# Static Program Analysis

## Part 5 – widening and narrowing

https://cs.au.dk/~amoeller/spa/

Anders Møller & Michael I. Schwartzbach
Computer Science, Aarhus University

# Interval analysis

- Compute upper and lower bounds for integers
- Possible applications:
  - array bounds checking
  - integer representation
  - …
- Lattice of intervals:

$$Interval = lift(\{\ [l, h]\ \mid l,h \in N \land l \leq h\ \})$$

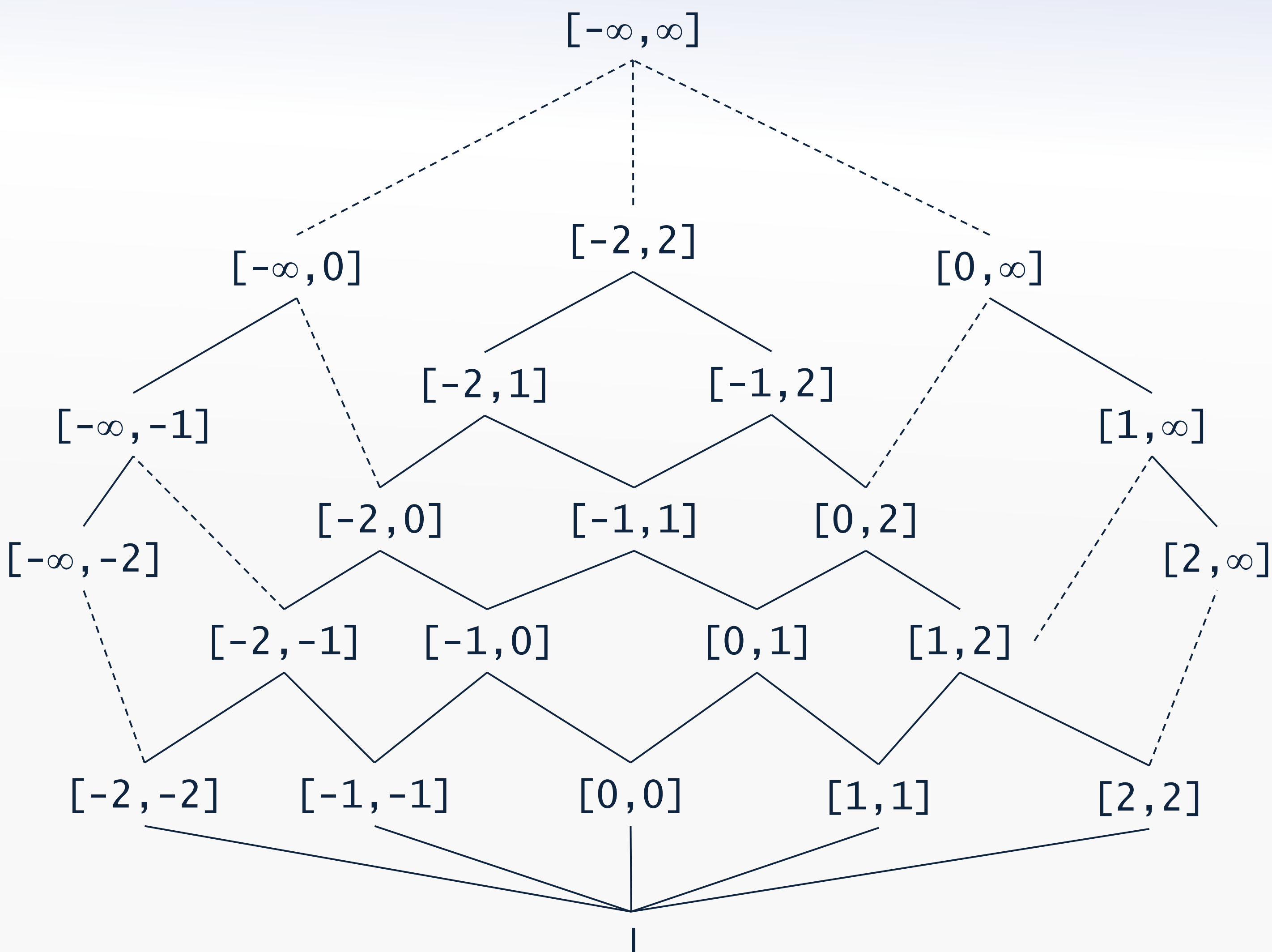
where

$$N = \{-\infty, \ldots, -2, -1, 0, 1, 2, \ldots, \infty\}$$

and intervals are ordered by inclusion:

