} \to \widehat{\Box}$.
Let $\mathcal{E}$ be Johnstone's topological topos.

Theorem (Cubical geometric realization)

There is a geometric morphism $r : \mathcal{E} \to \hat{\Box}$ defined using the flat distributive lattice $[0, 1]$.

We obtain the familiar formulas for both simplicial and topological realization.

Categorical models of Id-types

Application: We have an ETT with an internal 'interval' $\mathbb{I}$.
van den Berg, Garner path object categories.
Usual path composition is only h-associative.
Moore paths can have arbitrary length.
category freely generated from paths of length one.
Moore paths: strict associativity, but non-strict involution.
Docherty: Id-types in cubical sets with $\vee$, but no diagonals.

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Apply vdB/G-construction. However, work internally in the topos of cubical sets using the generic DL-algebra $\mathbb{I}$.
Simplifies computation substantially.

Categorical models of Id-types

Coquand (MFPS): much of the cubical model can be carried out in the internal logic of $\hat{\square}$.
Observation (WIP):
Need a topos with an DL with disjunction property and $\forall : \mathbb{I}^{\mathbb{I}} \to \mathbb{I}$.
E.g. sSets, $\widehat{\square \times \omega}$
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$PathA := A^{\mathbb{I}}$. Univalent model.
No judgmental computation rule for J (Path-recursion).
No: $J_{A,D}(d, a, a, r_A(a)) = d(a) : D(a, a, r_A(A))$
only: $Path_{D(a,a,r_A(A))}(J_{A,D}(d, a, a, r_A(a)), d(a))$
Coquand/Swan: Id
Coquand: Univalence for Id
Prop: Path, Id, Moore all equivalent.
Premodel structure on cubical sets, i.e. fits with abstract homotopy.

Conclusion

- Cubical sets as a classifying topos.
- Cubical model in the internal logic.
- Premodel structure in the internal logic
- Cubical geometric realization