Parametric Domain-theoretic Models of Linear/Intuitionistic Polymorphic Lambda Calculus

Lars Birkedal

The IT University of Copenhagen

Joint work with Rasmus Møgelberg and Rasmus Lerchedahl Petersen

Second-order LNL

- Types: $\sigma ::= \alpha \mid 1 \mid \sigma \otimes \tau \mid \sigma \multimap \tau \mid !\sigma \mid \prod \alpha.\, \sigma$
- Expressions of form

$$\alpha_1, \ldots, \alpha_l \mid x_1 \colon \sigma_1, \ldots, x_n \colon \sigma_n ; a_1 \colon \sigma'_1, \ldots, a_m \colon \sigma'_m \vdash t \colon \tau$$

- Terms: LNL + abstraction over types, type application, and fixed point combinator Y

$$\frac{}{\vec{\alpha} \mid \Gamma; - \vdash Y \colon \prod \alpha.\, (\alpha \to \alpha) \to \alpha}$$

where $\sigma \to \tau = !\sigma \multimap \tau$.

Parametricity and Domain Theory

- Important for data abstraction
- Useful metalanguage for domain-theory [Plotkin 1994]
  - In particular: Not only inductive and coinductive datatypes, but also *recursive* types

Encoding

- Inductive:

$$\mu\alpha.\,\sigma(\alpha) = \prod \alpha.\,(\sigma(\alpha) \multimap \alpha) \to \alpha.$$

- Coinductive:

$$\nu\alpha.\,\sigma(\alpha) = \prod \beta.\prod \alpha.\,(!(\alpha \multimap \sigma(\alpha)) \otimes \alpha \multimap \beta) \multimap \beta$$

- Recursive: given $\sigma(\alpha, \beta)$
  - $\omega(\alpha) = \mu\beta.\sigma(\alpha, \beta)$
  - $\tau' = \nu\alpha.\sigma(\omega(\alpha), \alpha)$
  - $\tau = \omega(\tau')$
  - $\tau \cong \sigma(\tau, \tau)$

Linear Abadi & Plotkin logic, I

- Abadi & Plotkin (1993) proposed a logic for reasoning about parametricity.
- Plotkin (1994) suggested it could be extended to reason about second-order LNL
- Allows to formulate parametricity as a schema and prove expected consequences, including recursive types for definable functors.
- We will define parametric models to be models of Linear Abadi & Plotkin logic satisfying parametricity.

Linear Abadi & Plotkin logic, II

- Expressions:

$$\vec{\alpha} \mid \vec{x} \colon \vec{\sigma} \mid \vec{R} \subset \vec{\tau} \times \vec{\tau}' \mid \vec{S} \subset^{\mathbf{Adm}} \vec{\omega} \times \vec{\omega}' \vdash \phi \colon \mathsf{Prop}$$

  where $\vec{\alpha} \mid \vec{x} \colon \vec{\sigma}; -$ is a context of second-order LNL

- Propositions:

$$\begin{aligned} \phi \quad ::= \quad & (t =_\sigma u) \mid \rho(t, u) \mid \phi \supset \psi \mid \bot \mid \top \mid \phi \wedge \psi \mid \phi \vee \psi \mid \\ & \forall \alpha \colon \mathsf{Type}.\, \phi \mid \forall x \colon \sigma.\, \phi \mid \forall R \subset \sigma \times \tau.\, \phi \mid \\ & \forall S \subset^{\mathbf{Adm}} \sigma \times \tau.\, \phi \mid \exists \alpha \colon \mathsf{Type}.\, \phi \mid \exists x \colon \sigma.\, \phi \mid \\ & \exists R \subset \sigma \times \tau.\, \phi \mid \exists S \subset^{\mathbf{Adm}} \sigma \times \tau.\, \phi \end{aligned}$$

  for ρ a definable relation.