$$[l_1, h_1] \sqsubseteq [l_2, h_2] \text{ iff } l_2 \leq l_1 \land h_1 \leq h_2$$

# The interval lattice




bottom element here interpreted as “not an integer”

# Interval analysis lattice

- The total lattice for a program point is

  $$\mathit{Var} \rightarrow \mathit{Interval}$$

  that provides bounds for each (integer) variable

- If using the worklist solver that initializes the worklist with only the *entry* node, use the lattice *lift*($\mathit{Var} \rightarrow \mathit{Interval}$)
  - bottom value of *lift*($\mathit{Var} \rightarrow \mathit{Interval}$) represents "unreachable program point"
  - bottom value of $\mathit{Var} \rightarrow \mathit{Interval}$ represents "maybe reachable, but all variables are non-integers"

- This lattice has *infinite height*, since the chain

  $$[0,0] \sqsubseteq [0,1] \sqsubseteq [0,2] \sqsubseteq [0,3] \sqsubseteq [0,4] \ldots$$

  occurs in *Interval*

# Interval constraints

- For assignments:

$$[\![ \, x = E \, ]\!] = JOIN(\mathrm{v})[x \rightarrow eval(JOIN(\mathrm{v}), E)]$$

- For all other nodes:

$$[\![\mathrm{v}]\!] = JOIN(\mathrm{v})$$

where $JOIN(\mathrm{v}) = \bigsqcup_{\mathrm{w} \in pred(\mathrm{v})} [\![\mathrm{w}]\!]$

# Evaluating intervals

- The *eval* function is an *abstract evaluation*:
  - $eval(\sigma, x) = \sigma(x)$
  - $eval(\sigma, intconst) = [intconst, intconst]$
  - $eval(\sigma, E_1 \text{ op } E_2) = \overline{\text{op}}(eval(\sigma,E_1),eval(\sigma,E_2))$
- Abstract operators:
  - $\overline{\text{op}}([l_1, h_1],[l_2, h_2]) =$

$$\Big[\min_{x\in[l_1,h_1],\, y\in[l_2,h_2]} x \text{ op } y,\ \max_{x\in[l_1,h_1],\, y\in[l_2,h_2]} x \text{ op } y\Big]$$

not trivial to implement!

# Fixed-point problems

- The lattice has infinite height,
  so the fixed-point algorithm does not work ☹

- The sequence of approximants

  $f^i(\bot)$ for $i = 0, 1, \ldots$

  is not guaranteed to converge
- (Exercise: give an example of a program where this happens)

- Restricting to 32 bit integers is not a practical solution
- *Widening* gives a useful solution…

# Does the least fixed point exist?

- The lattice has infinite height,
  so Kleene's fixed-point theorem does not apply ☹

- **Tarski's fixed-point theorem:**

In a complete lattice L, every monotone function $f: L \to L$ has a unique least fixed point given by

$$\mathrm{lfp}(f) = \sqcap\{\, x \in L \mid f(x) \sqsubseteq x \}$$

(Proof?)

