Assignment 1
Hand in date: February 17, 2017

Exercise 1. Let $T$ be the following functor on $\text{Sets}$. It maps the set $X$ to its power set $\mathcal{P}(X)$ and it maps the function $f : X \to Y$ to the image function

$$T(f) : \mathcal{P}(X) \to \mathcal{P}(Y),$$

which is defined as

$$T(f)(A) = \{f(x) \mid x \in A\}.$$

• Show that $T$ is a functor from $\text{Sets}$ to $\text{Sets}$.

Exercise 2. Define the category $\mathbb{K}$ as follows. Its objects are sets. Morphisms $X \to Y$ in $\mathbb{K}$ are morphisms $X \to T(Y)$ in $\text{Sets}$, i.e.,

$$\text{Hom}_{\mathbb{K}}(X, Y) = \text{Hom}_{\text{Sets}}(X, T(Y)).$$

Composition is defined as follows: if $f : X \to Y$ and $g : Y \to Z$ are two morphisms in $\mathbb{K}$ then

$$(g \circ f)(x) = \bigcup_{y \in f(x)} g(y) = \{z \mid \exists y \in f(x), z \in g(y)\}.$$

• Show that $\mathbb{K}$ is a category.

• Show that it is isomorphic to the category of sets and relations $\text{Rel}$. Hint: Any subset $R \subseteq X \times Y$ can be represented as a function

$$F(R) : X \to \mathcal{P}(Y)$$

defined as

$$F(R)(x) = \{y \mid (x, y) \in R\}.$$

Exercise 3. Let $\mathcal{C}$ be a category with binary products.

• Is the projection $\pi_X : X \times Y \to X$ an epimorphism in general? Is it a monomorphism?

• Let $f : Z \to X$, $g : Z \to Y$, and $h : W \to Z$ be three morphisms. Show

$$\langle f, g \rangle \circ h = \langle f \circ h, g \circ h \rangle$$

as morphisms $W \to X \times Y$.

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• Let \( f : Z \to X \) and \( g : W \to Y \) be two morphisms. Show there exists a unique morphism \( u : Z \times W \to X \times Y \) such that for all objects \( A \) and morphisms \( h_Z : A \to Z \) and \( h_W : A \to W \)

\[ u \circ \langle h_Z, h_W \rangle = \langle f \circ h_Z, g \circ h_W \rangle. \]

as morphisms \( A \to X \times Y \). This unique morphism \( u \) is typically written as \( f \times g \).

• Using the notation from the previous item, show that for any morphisms \( f : Z \to X, g : W \to Y, h : A \to Z, \) and \( k : B \to W \) we have

\[ (f \times g) \circ (h \times k) = (f \circ h) \times (g \circ k) \]

as morphisms \( A \times B \to X \times Y \).