# BI Hyperdoctrines and Higher-Order Separation Logic

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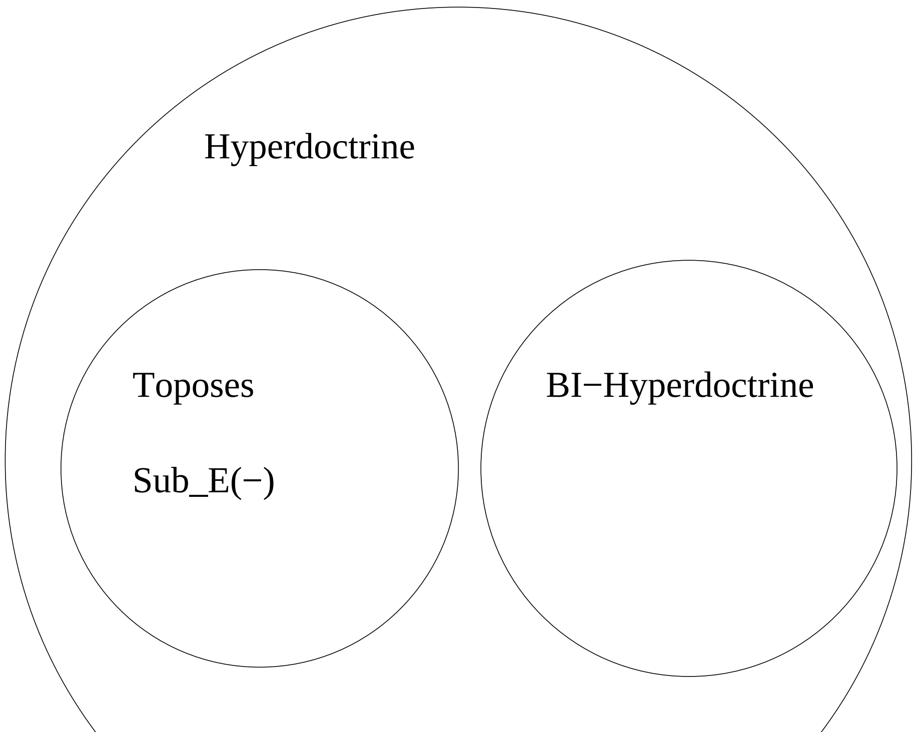
# Overview

- Earlier work [Pym, O'Hearn, et. al.] has established correspondence between a part of separation logic and propositional BI
- We extend the correspondence to full separation logic and a simple version of *predicate* BI, and, moreover, to *higher-order*
  - define a class of sound and complete models: BI Hyperdoctrines
  - show that one cannot simply use toposes as models
  - argue that higher-order separation logic is useful for formalizations of separation logic and for data abstraction

# BI Hyperdoctrines — Overview

- A hyperdoctrine is a categorical formalization of a model of predicate logic [Lawvere 1969]. Sound and complete for IHOL.
- Toposes also sound and complete for IHOL.
- BI Hyperdoctrines sound and complete for IHOL + BI

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# First-order Hyperdoctrines, I

Let $\mathcal{C}$ be a category with finite products. A *first-order hyperdoctrine* $\mathcal{P}$ over $\mathcal{C}$ is a contravariant functor $\mathcal{P} : \mathcal{C}^{op} \to \mathrm{Poset}$ s.t.:

- Each $\mathcal{P}(X)$ is a Heyting algebra.
- Each $\mathcal{P}(f) : \mathcal{P}(Y) \to \mathcal{P}(X)$ is a Heyting algebra homomorphism.
- There is an element $=_X$ of $\mathcal{P}(X \times X)$ satisfying that for all $A \in \mathcal{P}(X \times X)$,

$$\top \leq \mathcal{P}(\Delta_X)(A) \quad \text{iff} \quad =_X \leq A.$$

# First-order Hyperdoctrines, II

- For each product projection $\pi : \Gamma \times X \to \Gamma$ in $\mathcal{C}$, $\mathcal{P}(\pi) : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma \times X)$ has both a left adjoint $(\exists X)_\Gamma$ and a right adjoint $(\forall X)_\Gamma$:

$$A \leq \mathcal{P}(\pi)(A') \quad \text{if and only if} \quad (\exists X)_\Gamma(A) \leq A'$$

$$\mathcal{P}(\pi)(A') \leq A \quad \text{if and only if} \quad A' \leq (\forall X)_\Gamma(A).$$

Natural in $\Gamma$.

