A constructive view on compact groups

constructive algebra applied to analysis

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- Constructive mathematics is a high-level programming language.
- Abstraction (algebra) and constructivity can be combined.

Weyl's concern

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Bishop showed that large parts of mathematics can naturally be reconstructed in a constructive way.
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In functional analysis this approach is natural.
Now we can prove constructively and naturally (an extension of) the Peter-Weyl theorem, one of Weyl’s most important contributions to mathematical physics.

Constructivism/Intuitionism

Weyl's concern was about Brouwer's intuitionism (INT). INT can be seen as an extension of Bishop's constructive mathematics (BISH).

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BISH

Another way of looking at this is: INT = BISH+'meta-theorems'. (Example: real induction.)

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$$\begin{tikzcd}
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 & \textbf{BISH} &
\end{tikzcd}$$

Another way of looking at this is: INT = BISH+'meta-theorems'. (Example: real induction.)

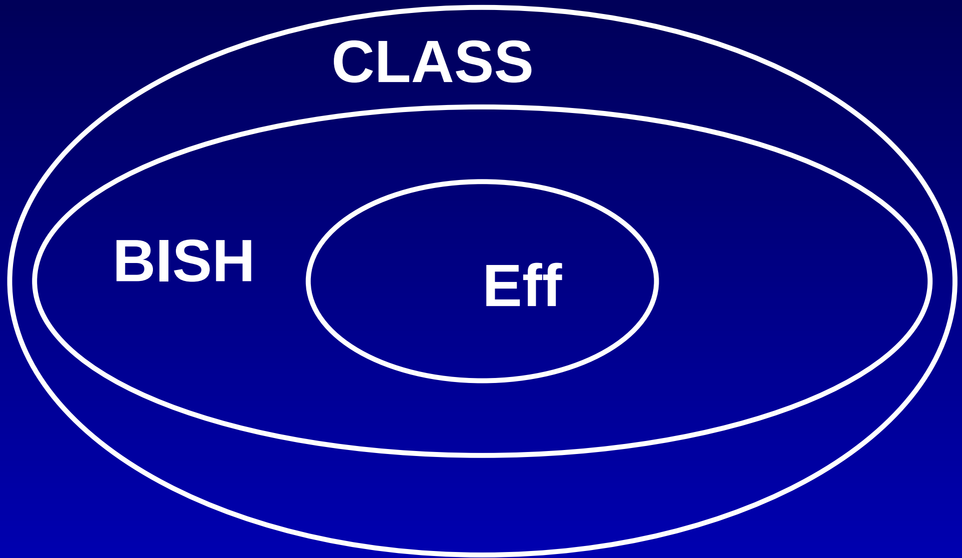
Basic objects of intuitionism: sequences of basic observables.

Only continuous functions.

Pointfree mathematics with points: Idealized objects (points, sequences,etc.) are only apparently present: a matter of speaking

Programming language

BISH can be formalized in constructive type theory. As such BISH is a very high level programming language. Does contain inefficient programs.



More importantly, usually the right picture for making actual computations possible. Makes clear which parts of a proof make non-computable decisions.
Interval arithmetic, exact real number computations.

Case-study: Peter-Weyl

G is a compact metric group.

Theorem Let π be a representation of a compact group G on a Hilbert space H. Then there are orthogonal finite dimensional subspaces H_i such that $H = \bigoplus_i H_i$ and π acts irreducibly on H_i.

non-commutative Fourier theorem.

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Need:

- Integration theory
- Haar measure
- Spectral theorem
- C*-algebras

Constructive integration theory

[Coquand/Palmgren] Boolean algebra $\mathcal{A}$ of basic observables with a measure μ.
The measure defines a metric on $\mathcal{A}$.
Complete this metric space to obtain a complete measure space. [metric completion instead of σ-completion.]

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One obtains L_1 as the completion of the simple functions $S(\mathcal{A})$.

$$\begin{array}{ccl} S(\mathcal{A}) & \longrightarrow & \mathcal{L}_1 \text{(all integrable functions)} \\ & & \downarrow \\ & & L_1 := \mathcal{L}_1/Null \end{array}$$

Alternatively, start with a positive linear functional on $C(X)$.