Definable relations, I

- Definable relations

$$\rho ::= R \mid (x\colon \sigma, y\colon \tau).\phi$$

- Formation rules

$$\frac{}{\Xi \mid \Gamma \mid \Theta, R \subset \sigma \times \tau \vdash R \subset \sigma \times \tau}$$

$$\frac{\Xi \mid \Gamma, x\colon \sigma, y\colon \tau \mid \Theta \vdash \phi\colon \mathsf{Prop}}{\Xi \mid \Gamma \mid \Theta \vdash (x\colon \sigma, y\colon \tau).\,\phi \subset \sigma \times \tau}$$

$$\frac{\Xi \mid \Gamma \mid \Theta \vdash \rho \subset^{\mathbf{Adm}} \sigma \times \tau}{\Xi \mid \Gamma \mid \Theta \vdash \rho \subset \sigma \times \tau}$$

Definable relations, III

$$\frac{\alpha_1, \ldots, \alpha_n \vdash \sigma(\vec{\alpha})\colon \mathsf{Type} \qquad \Xi \mid \Gamma \mid \Theta \vdash \rho_1 \subset^{\mathbf{Adm}} \tau_1 \times \tau_1', \ldots, \rho_n \subset^{\mathbf{Adm}} \tau_n \times \tau_n'}{\Xi \mid \Gamma \mid \Theta \vdash \sigma[\vec{\rho}] \subset^{\mathbf{Adm}} \sigma(\vec{\tau}) \times \sigma(\vec{\tau}')}$$

Definable relations, IV

- Instantiating definable relations

$$\frac{\Xi \mid \Gamma \mid \Theta \vdash \rho \subset \sigma \times \tau \quad \Xi \mid \Gamma; - \vdash t \colon \sigma, s \colon \tau}{\Xi \mid \Gamma \mid \Theta \vdash \rho(t, s) \colon \mathsf{Prop}}$$

- Also write $t \rho s$ for $\rho(t, s)$.

Definable relations, examples

- The graph of a function $f\colon \sigma \multimap \tau$

$$\langle f \rangle = (x\colon \sigma, y\colon \tau).\, f(x) =_\tau y$$

- The equality relation *eq*$_\sigma$ defined as the graph of the identity map.

Remarks

- In Abadi and Plotkins original article, $\sigma[\vec{\rho}]$ was defined inductively on the structure of σ.
- We would like to consider $\sigma[\vec{\rho}]$ as a construction in the internal language of the model (also work for types that are not inductively defined).
- In our version, we capture the inductive definition by axioms of the logic, e.g.,

$$\frac{\vec{\alpha} \vdash \sigma \multimap \sigma' : \mathsf{Type} \qquad \Xi \mid \Gamma \mid \Theta \vdash \vec{\rho} \subset^{\mathbf{Adm}} \vec{\tau} \times \vec{\tau}'}{\Xi \mid \Gamma \mid \Theta \mid \top \vdash (\sigma \multimap \sigma')[\vec{\rho}] \equiv (\sigma[\vec{\rho}] \multimap \sigma'[\vec{\rho}]),}$$

where $\rho \multimap \rho'$ is

$$(f : \sigma \multimap \sigma', g : \tau \multimap \tau'). \forall x : \sigma \forall y : \tau. \rho(x, y) \supset \rho'(fx, gy)$$

Admissible Relations

$$\frac{}{\Xi \mid \Gamma \mid \Theta \vdash \mathsf{eq}_\sigma \subset^{\mathbf{Adm}} \sigma \times \sigma}$$

$$\frac{\Xi \mid \Gamma \mid \Theta \vdash \rho \subset^{\mathbf{Adm}} \sigma \times \tau \qquad \Xi \mid \Gamma \mid \Theta \vdash t \colon \sigma' \multimap \sigma, u \colon \tau' \multimap \tau}{\Xi \mid \Gamma \mid \Theta \vdash (x \colon \sigma', y \colon \tau').\, \rho(t\,x, u\,y) \subset^{\mathbf{Adm}} \sigma' \times \tau'}$$

$$\frac{\begin{array}{c}\Xi \mid \Gamma \mid \Theta \vdash \rho \subset^{\mathbf{Adm}} \sigma \times \tau \\ \Xi \mid \Gamma \mid \Theta \vdash \phi \colon \mathsf{Prop} \qquad x, y \notin \mathrm{Dom}(\Gamma)\end{array}}{\Xi \mid \Gamma \mid \Theta \vdash (x \colon \sigma, y \colon \tau).\, \phi \supset \rho(x, y) \subset^{\mathbf{Adm}} \sigma \times \tau}$$

$$\frac{\begin{array}{c}\Xi, \alpha \mid \Gamma \mid \Theta \vdash \rho \subset^{\mathbf{Adm}} \sigma \times \tau \qquad \Xi \mid \Gamma \mid \Theta \\ \Xi \vdash \sigma \colon \mathsf{Type} \qquad \Xi \vdash \tau \colon \mathsf{Type}\end{array}}{\Xi \mid \Gamma \mid \Theta \vdash (x \colon \sigma, y \colon \tau).\, \forall \alpha \colon \mathsf{Type}.\, \rho(x, y) \subset^{\mathbf{Adm}} \sigma \times \tau}$$