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# Interpretation in Hyperdoctrines

- Types and terms interpreted by objects and morphisms of $\mathcal{C}$
- Each formula $\phi$ with free variables in $\Gamma$ is interpreted as a $\mathcal{P}$-predicate $[\![\phi]\!] \in \mathcal{P}([\![\Gamma]\!])$ by induction on the structure of $\phi$ using definining properties of hyperdoctrine.
- A formula $\phi$ with free variables in $\Gamma$ is *satisfied* if $[\![\phi]\!]$ is $\top$ in $\mathcal{P}([\![\Gamma]\!])$.
- Sound and complete for intuitionistic predicate logic.
- A first-order hyperdoctrine is sound for *classical* predicate logic in case all the fibres $\mathcal{P}(X)$ are Boolean algebras and all the reindexing functions $\mathcal{P}(f)$ are Boolean algebra homomorphisms.

# Hyperdoctrines

A (general) *hyperdoctrine* is a first-order hyperdoctrine with the following additional properties:

- $\mathcal{C}$ is cartesian closed; and
- there is $H \in \mathcal{C}$ and a natural bijection $\Theta_X : Obj(\mathcal{P}(X)) \simeq \mathcal{C}(X, H)$.

Cartesian closure interprets higher types.

Type of propositions is interpreted by $H$.

# BI Hyperdoctrines

- Recall: A *BI algebra* is a Heyting algebra, which has an additional symmetric monoidal closed structure $(\mathrm{I}, *, -\!*)$
- Define: A first-order hyperdoctrine $\mathcal{P}$ over $\mathcal{C}$ is a *first-order BI hyperdoctrine* in case
  - all the fibres $\mathcal{P}(X)$ are BI algebras, and
  - all the reindexing functions $\mathcal{P}(f)$ are BI algebra homomorphisms
- Likewise for general BI hyperdoctrines.

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# First-order Predicate BI, I

- Predicate logic with equality extended with $\mathrm{I}$, $\phi * \psi$, $\phi \mathrel{-\!\!*} \psi$ satisfying the usual rules for BI (in any context $\Gamma$):

$$(\phi * \psi) * \theta \vdash_\Gamma \phi * (\psi * \theta) \qquad \phi * (\psi * \theta) \vdash_\Gamma (\phi * \psi) * \theta$$

$$\vdash_\Gamma \phi \leftrightarrow \phi * \mathrm{I} \qquad \phi * \psi \vdash_\Gamma \psi * \phi$$

$$\frac{\phi \vdash_\Gamma \psi \qquad \theta \vdash_\Gamma \omega}{\phi * \theta \vdash_\Gamma \psi * \omega} \qquad \frac{\phi * \psi \vdash_\Gamma \theta}{\phi \vdash_\Gamma \psi \mathrel{-\!\!*} \theta}$$

# First-Order Predicate BI, II

Notice

- No BI structure on contexts (in [Pym:2002] there is)
- In particular, weakening on the level of variables is always allowed

$$\frac{\phi \vdash_\Gamma \psi}{\phi \vdash_{\Gamma \cup \{x:X\}} \psi}$$

- Fine because simple and what we need for separation logic
- Can be interpreted in first-order BI hyperdoctrines
- **Theorem** The interpretation of first-order predicate BI is sound and complete.
- Also for classical predicate BI, of course

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# Higher-order Predicate BI

- Higher-order predicate logic extended with BI as above.
- BI hyperdoctrines sound and complete class of models.

# Example of BI hyperdoctrine

Let $B$ be a complete BI algebra. Define $\mathbf{Set}$-indexed BI hyperdoctrine:

- $\mathcal{P}(X) = B^X$, functions from $X$ to $B$, ordered pointwise
- For $f : X \to Y$, $\mathcal{P}(f) : B^Y \to B^X$ is comp. with $f$.
- $=_X (x, x')$ is $\top$ if $x = x'$, otherwise $\bot$.
- Quantification: for $A \in B^{\Gamma \times X}$

$$(\exists X)_\Gamma(A) \stackrel{def}{=} \lambda i \in \Gamma. \bigvee_{x \in X} A(i, x)$$
$$(\forall X)_\Gamma(A) \stackrel{def}{=} \lambda i \in \Gamma. \bigwedge_{x \in X} A(i, x)$$

in $B^\Gamma$.

