Algoritmer og Datastrukturer 2

Gerth Stølting Brodal

AARHUS UNIVERSITET

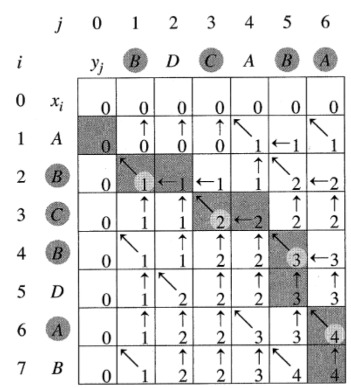
Algoritmer og Datastrukturer 2

Algoritme Design Teknikker (2 uger)

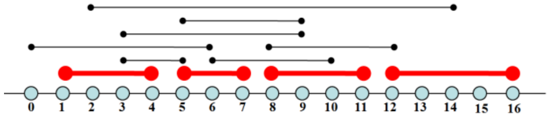
Del-og-kombiner

$$\begin{pmatrix} I & J \\ K & L \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} E & F \\ G & H \end{pmatrix}$$

Dynamisk programmering

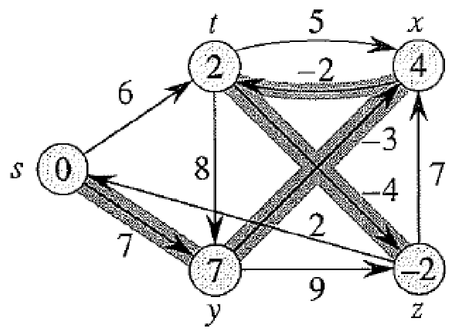


Grådige algoritmer

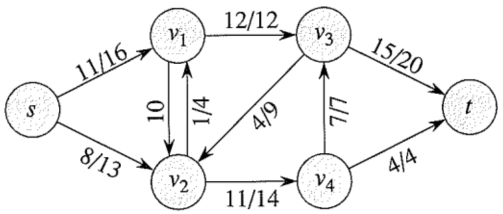


Graf-algoritmer (3 uge)

Korteste veje

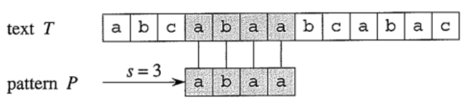


Maksimale strømninger

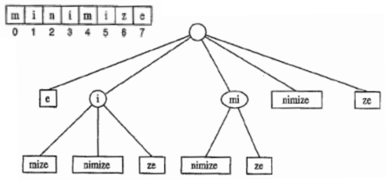


Streng-algoritmer (1 uge)

Mønstergenkendelse



Suffix-træer



Suffix arrays

Algoritmer og Datastrukturer 2

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Del-og-kombiner
[CLRS, kapitel 2.3, 4.2-4.5, problem 30.1.c]

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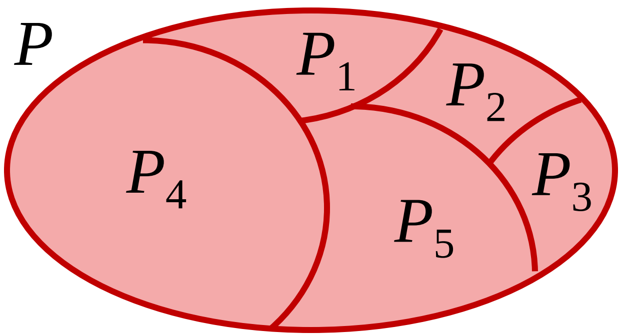
Del-og-Kombiner

Algoritme design teknik

Virker for mange problemer (men langt fra alle)

- **Opdel** et problem P i mindre problemer $P_1,..,P_k$, der kan løses uafhængigt
  (små problemer løses direkte)
- Løs delproblemerne $P_1,..,P_k$ **rekursivt**
- **Kombiner** løsningerne for $P_1,..,P_k$ til en løsning for P



Eksempel: Merge-Sort

MERGE-SORT(A, p, r)

if $p < r$

$q = \lfloor (p + r)/2 \rfloor$

MERGE-SORT(A, p, q)