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Measurable functions

How to get the measurable functions?
[S] Complete the simple functions wrt metric

$$d(f,g) = \int |f-g| \wedge 1.$$

Again limit of basic observables.
We have:

- d-convergence is convergence in measure

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- etc

Haar measure

Theorem [Haar] There is a unique translation invariant measure on G s.t. $\mu(G) = 1$.

Proof [von Neumann/Coquand]
Let $C(G)$ be the space of continuous functions.
Define T_s by $(T_s f)(x) := f(sx)$, the left translation and $S_f := \{T_s f : s \in G\}$.
coS_f is totally bounded and the function $\sup : C(G) \to \mathbf{R}$ is continuous, so $m_f := \inf\{\sup g : g \in \text{co}S_f\}$ exists.
There is a unique constant function in the closure of coS_f, its value is m_f.
One can check that $\mu(f) := m_f$ defines the Haar measure.
$\square$

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- C*-algebras

Spectral theorems

$\mathcal{A}$ algebra, $a \in \mathcal{A}$, $\mathcal{F}$ set of functions $\mathbf{C} \to \mathbf{C}$, $f \in \mathcal{F}$.
Is it possible to define $f(a)$
s.t.

- $(\sum b_n z^n)(a) = \sum b_n a^n$?
- "continuous" in f

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Banach Algebra		Analytical functions
C*-algebra	$a^*a = aa^*$	Continuous functions

More spectral theorems

1. There is a basis of eigenvectors.
   Does not hold constructively (the eigenvectors do not depend continuously on the matrix elements).
2. Gelfand: Every separable Abelian C*-algebra $\mathcal{A}$ is isomorphic to $C(\sigma_A)$.
   σ_A is compact metric space, the spectrum of $\mathcal{A}$.

1 is a pointwise version and was used by Peter-Weyl.
2 was used by Bishop to prove the Fourier theorem.

Fourier theorem

For every compact Abelian group there is a dual group (its group of characters), denoted G^*. When G is compact, G^* is discrete.

Theorem [Fourier] There is an isometric isomorphism between $(L_2(G), *)$ and $(L_2(G^*), \cdot)$: the Fourier Transform.

Proof The closure of $(L_2(G), *)$ is a C*-algebra. Use Gelfand theorem. etc. $\square$

Corollary Every complex periodic function on the reals is a sum the functions $z \mapsto e^{cnz}$.

Proof A periodic function on the reals is a function on the circle Γ, which is a compact group. $\Gamma^* = \{z \mapsto e^{cnz} \,|\, n \in \mathbf{Z}\}$ $\square$

Convolution operators

Theorem The convolution operators $T(f) := f * g$ for $f \in L_1$ on L_2 are compact operators.

Proof Short intuitionistic proof, longer proof in BISH. $\square$

The norm of a compact operator can be computed. So the closure of the group algebra

$$\{T(f) \mid f \in L_1\}$$

is a C*-algebra.

Theorem The center of the group algebra of a cpt group is an Abelian C*-algebra. Its spectrum is a discrete countable group.

There is a continuous projection onto the center.

Peter-Weyl

Theorem [Peter-Weyl]Let π be a representation of a compact group G on a Hilbert space H. Then there are orthogonal finite dimensional subspaces H_i such that $H = \oplus_i H_i$ and π acts irreducibly on H_i.

Follows from:

Theorem The characters $\{\chi_i : i \in \mathbf{Z}\}$ form a complete orthogonal set in the center of $L_2(G)$. The maps $f \mapsto \chi_i * f$, are orthogonal projections on finite dimensional subspaces H_i and $L_2(G) = \oplus_i H_i$ and the spaces H_i are minimal twosided ideals.

Theorem The two-sided ideals are isomorphic to matrix algebras.

Proof Uses the spectral theorem again. $\square$

Application

- Representation theorem for almost periodic functions.

Main result of Brom's thesis, Bishop's student.
A function is *almost periodic* if $S_f = \{T(s)f : s \in \mathbf{R}\}$ is totally bounded.

Theorem Every almost periodic function $f : \mathbf{R} \to \mathbf{C}$ can be approximated (L_2 or uniform) by a finite linear combination of characters $\Gamma^* = \{z \mapsto e^{cnz} \mid n \in \mathbf{Z}\}$.

Proof Define the metric $d_f(a,b) := \sup_x |f(a+x) - f(b+x)|$.
Then the closure of $(\mathbf{R}, d_f)$ is a compact group.
Now apply Fourier theory this group. $\square$

Also non-commutative version.