# Widening

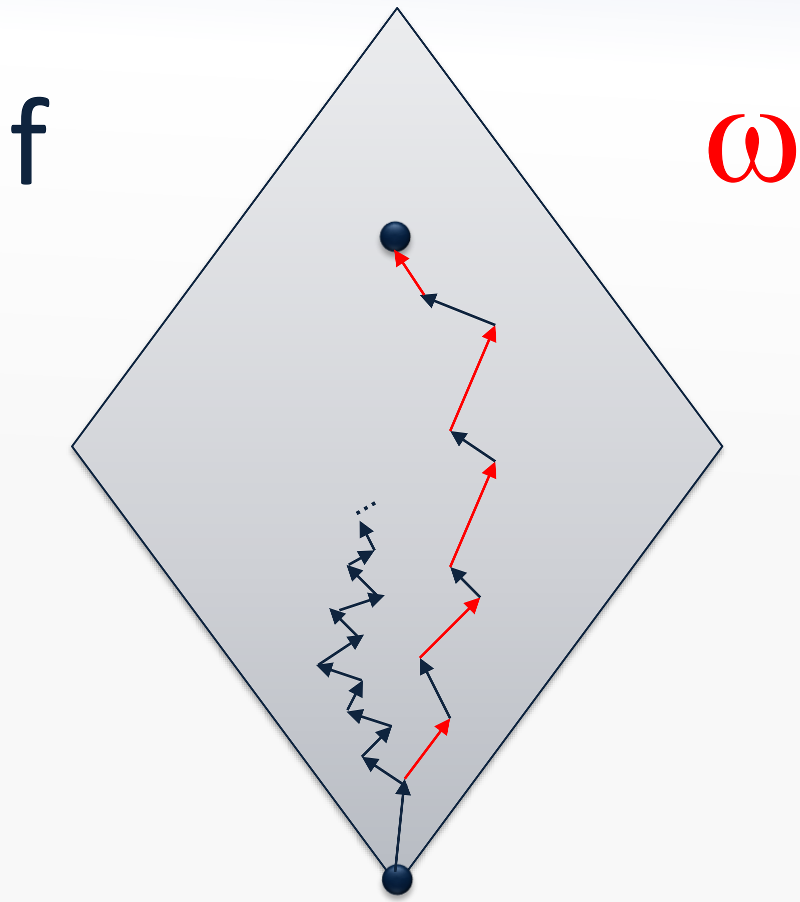
- Introduce a *widening* function $\omega$: L $\rightarrow$ L so that

  $(\omega \circ f)^i(\perp)$ for i = 0, 1, ...

  converges on a fixed point that is a safe approximation of each $f^i(\perp)$

- i.e. the function $\omega$ coarsens the information

# Turbo charging the iterations



# Simple widening for intervals

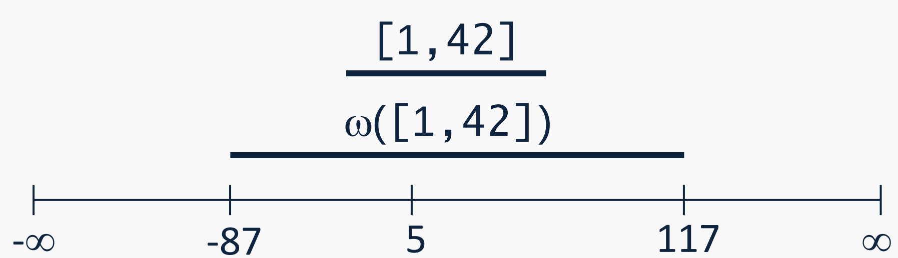
- The function $\omega$: L $\rightarrow$ L is defined pointwise on

  L = (*Var* $\rightarrow$ *Interval*)$^n$

- Parameterized with a fixed finite set *B*
  - must contain $-\infty$ and $\infty$ (to retain the $\top$ element)
  - typically seeded with all integer constants occurring in the given program
- Idea: Find the nearest enclosing allowed interval
- On single elements from *Interval*:

$$\omega([a,b]) = [\ max\{i\in B \mid i\leq a\},\ min\{i\in B \mid b\leq i\}\ ]$$

$$\omega(\bot) = \bot$$



# Divergence in action

```
y = 0;
x = 7;
x = x+1;
while (input) {
  x = 7;
  x = x+1;
  y = y+1;
}
```

$[x \to \perp, y \to \perp]$
$[x \to [8,8], y \to [0,1]]$
$[x \to [8,8], y \to [0,2]]$
$[x \to [8,8], y \to [0,3]]$
…