Parametricity

- Identity extension:

$$\forall \vec{\alpha} \colon \mathsf{Type}.\, \sigma[\textit{\textsf{eq}}_{\vec{\alpha}}] \equiv \textit{\textsf{eq}}_{\sigma(\vec{\alpha})}$$

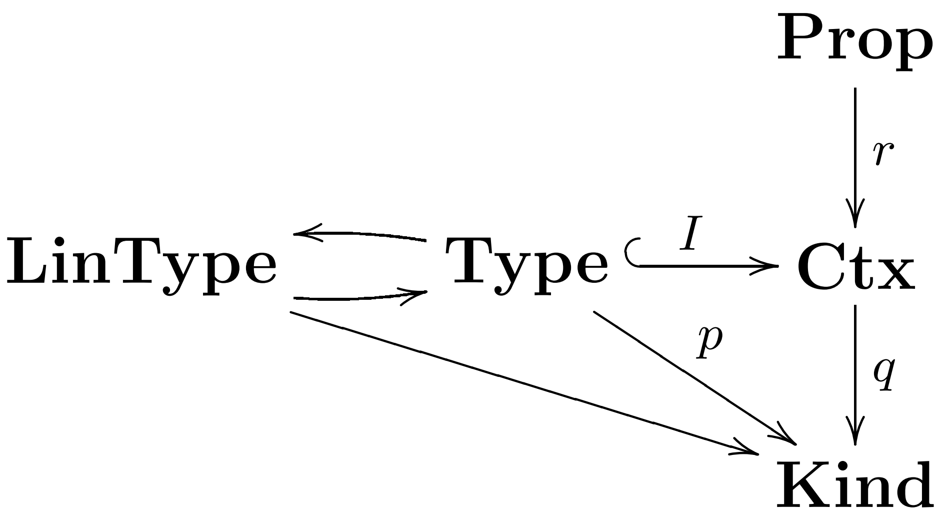
- Extensionality:

$$(\forall x \colon \sigma.\, t\, x =_{\tau} u\, x) \supset t =_{\sigma \multimap \tau} u$$

$$(\forall \alpha \colon \mathsf{Type}.\, t\, \alpha =_{\tau} u\, \alpha) \supset t =_{\prod \alpha \colon \mathsf{Type}.\tau} u.$$

LAPL-structures

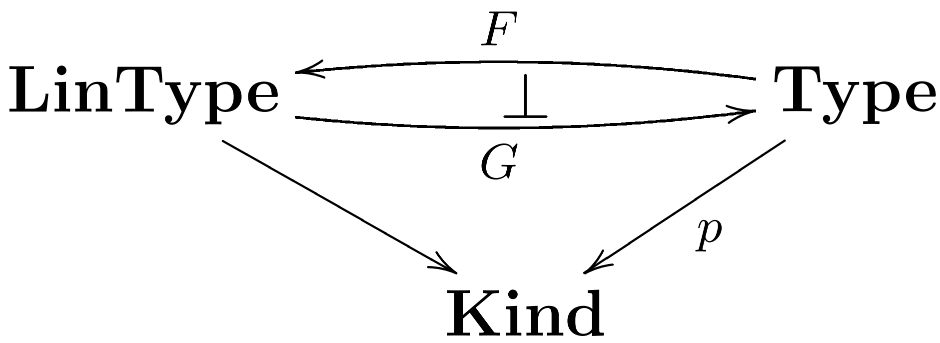
A LAPL-structure consists of categories and functors:



such that

LAPL-structures, I

- the diagram



is a model of second-order LNL + Y.