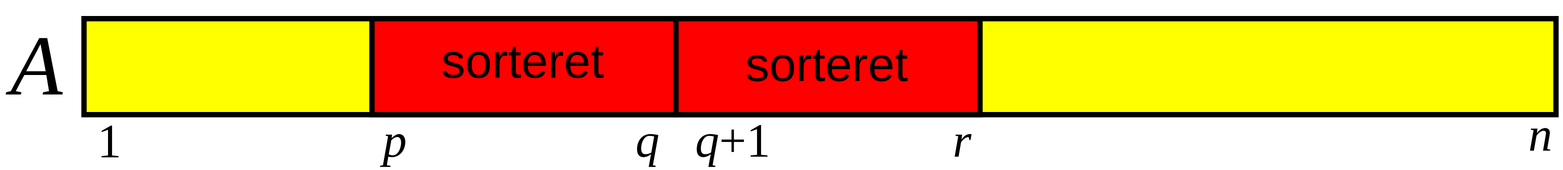
MERGE-SORT$(A, q + 1, r)$

MERGE(A, p, q, r)

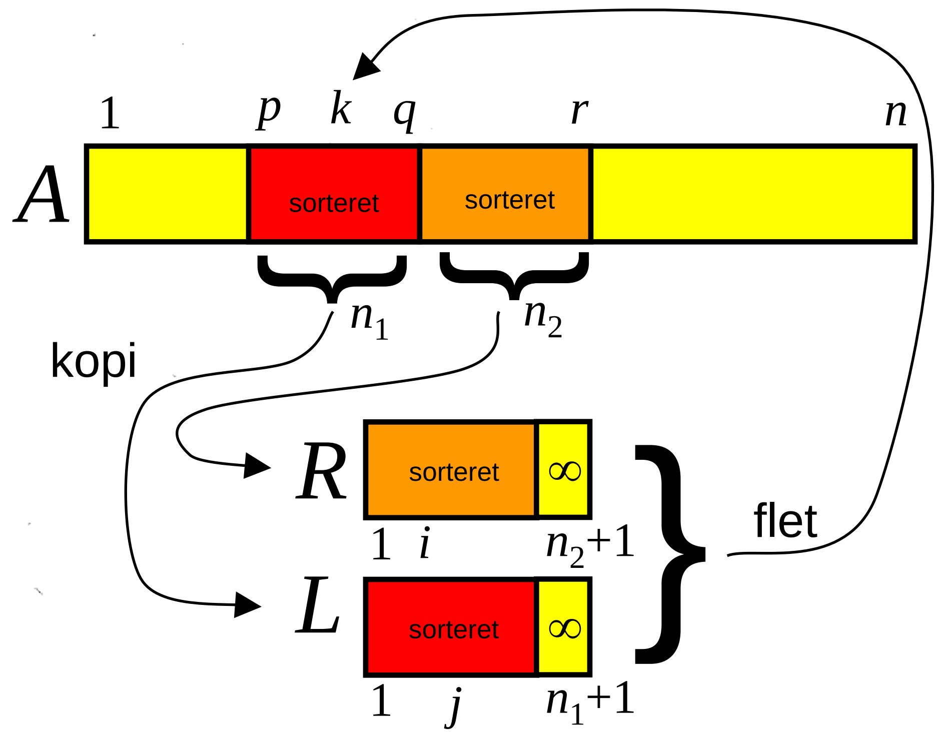
① To mindre delproblemer

② Løs rekursivt

③ Kombiner

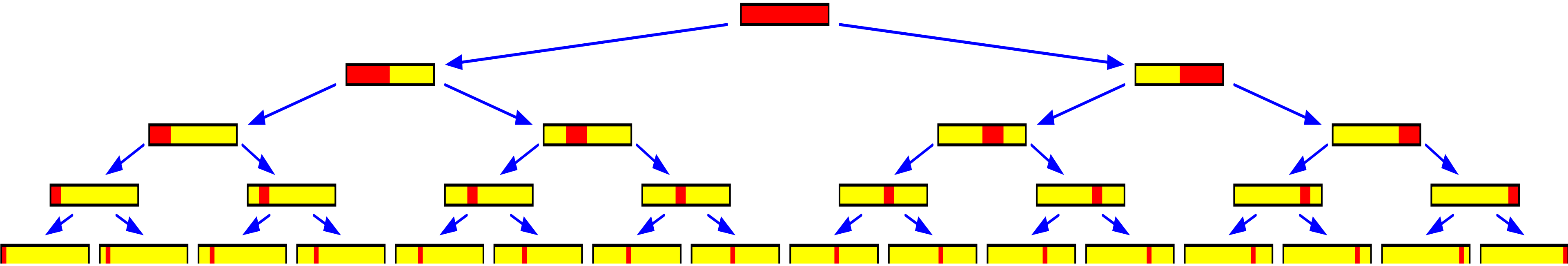


```
MERGE(A, p, q, r)
    n1 = q - p + 1
    n2 = r - q
    let L[1..n1 + 1] and R[1..n2 + 1] be new arrays
    for i = 1 to n1
        L[i] = A[p + i - 1]
    for j = 1 to n2
        R[j] = A[q + j]
    L[n1 + 1] = ∞
    R[n2 + 1] = ∞
    i = 1
    j = 1
    for k = p to r
        if L[i] ≤ R[j]
            A[k] = L[i]
            i = i + 1
        else A[k] = R[j]
            j = j + 1
```