# Simple widening in action

```
y = 0;
x = 7;
x = x+1;
while (input) {
  x = 7;
  x = x+1;
  y = y+1;
}
```

$[x \to \perp, y \to \perp]$
$[x \to [7,\infty], y \to [0,1]]$
$[x \to [7,\infty], y \to [0,7]]$
$[x \to [7,\infty], y \to [0,\infty]]$

$B = \{-\infty, 0, 1, 7, \infty\}$

# Correctness of simple widening

- This form of widening works when:
  - $\omega$ is an *extensive* and *monotone* function, and
  - the sub-lattice $\omega(L)$ has *finite height*

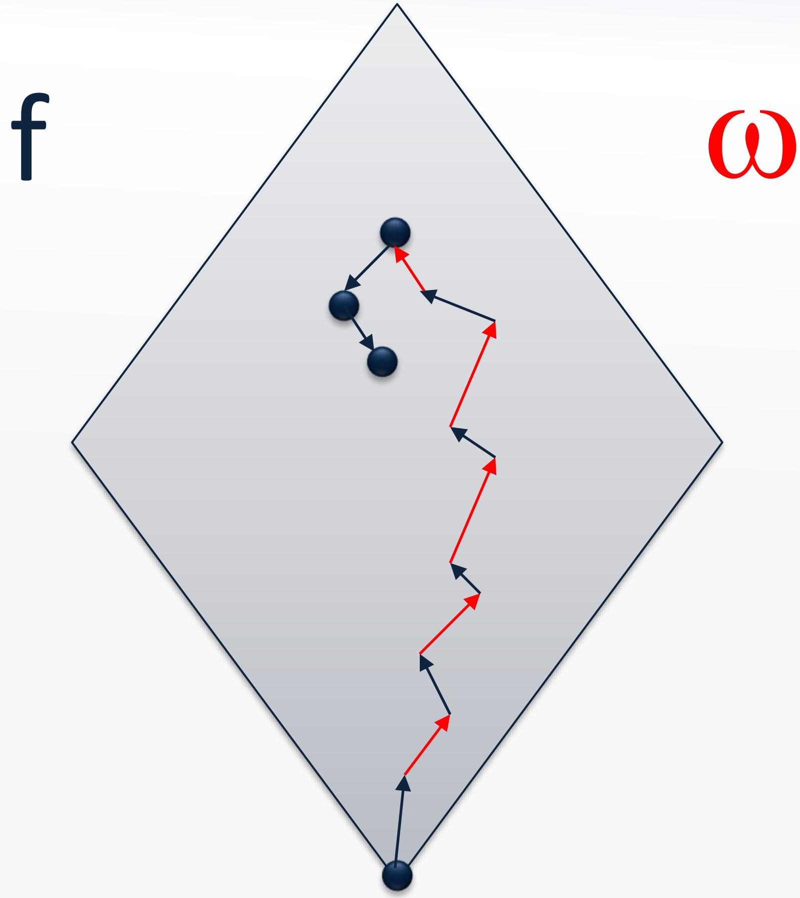
- $\omega \circ f$ is monotone and $\omega(L)$ has finite height, so $(\omega \circ f)^i(\bot)$ for $i = 0, 1, \ldots$ converges
- Let $f_\omega = (\omega \circ f)^k(\bot)$ where $(\omega \circ f)^k(\bot) = (\omega \circ f)^{k+1}(\bot)$
- $\mathrm{lfp}(f) \sqsubseteq f_\omega$ follows from Tarski's fixed-point theorem, i.e., $f_\omega$ is a safe approximation of $\mathrm{lfp}(f)$

# Narrowing

- Widening generally shoots over the target
- *Narrowing* may improve the result by applying f

- We have $f(f_\omega) \sqsubseteq f_\omega$ so applying f again may improve the result!
- And we also have $lfp(f) \sqsubseteq f(f_\omega)$ so it remains safe!
- This can be iterated arbitrarily many times
  - may diverge, but safe to stop anytime