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# Toposes and BI Hyperdoctrines

- Earlier work showed how to use some toposes to model propostional BI ($\mathrm{Sub}_{\mathcal{E}}(1)$ is a BI-algebra, for certain $\mathcal{E}$)
- Toposes model (higher-order) predicate logic, since $\mathrm{Sub}_{\mathcal{E}}$ is a hyperdoctrine.
- But, surprise, we cannot interpret predicate BI in toposes:

**Theorem** Let $\mathcal{E}$ be a topos and suppose $\mathrm{Sub}_{\mathcal{E}} : \mathcal{E}^{op} \to \mathrm{Poset}$ is a BI hyperdoctrine. Then the BI structure on each lattice $\mathrm{Sub}_{\mathcal{E}}(X)$ is trivial, i.e., for all $\varphi, \psi \in \mathrm{Sub}_{\mathcal{E}}(X)$, $\varphi * \psi \leftrightarrow \varphi \wedge \psi$.

# Higher-order Separation Logic

Next:

- Recall pointer model and interpretation of separation logic in pointer model
- Show how to view pointer model as a BI hyperdoctrine and that the standard interpretation therein coincides with standard interpretation of separation logic.
- Leads to obvious extension of separation logic to higher-order.
- Some implications thereof.

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# Pointer Model of Sep. Logic

- set $[\![\mathrm{Val}]\!]$ interpreting the type $\mathrm{Val}$
- set $[\![\mathrm{Loc}]\!]$ of locations with $[\![\mathrm{Loc}]\!] \subseteq [\![\mathrm{Val}]\!]$
- set of heaps $H = [\![\mathrm{Loc}]\!] \rightharpoonup_{fin} [\![\mathrm{Val}]\!]$, ordered discretely, with partial binary operation $*$ defined by

$$h_1 * h_2 = \begin{cases} h_1 \cup h_2 & \text{if } h_1 \# h_2 \\ \text{undefined} & \text{otherwise,} \end{cases}$$

- set $\mathrm{Var} \rightharpoonup_{fin} [\![\mathrm{Val}]\!]$ of stacks

# Standard Int. of Formulas

Given by a forcing relation $s, h \Vdash \phi$, where $\mathrm{FV}(\phi) \subseteq \mathrm{dom}(s)$:

$$
\begin{array}{lll}
s, h \Vdash t_1 = t_2 & \text{iff} & [\![t_1]\!]s = [\![t_2]\!]s \\
s, h \Vdash t_1 \mapsto t_2 & \text{iff} & \mathrm{dom}(h) = \{[\![t_1]\!]s\} \text{ and } h([\![t_1]\!]s) = [\![t_2]\!]s \\
s, h \Vdash \mathbf{emp} & \text{iff} & h = \emptyset \\
s, h \Vdash \phi * \psi & \text{iff} & \text{there exists } h_1, h_2 \in H.\, h_1 * h_2 = h \text{ and} \\
 & & s, h_1 \Vdash \phi \text{ and } s, h_2 \Vdash \psi \\
s, h \Vdash \phi \mathrel{-\!\!*} \psi & \text{iff} & \text{for all } h', h' \# h \text{ and } s, h' \Vdash \phi \text{ implies} \\
 & & \qquad s, h * h' \Vdash \psi \\
\ldots & & \\
s, h \Vdash \forall x.\phi & \text{iff} & \text{for all } v \in [\![\mathrm{Val}]\!].s[x \mapsto v], h \Vdash \phi
\end{array}
$$

# Separation Logic as a BI Hyp.

- $\mathcal{P}(H)$ is a complete Boolean BI algebra, ordered by inclusion.
- Let $S$ be the BI hyperdoctrine induced by the complete Boolean BI algebra
- **Theorem** $h \in [\![\phi]\!](v_1, \ldots, v_n)$ iff $[x_1 \mapsto v_1, \ldots, x_n \mapsto v_n], h \Vdash \phi$.
- (also works for other models of separation logic, e.g., intuitionistic and permissions models)