Merge-Sort : Analyse

Rekursionstræet



Observation

Samlet arbejde per lag er $O(n)$

Arbejde

$$O(n \cdot \#\ \text{lag}) = O(n \cdot \log_2 n)$$

Del-og-kombiner, dADS 1 eksempler:

- **MergeSort**
  - Del op i to lige store dele
  - Rekursiv sortering
  - Kombiner = fletning
- **QuickSort**
  - Opdel efter tilfældigt pivot (**tilfældig opdeling**)
  - Rekursiv sortering
  - Kombiner = ingen (konkatener venstre og højre)
- **QuickSelect**
  - Opdel efter tilfældigt pivot (**tilfældig opdeling**)
  - Rekursiv select
  - Kombiner = ingen

Analyse af Del-og-Kombiner

= analyse af en rekursiv procedure

Essentielt to forskellige måder:

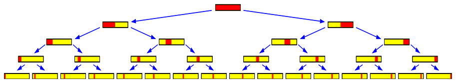
1. Argumenter direkte om **rekursionstræet** (analyser dybde, #knuder på hvert niveau, arbejde i knuderne/niveauerne/træet)
2. Løs en matematisk **rekursionsligning**, f.eks.

$$T(n) \leq a \quad \text{hvis} \quad n \leq c$$
$$T(n) \leq 2 \cdot T(n/2) + a \cdot n \quad \text{ellers}$$

Bevises f.eks. vha. induktion.

Løsning af rekursionsligninger

- Fold rekursionsligningen ud og argumenter om **rekursionstræet**



- Gæt en løsning og vis den ved induktion efter voksende n

$T(n) \leq a$	hvis $n \leq c$
$T(n) \leq 2 \cdot T(n/2) + a \cdot n$	ellers

Rekursionsligninger: Faldgrubber

- Ulige opdelinger glemmes (n ulige, så er de rekursive kald typisk $\lfloor n/2 \rfloor$ og $\lceil n/2 \rceil$)

[CLRS, kapitel 4.6.2]

- Analyserer typiske kun for $n = 2^k$

- Brug aldrig O-udtryk i rekursionsformlen – brug konstanter (~~$T(n)=O(n)+O(T(n/3))$~~)

$$T(n) \leq c \cdot n + a \cdot T(n/3)$$

Master Theorem
(Simplificering af [CLRS, Theorem 4.1])

Theorem

Hvis a, b, c, d, p er konstanter som opfylder $a, c, p > 0$, $d \geq 1$, og $b > 1$, så har rekursionsligningen

$$T(n) = \begin{cases} c & \text{hvis} \quad n \leq d \\ a \cdot T(n/b) + c \cdot n^p & \text{hvis} \quad n > d \end{cases}$$

følgende løsning

$$\begin{array}{lll} O(n^p) & \text{hvis} & a < b^p \\ O(n^p \log n) & \text{hvis} & a = b^p \\ O(n^{\log_b a}) & \text{hvis} & a > b^p \end{array}$$

Dybde	***i* = 0..log$_b$ (*n*/*d*) - 1**	**log$_b$ (*n*/*d*)**
# delproblemer	a^i	$a^{\log_b (n/d)}$
Størrelse af delproblemer	n/b^i	d
Tid per delproblem	$c \cdot (n/b^i)^p$	c
Tid per lag	$a^i \cdot c \cdot (n/b^i)^p$	$c \cdot a^{\log_b (n/d)}$

$$T(n) \le c \cdot a^{\log_b (n/d)} + \sum_{i=0}^{\log_b (n/d)-1} a^i \cdot c \cdot (n/b^i)^p$$