# Backing up



# Narrowing in action

```
y = 0;
x = 7;
x = x+1;
while (input) {
  x = 7;
  x = x+1;
  y = y+1;
}
```

$[x \to \bot, y \to \bot]$
$[x \to [7,\infty], y \to [0,1]]$
$[x \to [7,\infty], y \to [0,7]]$
$[x \to [7,\infty], y \to [0,\infty]]$
…
$[x \to [8,8], y \to [0,\infty]]$

$B = \{-\infty, 0, 1, 7, \infty\}$

# Correctness of (repeated) narrowing

Claim: $\mathrm{lfp}(f) \sqsubseteq \ldots \sqsubseteq f^i(f_\omega) \sqsubseteq \ldots \sqsubseteq f(f_\omega) \sqsubseteq f_\omega$

- $f(f_\omega) \sqsubseteq \omega(f(f_\omega)) = (\omega \circ f)(f_\omega) = f_\omega$ since $\omega$ is extensive
  - by monotonicity of f and induction we also have, for all i:

    $f^{i+1}(f_\omega) \sqsubseteq f^i(f_\omega) \sqsubseteq f_\omega$
  - i.e. $f^{i+1}(f_\omega)$ is at least as precise as $f^i(f_\omega)$
- $f(f_\omega) \sqsubseteq f_\omega$ so $f(f(f_\omega)) \sqsubseteq f(f_\omega)$ by monotonicity of f, hence $\mathrm{lfp}(f) \sqsubseteq f(f_\omega)$ by Tarski's fixed-point theorem
  - by induction we also have, for all i:

    $\mathrm{lfp}(f) \sqsubseteq f^i(f_\omega)$
  - i.e. $f^i(f_\omega)$ is a safe approximation of $\mathrm{lfp}(f)$

# Some observations

- The simple notion of widening is a bit naive…
- Widening happens at *every* interval and at *every* node

- There’s no need to widen intervals
  that are not “unstable”
- There’s no need to widen
  if there are no “cycles” in the dataflow

# More powerful widening

- A *widening* is a function $\nabla: L \times L \to L$ that is extensive in both arguments and satisfies the following property:
  for all increasing chains $z_0 \sqsubseteq z_1 \sqsubseteq \ldots$,
  the sequence $y_0 = z_0, \ldots, y_{i+1} = y_i \nabla z_{i+1}, \ldots$ converges
  (i.e. stabilizes after a finite number of steps)

- Now replace the basic fixed point solver by computing $x_0 = \bot$ and $x_{i+1} = x_i \nabla f(x_i)$ until convergence

- Theorem: $x_{k+1} = x_k$ and $\mathrm{lfp}(f) \sqsubseteq x_k$ for some k

(Proof: similar to the correctness proof for simple widening)

# More powerful widening for interval analysis

Extrapolates unstable bounds to *B*:

$\perp \nabla y = y$

$x \nabla \perp = x$

$[a_1, b_1] \nabla [a_2, b_2] =$

$\quad [\text{if } a_1 \leq a_2 \text{ then } a_1 \text{ else } \mathit{max}\{i \in B \mid i \leq a_2\},$

$\quad \text{if } b_2 \leq b_1 \text{ then } b_1 \text{ else } \mathit{min}\{i \in B \mid b_2 \leq i\}]$

The $\nabla$ operator on L is then defined pointwise down to individual intervals

For the small example program, we get the same result as with simple widening plus narrowing (but now without using narrowing)

# Yet another improvement

- Divergence (e.g. in the interval analysis without widening) can only appear in presence of recursive dataflow constraint
- Sufficient to “break the cycles”, perform widening only at, for example, loop heads in the CFG

# Choosing the set *B*

- Defining the widening function based on constants occurring in the given program may not work well

```
f(x) { // "McCarthy's 91 function"
  var r;
  if (x > 100) {
    r = x - 10;
  } else {
    r = f(f(x + 11));
  }
  return r;
}
```

https://en.wikipedia.org/wiki/McCarthy_91_function

- (This example requires interprocedural and control-sensitive analysis)