LAPL-structures, II

- (r, q) is an indexed first-order logic fibration which has products and coproducts with respect to projections $\Xi \times \Omega \to \Xi$ in $\mathbf{Kind}$, where Ω is the generic object of p.
- I is a faithful product-preserving map of fibrations.
- q is a fibration with fibred products

LAPL-structures, III

a contravariant morphism of fibrations:

$$\begin{array}{ccc} \mathbf{LinType} \times_{\mathbf{Kind}} \mathbf{LinType} & \xrightarrow{U} & \mathbf{Ctx} \\ & \searrow & \swarrow \\ & \mathbf{Kind} & \end{array}$$

such that

- $U(\sigma, \tau)$ is powerset of σ and τ in $\mathbf{LinType}_\Xi$, i.e.,

$$\Psi_\Xi : \mathrm{Hom}_{\mathbf{Ctx}_\Xi}(\xi, U(\sigma, \tau)) \to \mathrm{Obj}\left(\mathbf{Prop}_{\xi \times I(G(\sigma) \times G(\tau))}\right)$$

LAPL-structures, IV

To interpret admissible relations, we require subfunctor V of U, which we think of as the subset of all admissible relations, closed under all the rules for creating admissible relations.

LAPL-structures, V

A context

$$\Xi \mid x_1 : \sigma_1, \ldots x_n : \sigma_n \mid \; R_1 \subset \tau_1 \times \tau_1', \ldots, R_m \subset \tau_m \times \tau_m' \mid$$
$$S_1 \subset^{\mathbf{Adm}} \omega_1 \times \omega_1', \ldots, S_k \subset^{\mathbf{Adm}} \omega_k \times \omega_k'$$

is interpreted as

$$\prod_i IG([\![\sigma_i]\!]) \times \prod_j U([\![\tau_j]\!], [\![\tau_j']\!]) \times \prod_l V([\![\omega_l]\!], [\![\omega_l']\!])$$

where the interpretations of the types is the usual interpretation of types in $\mathbf{LinType} \rightarrow \mathbf{Kind}$.

Fibration of relations, I

- Given the above data, one can construct a fibration

$$\begin{array}{c} \mathbf{AdmRelations} \\ \downarrow \\ \mathbf{AdmRelCtx}. \end{array}$$

- $\mathbf{AdmRelCtx}$ is given as the pullback

$$\begin{array}{ccc} \mathbf{AdmRelCtx} & \longrightarrow & \mathbf{Ctx} \\ {\scriptstyle \langle \partial_0, \partial_1 \rangle}\downarrow & \lrcorner & \downarrow \\ \mathbf{Kind} \times \mathbf{Kind} & \xrightarrow{\times} & \mathbf{Kind} \end{array}$$

We write an object Θ in $\mathbf{AdmRelCtx}$ over (Ξ, Ξ') as $\Xi, \Xi' \mid \Theta$.

Fibrations of relations, II

- Objects of $\mathbf{AdmRelations}$ over an object $\Xi, \Xi' \mid \Theta$ are

$$\Xi; \Xi' \mid \Theta \vdash \rho \subset^{\mathbf{Adm}} \sigma \times \tau.$$

- A vertical morphims in $\mathbf{AdmRelations}$ from $\rho \subset^{\mathbf{Adm}} \sigma \times \tau$ to $\rho' \subset^{\mathbf{Adm}} \sigma' \times \tau'$ is a pair of morphisms $f \colon \sigma \multimap \tau$, $g \colon \sigma' \multimap \tau'$ in $\mathbf{LinType}$ such that in the internal language the formula

$$\forall x \colon \sigma, y \colon \tau.\, \rho(x, y) \supset \rho'(f\ x, g\ y)$$

holds.

Relational interpretation of types

To interpret $\sigma[\rho]$ we will require that the graph

$$\left(\begin{array}{c}\mathbf{AdmRelations}\\ \downarrow \\ \mathbf{AdmRelCtx}\end{array}\right) \overset{\partial_0}{\underset{\partial_1}{\rightrightarrows}} \left(\begin{array}{c}\mathbf{LinType}\\ \downarrow \\ \mathbf{Kind}\end{array}\right).$$