(bunden af rekursionen) (lag i = 0..log$_b$ (n/d) - 1)

$$\le c \cdot a^{\log_b n} + c \cdot n^p \cdot \sum_{i=0}^{\log_b n - 1} (a/b^p)^i$$

$$\frac{(a/b^p)^{\log_b n} - 1}{a/b^p - 1} \quad \text{for} \quad a \ne b^p$$

$$= O\left(n^{\log_b a} + n^p \cdot \begin{cases} 1 & \text{for } a < b^p \\ \log n & \text{for } a = b^p \\ (a/b^p)^{\log_b n} & \text{for } a > b^p \end{cases} \right) = O\left(\begin{cases} n^p & \text{for } a < b^p \\ n^p \cdot \log n & \text{for } a = b^p \\ n^{\log_b a} & \text{for } a > b^p \end{cases} \right)$$

$$a^{\log_b n} = n^{\log_b a}$$

$$(b^p)^{\log_b n} = n^p$$

Multiplikation af lange heltal

[CLRS, problem 30.1.c]

Karatsuba 1960

- I og J hver heltal med n bits
- Naive implementation kræver $O(n^2)$ bit operationer
- Lad $I = I_h \cdot 2^{n/2} + I_l$ og $J = J_h \cdot 2^{n/2} + J_l$
- $I \cdot J = I_h \cdot J_h \cdot 2^n + ((I_h - I_l) \cdot (J_l - J_h) + I_l \cdot J_l + I_h \cdot J_h) \cdot 2^{n/2} + I_l \cdot J_l$

$$T(n) \leq 3 \cdot T(n/2) + c \cdot n \qquad \text{for } n \geq 2$$

$$T(n) \leq c \qquad \text{for } n = 1$$

- $T(n) = O(n^{\log_2 3}) = O(n^{1.58})$

Multiplikation af lange heltal

Del-og-kombiner Karatsuba 1960	$O(n^{\log_2 3})$
Schönhage-Strassen, 1971	$O(n \cdot \log n \cdot \log\log n)$
Fürer, 2007	$O(n \cdot \log n \cdot 2^{O(\log^* n)})$

Matrix Multiplication

$$\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{np} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mp} \end{pmatrix}$$

$$c_{ij} = \Sigma_{k=1..m} \, a_{ik} \cdot b_{kj}$$

Matrix Multiplication

$$\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{np} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mp} \end{pmatrix}$$

```
MATRIX-MULTIPLY(A, B)
if A.columns ≠ B.rows
    error "incompatible dimensions"
else let C be a new A.rows × B.columns matrix
    for i = 1 to A.rows
        for j = 1 to B.columns
            c_ij = 0
            for k = 1 to A.columns
                c_ij = c_ij + a_ik · b_kj
    return C
```

Naive implementation: tid $O(npm)$

(Kvadratisk) Matrix Multiplikation

[CLRS, kapitel 4.2]

$$\begin{pmatrix} I & J \\ K & L \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} E & F \\ G & H \end{pmatrix}$$

$$I = A \cdot E + B \cdot G$$
$$J = A \cdot F + B \cdot H$$
$$K = C \cdot E + D \cdot G$$
$$L = C \cdot F + D \cdot H$$

- $A,B,\ldots,K,L$ er $n/2$ x $n/2$-matricer
- I,J,K,L kan beregnes med 8 rekursive multiplication og 4 matrix additioner på $n/2$ x $n/2$ -matricer
- $T(n) \leq 8 \cdot T(n/2) + c \cdot n^2$ for $n \geq 2$

