Formal topology applied to Riesz spaces

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[0] mostly jww Thierry Coquand

Problem 1

Gian-Carlo Rota (similar remarks by Kolmogorov)
('Twelve problems in probability no one likes to bring up')
Number 1: 'The algebra of probability'
About the pointwise definition of probability:
'The beginning definitions in any field of mathematics are always misleading, and the basic definitions of probability are perhaps the most misleading of all.'
Problem: Probability should not be build up from points: impossible events! → develop 'pointless probability' (work by Caratheory and von Neumann)
von Neumann - towards Quantum Probability

Constructive maths

Constructive mathematics
Two important interpretations:

1. Computational: type theory, realizability, Eff, ...
2. Geometrical: (sheaf) toposes, ...

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Problem 2

Develop constructive maths without (countable) choice

Richman

'Measure theory and the spectral theorem are major challenges for a choiceless development of constructive mathematics and I expect a choiceless development of this theory to be accompanied by some surprising insights and a gain of clarity.'

We will address both of these problems simultaneously.

Point Free Topology

Choice is used to construct
ideal points (real numbers, max. ideals).
Avoiding points one can avoid
choice and non-constructive reasoning

- Pointfree topology aka locale theory, formal topology (formal opens)

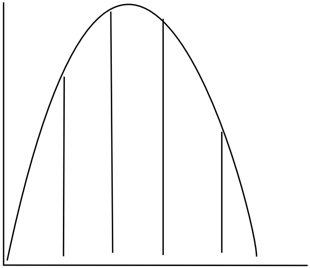
These formal objects model basic observations
Topology: lattice of sets closed under unions and finite intersections
Pointfree topology: lattice closed under joins and finite meets
pointfree topology=complete Heyting algebra

Constructive integration theory

See Palmgren's talk.

- Riemann
- Lebesgue
- Daniell - Positive linear functionals
  - Bishop integration spaces

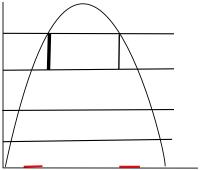
Riemann considered partions of the domain



$$\int f = \lim \sum f(x_i)|x_{i+1} - x_i|$$

Lebesgue

Lebesgue considered partitions of the range



Need measure on the domain:

$$\int f = \lim \sum s_i \mu(s_i \leq f < s_{i+1})$$

Daniell

Consider integrals on algebras of *functions*.
Classical Daniell theory
integration for positive linear functionals on space of continuous functions on a topological space
Prime example: Lebesgue integral $\int$
Linear: $\int af + bg = a \int f + b \int g$
Positive: If $f(x) \geq 0$ for all x, then $\int f \geq 0$.

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Other example: Dirac measure $\delta_t(f) := f(t)$.
Can be extended to a quite general class of underlying topological spaces

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$\mathcal{L}_1$: concrete functions
L_1: $\mathcal{L}_1$ module equal almost everywhere
Work with $\mathcal{L}_1$ because functions 'are easy'.
Secretly we work with L_1.
Do this overtly with an abstract space of functions, see later.

Integral on Riesz space

We generalize Bishop/Cheng and metric Boolean algebras
Integral on Riesz space

Definition

A *Riesz space* (vector lattice) is a vector space with 'compatible' lattice operations $\vee, \wedge$.
E.g. $f \vee g + f \wedge g = f + g$.

Prime ('only') example:
vector space of real functions with pointwise $\vee, \wedge$.
Also: the simple functions.

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We assume that Riesz space R has a strong unit 1: $\forall f \exists n. f \leq n \cdot 1$.
An integral on a Riesz space is a positive linear functional I

Integrals on Riesz space

Most of Bishop's results generalize to Riesz spaces!
However, we first need to show how to handle multiplication.
Once we know how to do this we can treat:

1. integrable, measurable functions, L_p-spaces
2. Riemann-Stieltjes
3. Dominated convergence
4. Radon-Nikodym
5. Spectral theorem

Profile theorem

The profile theorem is crucial is Bishop's development
However, it implies that the reals are uncountable.

Theorem (Rosolini/S)

The (Dedekind) reals are not uncountable (in Sh($\mathbb{R}$)).