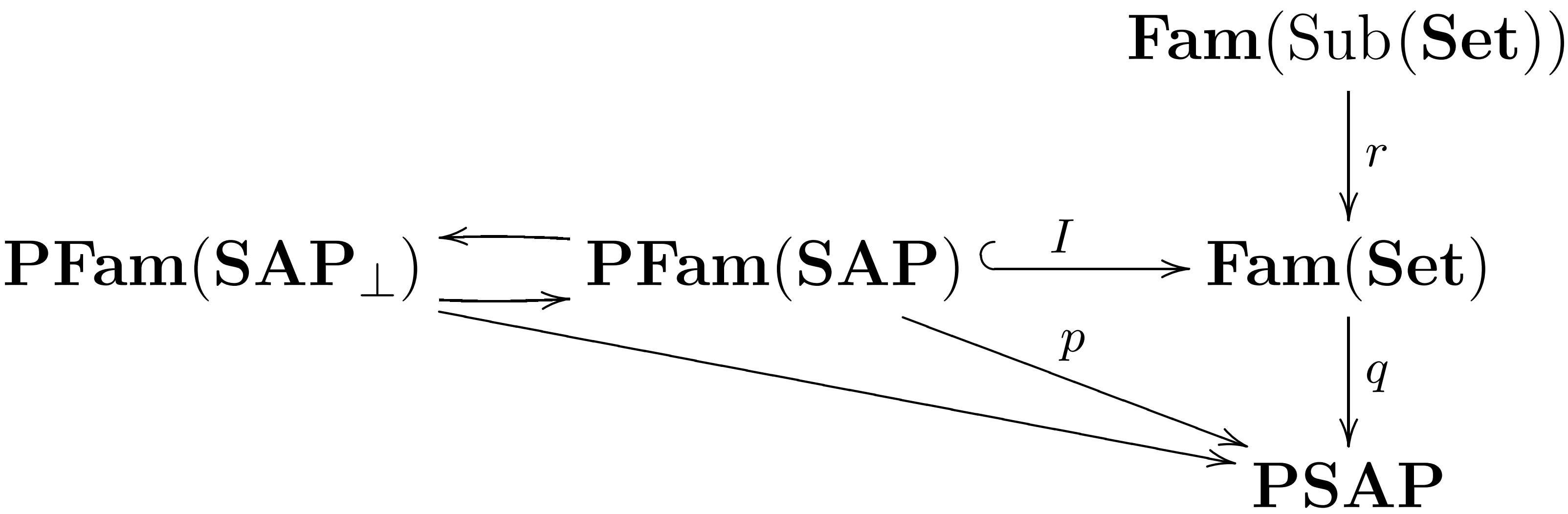
has an extension to a reflexive graph of linear λ_2-fibrations.

Soundness and completeness

- The presented model is sound.
- Completeness is proved by constructing a syntactical model.
- A *parametric LAPL-structure* is an LAPL-structure, with very strong equality in which identity extension and extensionality holds in the internal logic.
- **Theorem:** In a parametric LAPL-structure, for any polymorphically strong fibred functor $F : \mathbf{LinType}^{\mathrm{op}} \times \mathbf{LinType} \to \mathbf{LinType}$, there exists a closed type τ such that $F(\tau, \tau) \cong \tau$.

A concrete model, I

- The expected model does indeed work:



A concrete model, II

- $\mathbf{PSAP}$ has natural numbers as objects, and a morphism $n \to m$ is an m-vector of objects in $\mathbf{PFam}(\mathbf{SAP}_\perp)$.
- Over 1, objects of $\mathbf{PFam}(\mathbf{SAP})$ are functions
  - $f^p \colon \mathbf{SAP}_\perp \to \mathbf{SAP}_\perp$
  - $f^r \colon \prod_{\sigma,\tau \in \mathbf{SAP}_\perp} \big(\mathrm{RegSub}(\sigma \times \tau) \to \mathrm{RegSub}(f^p(\sigma) \times f^p(\tau)) \big)$

  satisfying $f^r(eq_\sigma) = eq_{(f^p(\sigma))}$, for all $\sigma \in \mathbf{SAP}_\perp$.
- Morphisms in $\mathbf{PFam}(\mathbf{SAP})$ have continous realizers, preserving relations
- Morphisms in $\mathbf{PFam}(\mathbf{SAP}_\perp)$ have strict continous realizers, preserving relations

Other concrete models

(back-of-the-envelope calculations only)

- Rosolini and Simpson model (IST)
- Model based on Lily (Pitts et. al.)

Internal models

- the definition of LAPL-structure is fairly long
- for an internal LNL-model in a quasi-topos with a Lawvere-Tierney topology j, much of the LAPL-structure can be derived

Summary

- Precise definition of LAPL and proofs of expected properties
- Definition of parametric LAPL-structure, sound and complete
- Concrete model
- Ongoing and future work:
  - Adequacy of Lily-like language a la Rosolini and Simpson
  - Completion process
  - References and other computational effects