  $T(n) \leq c$ for $n = 1$
- $T(n) = O(n^{\log_2 8}) = O(n^3)$

Strassen's Matrix Multiplikation

1969

$$\begin{pmatrix} I & J \\ K & L \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} E & F \\ G & H \end{pmatrix}$$

$$\begin{aligned} I &= S_5 + S_6 + S_4 - S_2 \\ &= (A+D)(E+H) + (B-D)(G+H) + D(G-E) - (A+B)H \\ &= AE + DE + AH + DH + BG - DG + BH - DH + DG - DE - AH - BH \\ &= AE + BG. \end{aligned}$$

$$\begin{aligned} J &= S_1 + S_2 \\ &= A(F-H) + (A+B)H \\ &= AF - AH + AH + BH \\ &= AF + BH. \end{aligned}$$

$$\begin{aligned} K &= S_3 + S_4 \\ &= (C+D)E + D(G-E) \\ &= CE + DE + DG - DE \\ &= CE + DG. \end{aligned}$$

$$\begin{aligned} L &= S_1 - S_7 - S_3 + S_5 \\ &= A(F-H) - (A-C)(E+F) - (C+D)E + (A+D)(E+H) \\ &= AF - AH - AE + CE - AF + CF - CE - DE + AE + DE + AH + DH \\ &= CF + DH. \end{aligned}$$

$$\begin{aligned} S_1 &= A \cdot (F - H) \\ S_2 &= (A + B) \cdot H \\ S_3 &= (C + D) \cdot E \\ S_4 &= D \cdot (G - E) \\ S_5 &= (A + D) \cdot (E + H) \\ S_6 &= (B - D) \cdot (G + H) \\ S_7 &= (A - C) \cdot (E + F) \end{aligned}$$

7 rekursive multiplikationer

Strassen's Matrix Multiplikation

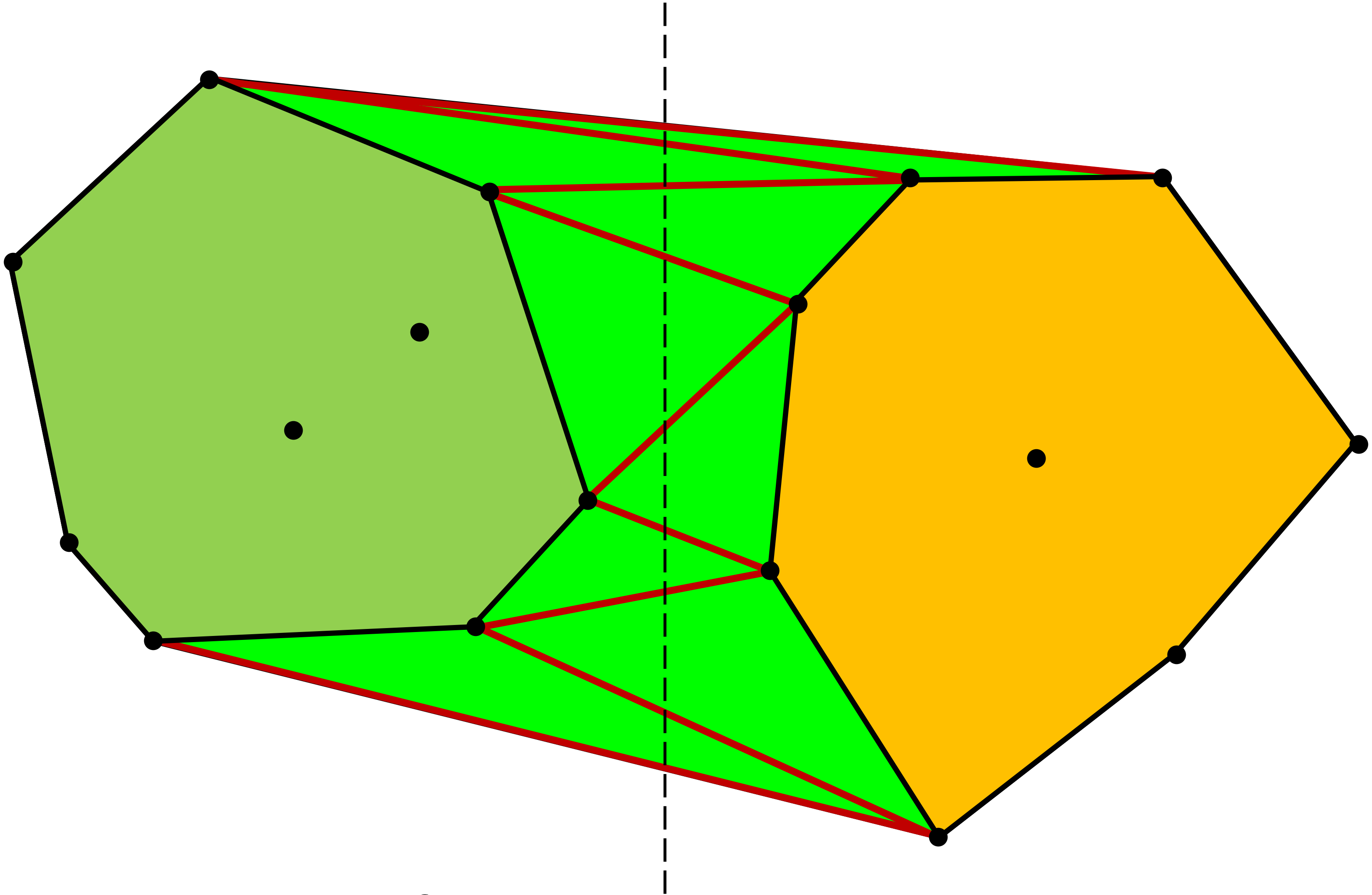
- Bruger 18 matrix additioner (tid $O(n^2)$) og
  7 rekursive matrix multiplikationer