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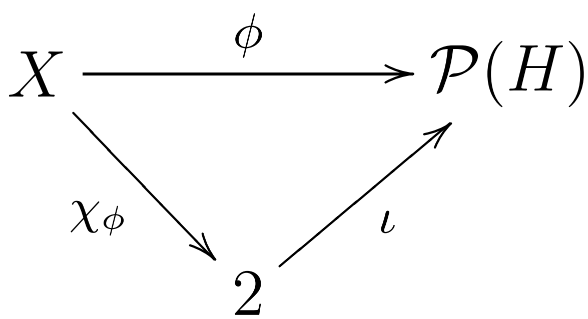
# Higher-order Sep. Logic

- The BI hyperdoctrine $S$ also gives a model of *higher-order* separation logic, with $\mathcal{P}(H)$ the set of truth values.
- Now consider some applications of higher-order.

# Formalization of Sep. Logic, I

- Applications of sep. logic have used various extensions, with sets of lists, trees, relations, etc.
- Our point here is that they can be seen as trivial definitional extensions, since they are all definable in higher-order logic.
- Let $2 = \{\bot, \top\}$. There is a canonical map $\iota : 2 \to \mathcal{P}(H)$. Say $\phi : X \to \mathcal{P}(H)$ is *pure* if there is a map $\chi_\phi : X \to 2$ s.t.




commutes.

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# Formalization of Sep. Logic, I

- The sub-logic of pure predicates is simply the standard classical higher-order logic of $\mathbf{Set}$.
- Allows to use classical higher-order logic for defining lists, trees, etc.
- In particular, recursive definitions of predicates, earlier done at the meta-level, can now be done inside the higher-order logic itself.

# Logical Characterizations. . .

of classes of formulas:

- Traditional definition of a *precise*: $q$ is precise iff, for $s, h$, there is at most one subheap $h_0$ of $h$ such that $s, h_0 \Vdash q$.
- **Prop.** $q$ is precise iff

$$\forall p_1, p_2 : \mathsf{prop} \, . \, (p_1 * q) \wedge (p_2 * q) \longrightarrow (p_1 \wedge p_2) * q$$

  is valid in the BI hyperdoctrine $S$.
- Thus: can make *logical* proofs about precise formulas.

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# Characterizations, II

- Traditional: $q$ is *monotone* iff whenever $h \in [\![q]\!]$ then also $h' \in [\![q]\!]$, for all extensions $h' \supseteq h$.
- **Prop.** $q$ is *monotone* iff

$$\forall p : \mathsf{prop} \,.\, p * q \longrightarrow q$$

is valid in the BI hyperdoctrine $S$.

- **Prop.** $q$ is *pure* iff

$$\forall p_1, p_2 : \mathsf{prop} \,.\, (q \wedge p_1) * p_2 \leftrightarrow q \wedge (p_1 * p_2)$$

is valid in the BI hyperdoctrine $S$.

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# Applications in Program Proving

- one can use existential quantification over hidden (abstract) resource invariants to reason about programs using abstract data types

$$\exists \alpha : \mathsf{prop}^{\mathsf{Int} \times \mathsf{Int}} . \{P\}\, k \{Q\} \vdash \{P'\}\; c' \; \{Q'\}$$

- polymorphic types using universal quantification (generic reasoning)
- more details in forthcoming paper

`www.itu.dk/people/birkedal/papers/hosl.ps.gz`

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# Strengthening

**Theorem** Let $\mathcal{P}$ be an indexed preorder, fibres all BI algebras, preserved under reindexing, with *full* subset types. Then the BI structure on each lattice $\mathcal{P}(X)$ is trivial, i.e., for all $\varphi, \psi \in \mathcal{P}(X)$, $\varphi * \psi \leftrightarrow \varphi \wedge \psi$.

- The BI hyperdoctrine for separation logic has subset types, but not *full* subset types.
- Full subset types:

$$\frac{y : \{x : X \mid \varphi\} \mid \theta \vdash \psi}{x : X \mid \theta, \varphi \vdash \psi}$$

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