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Substitute for the profile theorem...
'Every Riesz space can be embedded in an algebra of continuous functions'

Theorem (Classical Stone-Yosida)

Let R be a Riesz space. Let $\mathrm{Max}(R)$ be the space of representations. The space $\mathrm{Max}(R)$ is compact Hausdorff and there is a Riesz embedding $\hat{\cdot} : R \to C(\mathrm{Max}(R))$. The uniform norm of $\hat{a}$ equals the norm of a.

We will replace $\mathrm{Max}(R)$ by a formal space.

- Substitute for the profile theorem
- Towards spectral theorem
- To define multiplication

Entailment

Pointfree definition of a space using entailment relation $\vdash$
Used to represent distributive lattices
Write $A \vdash B$ iff $\wedge A \leq \bigvee B$
Conversely, given an entailment relation define a lattice:
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Stone's duality :
Boolean algebras and Stone spaces
distributive lattices and coherent T_0 spaces
Points are models
space is theory, open is formula
model theory $\rightarrow$ proof theory

Spectral theorem

Pointfree Stone-Yosida implies Bishop's version of the Gelfand representation theorem (Coquand/S:2005) answering Richman's challenge.
... and the classical theorem (by a direct application of AC).
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We avoid such excursions.
We have proved the Stone-Yosida representation theorem:

Theorem

Every Riesz space can be embedded in an algebra of continuous functions on its spectrum qua formal space.

Any integral can be extended to all the continuous functions. Thus we are in a formal Daniell setting!
We can now develop much of Bishop's integration theory in this abstract setting.

Another application

An f-algebra is a Riesz space with multiplication.

Theorem

Every f-algebra is commutative.

Several proofs using AC.
'Constructive' (i.e. no AC) proof by Buskens and van Rooij.
Mechanically translation to a *simpler constructive* proof (no PEM, AC) which is entirely internal to the theory of Riesz spaces.

Summary

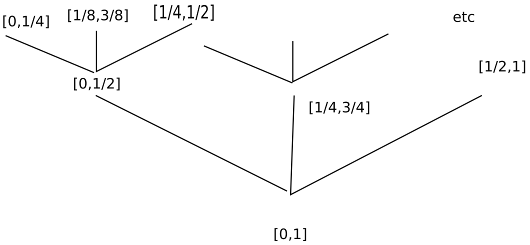
- Observational mathematics
  - Topology
  - Measure theory
- Integration on Riesz spaces (towards Richman's challenge).
  - 'functions' instead of 'opens'
  - Most of Bishop's results can be generalized to this setting!
- New (easier) proof of Bishop's spectral theorems using Coquand's Stone representation theorem (pointfree topology)
- The reals are not uncountable.
- Pointfree is natural in constructive maths without choice
- There's more...

Literature

- Formal Topology and Constructive Mathematics: the Gelfand and Stone-Yosida Representation Theorems (with Coquand)
- Constructive algebraic integration theory without choice
- Constructive algebraic Integration theory
- Coquand - About Stone's notion of spectrum J. Pure Appl. Algebra, 197(1-3):141-158, 2005

Varieties

Constructive mathematics (Brouwer, Markov, Bishop, ...) mostly deals with complete separable metric spaces,
images of Baire space ($\mathbb{N}^{\mathbb{N}}$ with product topology)
Example: $[0, 1]$ limits of Cauchy sequences/ image of $3^{\mathbb{N}}$



see also reverse maths, explicit maths, Weihrauch's TTE
Has surprisingly large range, but invites sequential reasoning (representation dependent)

Sequences, trees, spaces

Richman: DC is often used to pick a path (choice sequence) in a tree/ subset of Baire space.
Proposal: consider the trees of *all* paths directly.
Example: construction of *all* zeros of a polynomial in the FTA.

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Proposal: consider the trees of *all* paths directly.
Example: construction of *all* zeros of a polynomial in the FTA.
The tree represents a topological space.
Here we give a formal description of this space.
Basic opens for finite paths.
Now: consider the formal space of 'all' choices.
Again the idea was obtained in both worlds:
Brouwer's theory of spreads and in topos theory