$T(n) \leq 7 \cdot T(n/2) + c \cdot n^2$	for $n \geq 2$
$T(n) \leq c$	for $n = 1$

- $T(n) = \mathrm{O}(n^{\log_2 7}) = O(n^{2.81})$

Bedste resultat for matix multiplikation $\mathrm{O}(n^{2.3727})$: Virginia Vassilevska Williams, *Multiplying matrices faster than Coppersmith-Winograd*, STOC 2012

Konveks Hylster



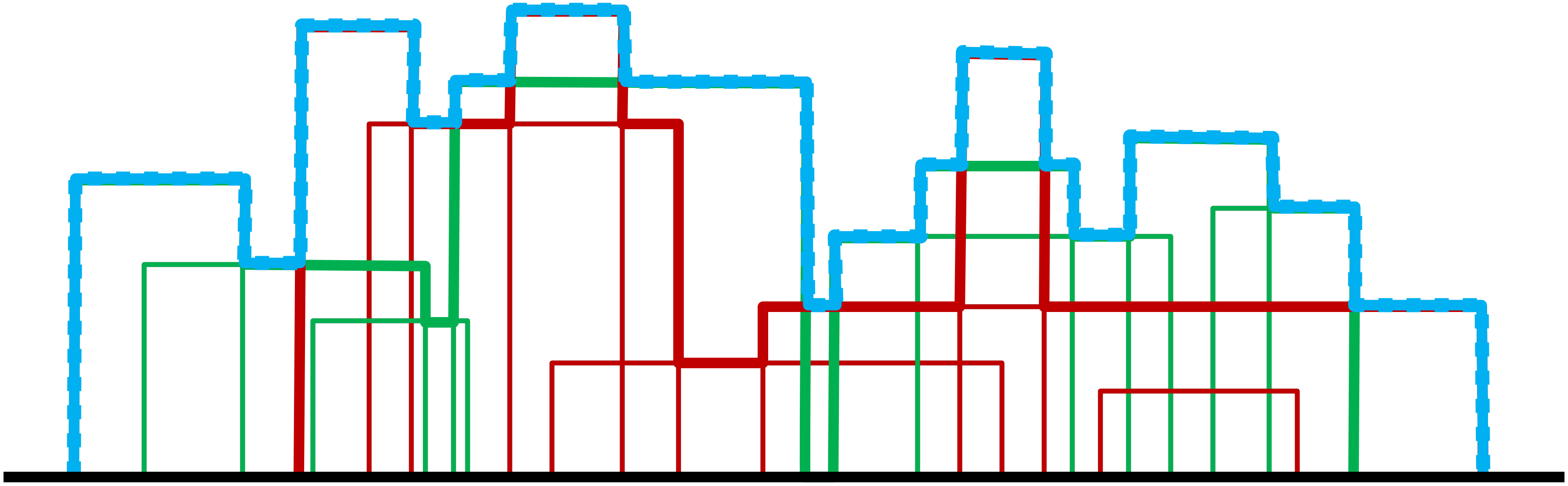
$T(n) \leq 2 \cdot T(n/2) + c \cdot n$ for $n \geq 2$

$T(n) \leq c$ for $n = 1$

$T(n) = O(n \cdot \log n)$

Silhuet

(afleveringsopgave)



$T(n) \leq ? \cdot T(n/?) + ?$ for $n \geq 2$

$T(n) \leq c$ for $n = 1$